Deriving Natural Deduction Rules from Truth Tables

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Abstract. We develop a general method for deriving natural deduction rules from the truth table for a connective. The method applies to both constructive and classical logic. This implies we can derive "constructively valid" rules for any classical connective. We show this constructive validity by giving a general Kripke semantics, that is shown to be sound and complete for the constructive rules. For the well-known connectives ($\vee$, $\wedge$, $\rightarrow$, $\neg$) the constructive rules we derive are equivalent to the natural deduction rules we know from Gentzen and Prawitz. However, they have a different shape, because we want all our rules to have a standard "format", to make it easier to define the notions of cut and to study proof reductions. In style they are close to the "general elimination rules" studied by Von Plato [13] and others. The rules also shed some new light on the classical connectives: e.g. the classical rules we derive for $\rightarrow$ allow to prove Peirce's law. Our method also allows to derive rules for connectives that are usually not treated in natural deduction textbooks, like the "if-then-else", whose truth table is clear but whose constructive deduction rules are not. We prove that "if-then-else", in combination with $\bot$ and $\top$, is functionally complete (all other constructive connectives can be defined from it). We define the notion of cut, generally for any constructive connective and we describe the process of "cut-elimination".

1 Introduction

Natural deduction rules come in various forms, where one either uses formulas A, or sequents $\Gamma \vdash A$ (where Γ is a sequence or a finite set of formulas). Other formalisms use a linear format, using flags or boxes to explicitly manage the open and discharged assumptions.

We use a tree format with sequents, where all rules have a special form:

$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n \qquad \Gamma, B_1 \vdash D \quad \dots \quad \Gamma, B_m \vdash D}{\Gamma \vdash D}$$

So if the conclusion of a rule is $\Gamma \vdash D$, then the hypotheses of the rule can be of one of two forms:

1. $\Gamma \vdash A$: instead of proving D from Γ, we now need to prove A from Γ. We call A a Lemma.

2. $\Gamma, B \vdash D$: we still need to prove D from Γ, but we are now also allowed to use B as additional assumption. We call B a Casus.

One obvious advantage is that we don't have to give the Γ explicitly, as it can be retrieved from the other information in a deduction. So, we will present the deduction rules without the Γ in the format

$$\frac{\vdash A_1 \quad \dots \quad \vdash A_n \qquad B_1 \vdash D \quad \dots \quad B_m \vdash D}{\vdash D}$$

For every connective we have elimination rules and introduction rules. The elimination rules have the following form, where φ is the formula that is eliminated and A_i, B_j are direct subformulas of φ.

$$\frac{\vdash \varphi \quad \vdash A_1 \quad \dots \quad \vdash A_n \qquad B_1 \vdash D \quad \dots \quad B_m \vdash D}{\vdash D}\,\mathrm{el}$$

The introduction rules have a classical and an intuitionistic form; the following form is the classical one. (φ is the formula that is "introduced" and A_i, B_j are direct subformulas of φ.) The classical duality between elimination and introduction is clearly visible from these rules.

$$\frac{\varphi \vdash D \quad \vdash A_1 \quad \dots \quad \vdash A_n \qquad B_1 \vdash D \quad \dots \quad B_m \vdash D}{\vdash D}\,\mathrm{in}^c$$

The intuitionistic introduction rules have the following form

$$\frac{\vdash A_1 \quad \dots \quad \vdash A_n \qquad B_1 \vdash \varphi \quad \dots \quad B_m \vdash \varphi}{\vdash \varphi}\,\mathrm{in}^i$$

We see that, compared to the classical rule, the D has been replaced by φ, the formula we introduce, and we have omitted the first premise, which is $\varphi \vdash \varphi$, because it is trivial. For each connective, we extract the introduction and elimination rules from a truth table as described in the following Definition.

