5. From another quite different starting point, *Euler* obtained an interesting expansion for the cotangent which we proceed to deduce, especially as it is of great importance for many problems in series[24]. At the same time, it will give us the radius of convergence of the series **115** and **116** (v. **241**).

We have, as was just shewn,

$$\cot y = \frac{1}{2}\left(\cot\frac{y}{2} - \tan\frac{y}{2}\right)$$

or

$$(*) \qquad \cot\pi x = \frac{1}{2}\left\{\cot\frac{\pi x}{2} + \cot\frac{\pi(x\pm 1)}{2}\right\},$$

a formula in which we may, on the right, take either of the signs $\pm$. Let $x$ be *an arbitrary real number distinct from* $0, \pm 1, \pm 2, \ldots,$ whose value will reamain fixed in what follows. Then

$$\pi x\cot\pi x = \frac{\pi x}{2}\left\{\cot\frac{\pi x}{2} + \cot\frac{\pi(x+1)}{2}\right\}$$

and applying the formula (*) once more to both functions on the right had side, taking for the first the $+$ and for the second, the $-$ sign, we obtain

$$\pi x\cot\pi x = \frac{\pi x}{4}\left\{\cot\frac{\pi x}{4} + \left[\cot\frac{\pi(x+1)}{4} + \cot\frac{\pi(x-1)}{4}\right] + \cot\frac{\pi(x+2)}{4}\right\}.$$

A third similar step gives, for $\pi x\cot\pi x$, the value

$$\frac{\pi x}{8}\left\{\begin{array}{ccccc} & +\cot\frac{\pi(x+1)}{8} & +\cot\frac{\pi(x+2)}{8} & +\cot\frac{\pi(x+3)}{8} & \\ \cot\frac{\pi x}{8} & & \bullet & & +\cot\frac{\pi(x+4)}{8} \\ & +\cot\frac{\pi(x-1)}{8} & +\cot\frac{\pi(x-2)}{8} & +\cot\frac{\pi(x-3)}{8} & \end{array}\right\},$$

since here each pair of terms which occupy symmetrical positions relatively to the centre ($\bullet$) of the aggregate in the curly brackets give, except for a factor $\frac{1}{2}$, a term of the precdeing aggregate, in accordance with the formula (*). If we proceed thus through $n$ stages, we obtain for $n > 1$

$$(\dagger) \qquad \pi x\cot\pi x = \frac{\pi x}{2^n}\left\{\cot\frac{\pi x}{2^n} + \sum_{\nu=1}^{2^{n-1}-1}\left[\cot\frac{\pi(x+\nu)}{2^n} + \cot\frac{\pi(x-\nu)}{2^n}\right] - \tan\frac{\pi x}{2^n}\right\}.$$

Now by **115**,

$$\lim_{z\to 0} z\cot z = 1$$

and hence for each $\alpha \neq 0$

$$\lim_{n\to\infty}\frac{1}{2^n}\cot\frac{\alpha}{2^n} = \frac{1}{\alpha};$$

if in the above expression we let $n\to\infty$ and, *at first tentatively,* carry out the limiting process for each term separately, we obtain the expansion

$$\pi x\cot\pi x = 1 + x\sum_{\nu=1}^{\infty}\left(\frac{1}{x+\nu} + \frac{1}{x-\nu}\right) - 0 = 1 + 2x^2\sum_{\nu=1}^{\infty}\frac{1}{x^2-\nu^2}.$$

We proceed to show that this in general *faulty mode of passage to the limit* has, however, led in this case to a right result.

[24] The following considerable simplification of *Euler's* method for obtaining the expansion is due to *Schröter* (Ableitung der Partialbruch- unde Produktentwicklungen für die trigonometrischen Funktionen. Zeitschrift für Math. u. Phys., Vol. 13, p. 254. 1868).

We first note that the series converges absolutely for every $x \neq \pm 1, \pm 2, \ldots$, by **70**, 4, since the absolute values of its terms are asymptotically equal to those of the series $\sum \frac{1}{\nu^2}$. Now choose an arbitary integer $k > 6|x|$, to be kept provisionally fixed. If $n$ is so large that the number $2^{n-1} - 1$, which we will denote for short by $m$, is $> k$, we then split up the expression (†) for $\pi x \cot \pi x$, as follows[25]:

$$\pi x \cot \pi x = \frac{\pi x}{2^n} \left\{ \cot \frac{\pi x}{2^n} - \tan \frac{\pi x}{2^n} + \sum_{\nu=1}^{k} [\cdots] \right\} + \frac{\pi x}{2^n} \left\{ \sum_{\nu=k+1}^{m} [\cdots] \right\}.$$