Definition 1 *Suppose we have an n-ary connective c with a truth table t_c (with 2^n rows). We write $\varphi = c(A_1, \dots, A_n)$ for a formula with c as main connective and $A_1, \dots, A_n$ as immediate subformulas. Each row of t_c gives rise to an elimination rule or an introduction rule for c in the following way.*

$$\begin{array}{ccc|c} A_1 & \dots & A_n & \varphi \\ \hline p_1 & \dots & p_n & 0 \end{array} \mapsto \frac{\vdash \varphi \quad \dots \vdash A_j \ (\textit{if } p_j = 1) \dots \quad \dots A_i \vdash D \ (\textit{if } p_i = 0) \dots}{\vdash D}\,el$$

$$\begin{array}{ccc|c} A_1 & \dots & A_n & \varphi \\ \hline q_1 & \dots & q_n & 1 \end{array} \mapsto \frac{\dots \vdash A_j \ (\textit{if } q_j = 1) \dots \quad \dots A_i \vdash \varphi \ (\textit{if } q_i = 0) \dots}{\vdash \varphi}\,in^i$$

$$\begin{array}{ccc|c} A_1 & \dots & A_n & \varphi \\ \hline r_1 & \dots & r_n & 1 \end{array} \mapsto \frac{\varphi \vdash D \quad \dots \vdash A_j \ (\textit{if } r_j = 1) \dots \quad \dots A_i \vdash D \ (\textit{if } r_i = 0) \dots}{\vdash D}\,in^c$$

If $p_j = 1$ in t_c, then A_j occurs as a Lemma *in the rule; if $p_i = 0$ in t_c, then A_i occurs as a* Casus*. The rules are given in abbreviated form and it should be* *understood that all judgments can be used with an extended hypotheses set Γ. So the elimination rule in full reads as follows (where Γ is a set of formulas).*

$$\frac{\Gamma \vdash \varphi \quad \ldots \Gamma \vdash A_j \text{ (if } p_j = 1) \ldots \quad \ldots \Gamma, A_i \vdash D \text{ (if } p_i = 0) \ldots}{\Gamma \vdash D}\ el$$

Definition 2 *Given a set of connectives $\mathcal{C} := \{c_1, \ldots, c_n\}$, we define the* intuitionistic *and* classical *natural deduction systems for $\mathcal{C}$, $\mathsf{IPC}_\mathcal{C}$ and $\mathsf{CPC}_\mathcal{C}$ as follows.*

- *Both $\mathsf{IPC}_\mathcal{C}$ and $\mathsf{CPC}_\mathcal{C}$ have an* axiom rule

$$\frac{}{\Gamma \vdash A}\ axiom(\text{ if } A \in \Gamma)$$

- *$\mathsf{IPC}_\mathcal{C}$ has the elimination rules for the connectives in $\mathcal{C}$ and the intuitionistic introduction rules for the connectives in $\mathcal{C}$, as defined in Definition 1.*
- *$\mathsf{CPC}_\mathcal{C}$ has the elimination rules for the connectives in $\mathcal{C}$ and the classical introduction rules for the connectives in $\mathcal{C}$, as defined in Definition 1.*

Example 3 *From the truth table we derive the following intuitionistic rules for $\wedge$, 3 elimination rules and one introduction rule:*

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \wedge\text{-}el_a \qquad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D}\ \wedge\text{-}el_b$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D}\ \wedge\text{-}el_c \qquad \frac{\vdash A \quad \vdash B}{\vdash A \wedge B}\ \wedge\text{-}in$$

These rules are all intuitionistically correct, as one can observe by inspection. We will show that these are equivalent to the well-known intuitionistic rules. We will also show how these rules can be optimized and be reduced to 2 elimination rules and 1 introduction rule.

From the truth table we also derive the following rules for $\neg$, 1 elimination rule and 1 introduction rule, a classical and an intuitionistic one.