(In the square brackets we have of course to insert the same expression as occurs in (†).) The two parts of this expression we denote by $A_n$ and $B_n$. Since $A_n$ consists of a finite number of terms, the passage to the limit term by term is certainly allowed there, by **41**, 9, and we have

$$\lim_{n\to\infty} A_n = 1 + 2x^2 \sum_{\nu=1}^{k} \frac{1}{x^2 - \nu^2}.$$

Also $B_n$ is preciesely $\pi x \cot \pi x - A_n$, hence $\lim B_n$ certainly exists. Let $r_k$ denote its value, depending as it does upon the chosen value $k$ thus

$$\lim_{n\to\infty} B_n = r_k = \pi x \cot \pi x - \left[ 1 - 2x^2 \sum_{\nu=1}^{k} \frac{1}{x^2 - \nu^2} \right].$$

Bounds above for the numbers $B_n$, hence for their limit $r_k$ and so finally for the difference on the right hand side, may now quite easily be estimated:

We have

$$\cot(a+b) + \cot(a-b) = \frac{-2\cot a}{\frac{\sin^2 b}{\sin^2 a} - 1}$$

and hence

$$\cot \frac{\pi(x+\nu)}{2^n} + \cot \frac{\pi(x-\nu)}{2^n} = \frac{-2\cot \alpha}{\frac{\sin^2 \beta}{\sin^2 \alpha} - 1}$$

writing for the moment $\frac{\pi x}{2^n} = \alpha$ and $\frac{\pi \nu}{2^n} = \beta$, for short.

As $2^n > k > 6|x|$, we certainly have $|a| = \left|\frac{\pi x}{2^n}\right| < 1$ and so[26]

$$|\sin \alpha| = \left| \alpha - \frac{\alpha^3}{3!} + \cdots \right| \leq |\alpha| \left( 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots \right) < 2|\alpha|.$$

Since, further $0 < \beta < \frac{\pi}{2} < 2$, we have[27]

$$\sin \beta = \beta \left( 1 - \frac{\beta^2}{2 \cdot 3} \right) + \frac{\beta^5}{5!} \left( 1 - \frac{\beta^2}{6 \cdot 7} \right) + \cdots > \frac{\beta}{3}.$$

Hence

$$\left| \frac{\sin \beta}{\sin \alpha} \right| > \frac{\beta}{6|\alpha|} = \frac{\nu}{6|x|} > 1,$$

the latter, because $\nu > k > 6|x|$. It therefore follows that (for $\nu > k$)

$$\left| \cot \frac{\pi(x+\nu)}{2^n} + \cot \frac{\pi(x-\nu)}{2^n} \right| \leq \frac{2\left|\cot \frac{\pi x}{2^n}\right|}{\frac{\nu^2}{36\,x^2} - 1}$$

[25] Cf. Footnote 7, p. 194.

[26] For the sake of later applications we make these estimates in the above rough form.

[27] Cf. p. 200.

and hence

$$|B_n| \le \left|\frac{\pi x}{2^n} \cot \frac{\pi x}{2^n}\right| \cdot \sum_{\nu=k+1}^{m} \frac{72\,x^2}{\nu^2 - 36\,x^2}.$$

The factor outside the sign of summation is – quite roughly estimated – certainly $< 3$; for $\left|\frac{\pi x}{2^n}\right|$ was $< 1$, and for $|z| < 1$ we have[26]

$$|z \cot z| = \left|\frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \cdots}\right| < \frac{1 + \frac{1}{2!} + \frac{1}{4!} - + \cdots}{1 - \frac{1}{3!} - \frac{1}{5!} - + \cdots} < 3.$$

Accordingly,

$$|B_n| < 216\,x^2 \cdot \sum_{\nu=k+1}^{m} \frac{1}{\nu^2 - 36\,x^2} < 216\,x^2 \cdot \sum_{\nu=k+1}^{\infty} \frac{1}{\nu^2 - 36\,x^2}.$$

But this is a number quite independent of $n$, so that we may also write

$$|\lim B_n| = |r_k| \le 216\,x^2 \cdot \sum_{\nu=k+1}^{\infty} \frac{1}{\nu^2 - 36\,x^2}.$$

But the bound above which we have thus obtained for $r_k$ is equal to the remainder, after the $k^{\text{th}}$ term, of a convergent series[28]. Given $\varepsilon = 0$, we can therefore choose $k_0$ so large that, for every $k > k_0$, we have

$$|r_k| < \varepsilon.$$

If we refer back to the meaning of $r_k$, we see that this implies

$$\lim_{k\to\infty} \left\{\pi x \cot \pi x - \left[1 + 2x^2 \sum_{n=1}^{k} \frac{1}{x^2 - \nu^2}\right]\right\} = 0,$$

or as asserted,

$$\pi x \cot \pi x = 1 + 2x^2 \sum_{\nu=1}^{\infty} \frac{1}{x^2 - \nu^2}, \qquad \textbf{117.}$$

– a formula which is thus proved valid for every $x \ne 0, \pm 1, \pm 2, \cdots$.

[28] The convergence is obtained just as simply as, previously, that of the series $\sum \frac{1}{x^2 - \nu^2}$.