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\ \neg\text{-}el \qquad \frac{A \vdash \neg A}{\vdash \neg A}\ \neg\text{-}in^i \qquad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D}\ \neg\text{-}in^c$$

As an example of the classical derivation rules we show that $\neg\neg A \vdash A$ is derivable:

$$\frac{\dfrac{\neg\neg A, \neg A \vdash \neg\neg A \quad \neg\neg A, \neg A \vdash \neg A}{\neg\neg A, \neg A \vdash A}\ \neg\text{-}el \qquad \neg\neg A, A \vdash A}{\neg\neg A \vdash A}\ \neg\text{-}in^c$$

It can be proven that $\neg\neg A \vdash A$ is not derivable with the intuitionistic rules. As an example of the intuitionistic derivation rules we show that $A \vdash \neg\neg A$ is derivable:

$$\frac{\dfrac{A, \neg A \vdash \neg A \quad A, \neg A \vdash A}{A, \neg A \vdash \neg\neg A}\ \neg\text{-}el}{A \vdash \neg\neg A}\ \neg\text{-}in^i$$

In the intuitionistic case, there is an obvious notion of *cut*: an intro of φ immediately followed by an elimination of φ. In such case there is at least one k for which $p_k \neq q_k$. In case $p_k = 0, q_k = 1$, we have a sub-derivation Σ of $\vdash A_k$ and a sub-derivation Θ of $A_k \vdash D$ and we can "plug" Σ on top of Θ to obtain a derivation of $\vdash D$. In case $p_k = 1, q_k = 0$, we have a sub-derivation Σ of $A_k \vdash \varphi$ and a sub-derivation Θ of $\vdash A_k$ and we can "plug" Θ on top of Σ to obtain a derivation of $\vdash \varphi$. This is then used as a hypothesis for the elimination rule (that remains in this case) instead of the original one that was a consequence of the introduction rule (that now disappears). Note that in general there are more such k, so the cut-elimination procedure is non-deterministic. We view this non-determinism as a natural feature in natural deduction; the fact that for some connectives (or combination of connectives), cut-elimination is deterministic is an "emerging" property.

1.1 Contribution of the paper and related work

The main contributions of the paper are:

- A general construction of natural deduction rules for a logical connective from its truth table semantics, yielding natural deduction rules in a fixed structured format.
- The method applies to both a classical and a constructive (!) reading of the connectives, and applies to connectives of any arity.
- Soundness and completeness of the constructive connectives with respect to a general Kripke semantics that we define.
- Example of the if-then-else connective, which is shown to be constructively functionally complete, once the constants $\top$ and $\bot$ have been added.
- A general definition of "direct cut" and "elimination of a direct cut" for the generalized constructive connectives.

Natural deduction has been studied extensively, since the original work by Gentzen, both for classical and intuitionistic logic. Overviews can be found in [12] and [7]. Also the generalization of natural deduction to include other connectives or allow different derivation rules has been studied by various researchers. Notably, there is the work of Schroeder-Heister [10], Von Plato [13], Milne [6] and Francez and Dyckhoff [4, 3] is related to ours. Schroeder-Heister studies general formats of natural deduction where also rules may be discharged (as opposed to the normal situation where only formulas may be discharged). He also studies a general rule format for intuitionistic logic and shows that the connectives $\wedge, \vee, \rightarrow, \bot$ are complete for it. Von Plato, Milne, Francez and Dyckhoff study "generalized elimination rules", where the idea is that elimination rules arise naturally from the introduction rules, following Prawitz's [9] inversion principle: "the conclusion obtained by an elimination does not state anything more than what must have already been obtained if the major premiss of the elimination was inferred by an introduction".

A difference is that we focus not so much on the rules but on the fact that we can define different and new connectives constructively. In our work, we do not take the introduction rules as primary, with the elimination rules defined from them, but we derive elimination and introduction rules directly from the truth table. Then we optimize them, which can be done in various ways, where we adhere to a fixed format for the rules. Many of the generalized elimination rules, for example for $\wedge$, appear naturally as a consequence of our approach of deriving the rules from the truth table.

The idea of deriving deduction rules from the truth table also occurs in the work of Milne [6], for the classical case: from the introduction rules, a truth table is derived and then the elimination rules are derived from the truth table. For the if-then-else connective, this amounts to classical rules equivalent to ours (see Section 2.1), but less optimized. We start from the truth table and also derive rules for constructive logic.

In Section 3 we give a complete Kripke semantics for the constructive connectives. This is reminiscent of the *Lindenbaum construction* used in [6] to prove classical completeness. The Kripke semantics also allows us to prove some meta properties about the rules. For example, we give a generalization of the *disjunction property* in intuitionistic logic. In Section 4 we define cuts precisely, for the intuitionistic case.

2 Simple properties and examples

We first define precisely how the "plugging one derivation in another" works.

Lemma 4 *If $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$*

Proof. By a simple induction on the derivation of $\Delta, \varphi \vdash \psi$, using the fact that, in general (for all Γ, Γ' and φ): If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \varphi$. $\square$

We can be a bit more precise about what is happening in the proof of Lemma 4. If Π is the derivation of $\Delta, \varphi \vdash \psi$, due to the format of our rules, the only place in Π where the hypothesis φ can be used is at a leaf of Π, in an instance of the (axiom) rule. These leaves are of the shape $\Delta', \varphi \vdash \varphi$ for some $\Delta' \supseteq \Delta$.

If Σ is the derivation of $\Gamma \vdash \varphi$, then Σ is also a derivation of $\Delta', \Gamma \vdash \varphi$ (for any Δ). So, we can replace each leaf of Π that is an instance of an axiom $\Delta', \varphi \vdash \varphi$ by a derivation Σ of $\Delta', \Gamma \vdash \varphi$, to obtain a derivation of $\Gamma, \Delta \vdash \psi$. We introduce some notation to support this.

Notation 5 *If Σ is a derivation of $\Gamma \vdash \varphi$ and Π is a derivation of $\Delta, \varphi \vdash \psi$, then we have a derivation of $\Gamma, \Delta \vdash \psi$ that looks like this:*

$$\begin{array}{c} \begin{array}{ccc} \vdots\,\Sigma & & \vdots\,\Sigma \\ \Gamma, \Delta_1 \vdash \varphi & \dots & \Gamma, \Delta_n \vdash \varphi \end{array} \\ \vdots\,\Pi \\ \Gamma, \Delta \vdash \psi \end{array}$$

So in Π, every application of an (axiom) rule at a leaf, deriving $\Delta' \vdash \varphi$ for some $\Delta' \supseteq \Delta$ is replaced by a copy of a derivation Σ, which is also a derivation of $\Delta', \Gamma \vdash \varphi$.

In Definitions 1 and 2, we have given the precise rules for our logic, in intuitionistic and classical format. We can freely reuse formulas and weaken the context, so the structural rules of contraction and weakening are wired into the system. To reduce the number of rules, we can take a number of rules together and drop one or more hypotheses. We illustrate this by again looking at the example of the rules for $\wedge$ (Example 3).

Example 6 *From the truth table we have derived the* 3 *intuitionistic elimination rules of Example 3. These rules can be reduced to the following* 2 *equivalent elimination rules:*

$$\frac{\vdash A \wedge B \quad A \vdash D}{\vdash D}\wedge\text{-}el_1 \qquad \frac{\vdash A \wedge B \quad B \vdash D}{\vdash D}\wedge\text{-}el_2$$

The general method is that we can replace two rules that only differ in one hypothesis, which in one rule occurs as a Lemma and in the other as a Casus, by one rule where the hypothesis is removed. It will be clear that the Γ's above are not relevant for the argument, so we will not write these.

Lemma 7 *A system with two derivation rules of the form*

$$\frac{\vdash A_1 \ldots \vdash A_n \quad B_1 \vdash D \ldots B_m \vdash D \quad C \vdash D}{\vdash D} \qquad \frac{\vdash A_1 \ldots \vdash A_n \quad \vdash C \quad B_1 \vdash D \ldots B_m \vdash D}{\vdash D}$$

is equivalent to the system with these two rules replaced by

$$\frac{\vdash A_1 \ \ldots \ \vdash A_n \quad B_1 \vdash D \ \ldots \ B_m \vdash D}{\vdash D}$$

Proof. The implication from bottom to top is immediate. From top to bottom, suppose we have the two given rules. We now derive the bottom one. Assume we have derivations of $\vdash A_1, \ldots, \vdash A_n, B_1 \vdash D, \ldots, B_m \vdash D$. We now have the following derivation of $\vdash D$.

$$\frac{\vdash A_1 \ldots \vdash A_n \quad B_1 \vdash D \ldots B_m \vdash D \qquad \dfrac{C \vdash A_1 \ldots C \vdash A_n \quad C \vdash C \quad C, B_1 \vdash D \ldots C, B_m \vdash D}{C \vdash D}}{\vdash D}$$

Similarly, we can replace a rule which has only one Casus by a rule where the Casus is the conclusion. To illustrate this: the simplified elimination rules for $\wedge$, $\wedge$-el$_1$ and $\wedge$-el$_2$ have only one Casus. The rule $\wedge$-el$_1$ (left) can thus be replaced by the rule $\wedge$-el$'_1$ (right), which is the usual projection rule.

$$\frac{\vdash A \wedge B \quad A \vdash D}{\vdash D}\wedge\text{-el}_1 \qquad \frac{\vdash A \wedge B}{\vdash A}\wedge\text{-el}'_1$$

There is a general Lemma stating this simplification is correct. The proof is similar to the proof of Lemma 4.

Lemma 8 *A system with a derivation rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.*

$$\frac{\vdash A_1 \; \ldots \; \vdash A_n \quad \psi \vdash D}{\vdash D} \qquad\qquad \frac{\vdash A_1 \; \ldots \; \vdash A_n}{\vdash \psi}$$

Definition 9 *The derivation rules for the standard intuitionistic connectives are the following. These rules are derived from the truth tables and optimized following Lemmas 7 and 8. The rules for* $\wedge$ *are the intro rule of Example 3 and the elimination rules of Example 6. The rules for* $\neg$ *are given in Example 3. The rules for* $\vee$ *and* $\to$ *and* $\top$ *and* $\bot$ *are:*

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D}\,\vee\text{-}el \qquad \frac{\vdash A}{\vdash A \vee B}\,\vee\text{-}in_1 \qquad \frac{\vdash B}{\vdash A \vee B}\,\vee\text{-}in_2 \qquad \frac{}{\vdash \top}\,\top\text{-}in$$

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B}\to\text{-}el \qquad \frac{\vdash B}{\vdash A \to B}\to\text{-}in_1 \qquad \frac{A \vdash A \to B}{\vdash A \to B}\to\text{-}in_2 \qquad \frac{\vdash \bot}{\vdash D}\,\bot\text{-}el$$

Example 10 *As our only example for classical logic, we give the classical rules for implication. The elimination rule is the same (*$\to$ *-el above) and we also have the first introduction rule* $\to$ *-in*$_1$*, but in addition we have the rule* $\to$ *-in*$_2^c$*. We observe that this rule is classical in the sense that one can derive Peirce's law, without using negation. See the derivation below, of Peirce's law.*

$$\boxed{\frac{A \vdash D \quad A \to B \vdash D}{\vdash D}\to\text{-}in_2^c}$$

$$\dfrac{\dfrac{A \vdash A}{A \vdash ((A \to B) \to A) \to A} \qquad \dfrac{\dfrac{\dfrac{(A \to B) \to A \vdash (A \to B) \to A \quad A \to B \vdash A \to B}{A \to B, (A \to B) \to A \vdash A}}{A \to B, (A \to B) \to A \vdash ((A \to B) \to A) \to A}}{A \to B \vdash ((A \to B) \to A) \to A}}{\vdash ((A \to B) \to A) \to A}\to\text{-}in_2^c$$

2.1 If then else

We now give two examples of ternary connectives that we can treat by our method: if-then-else and most, which have the obvious (classical) truth table semantics given below. We look into if-then-else in further detail and we will say something about most in Section 3.

A	B	C	most(A, B, C)	$A \to B/C$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

Example 11 *The constructive rules for* if-then-else *we obtain, after optimization are the following four.*

$$\frac{\vdash A{\to}B/C \quad \vdash A}{\vdash B}\ \textit{then-el} \qquad \frac{\vdash A{\to}B/C \quad A \vdash D \quad C \vdash D}{\vdash D}\ \textit{else-el}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A{\to}B/C}\ \textit{then-in} \qquad \frac{A \vdash A{\to}B/C \quad \vdash C}{\vdash A{\to}B/C}\ \textit{else-in}$$

We now show in some detail that we can obtain these four optimized rules. (NB. other optimizations are possible, yielding a different set of rules.) From the lines in the truth table of $A{\to}B/C$ with a 0 we get the following four elimination rules:

$$\frac{\vdash A{\to}B/C \quad A \vdash D \quad B \vdash D \quad C \vdash D}{\vdash D} \qquad \frac{\vdash A{\to}B/C \quad A \vdash D \quad \vdash B \quad C \vdash D}{\vdash D}$$

$$\frac{\vdash A{\to}B/C \quad \vdash A \quad B \vdash D \quad C \vdash D}{\vdash D} \qquad \frac{\vdash A{\to}B/C \quad \vdash A \quad B \vdash D \quad \vdash C}{\vdash D}$$

Using Lemmas 7 and 8, these can be reduced. The two rules on the first line reduce to else-el, the two rules on the second line reduce to then-el.

Similarly, from the lines in the truth table of $A{\to}B/C$ with a 1 we get four introduction rules, which can consequently be reduced to else-in and then-in.

Example 12 *From the lines in the truth table of $A{\to}B/C$ with a 1 we get the following four classical introduction rules:*

$$\frac{A{\to}B/C \vdash D \quad A \vdash D \quad B \vdash D \quad \vdash C}{\vdash D} \qquad \frac{A{\to}B/C \vdash D \quad A \vdash D \quad \vdash B \quad \vdash C}{\vdash D}$$

$$\frac{A{\to}B/C \vdash D \quad \vdash A \quad \vdash B \quad C \vdash D}{\vdash D} \qquad \frac{A{\to}B/C \vdash D \quad \vdash A \quad \vdash B \quad \vdash C}{\vdash D}$$

Using Lemmas 7 and 8 these can be reduced to the following two. (The two rules on the first line reduce to else-in, the two rules on the second line reduce to then-in.)

$$\frac{A{\to}B/C \vdash D \quad A \vdash D \quad \vdash C}{\vdash D}\ \textit{else-in}^c \qquad \frac{\vdash A \quad \vdash B}{\vdash A{\to}B/C}\ \textit{then-in}$$

These are the classical rules for if-then-else. *Only the rule else-in^c is different from the constructive one, as given in Example 11.*

Constructively, $A{\to}B/C$ is equivalent to $(A \to B) \wedge (A \vee C)$. It can be shown that $A{\to}B/C$ is "in between" other constructive renderings of if-then-else:

$$(A \wedge B) \vee (\neg A \wedge C) \quad \underset{\vdash}{\not\dashv} \quad A{\to}B/C \quad \underset{\vdash}{\not\dashv} \quad (A \to B) \wedge (\neg A \to C)$$

The left-to-right can easily be derived, for the non-derivability of the reverse, we need a Kripke model (see Section 3).

If we compare with well-known classical rules for $\mathsf{if\text{-}then\text{-}else}$, we observe that two of them hold, while the other fails.

Fact 13 *1.* $\mathsf{if}\ A\ \mathsf{then}\ B\ \mathsf{else}\ B \vdash B$ *and* $B \vdash \mathsf{if}\ A\ \mathsf{then}\ B\ \mathsf{else}\ B$*,*

2. $\mathsf{if}\ (\mathsf{if}\ A\ \mathsf{then}\ B\ \mathsf{else}\ C)\ \mathsf{then}\ D\ \mathsf{else}\ E \not\vdash \mathsf{if}\ A\ \mathsf{then}\ (\mathsf{if}\ B\ \mathsf{then}\ D\ \mathsf{else}\ E)\ \mathsf{else}\ (\mathsf{if}\ C\ \mathsf{then}\ D\ \mathsf{else}\ E)$

3. $\mathsf{if}\ A\ \mathsf{then}\ (\mathsf{if}\ B\ \mathsf{then}\ D\ \mathsf{else}\ E)\ \mathsf{else}\ (\mathsf{if}\ C\ \mathsf{then}\ D\ \mathsf{else}\ E) \not\vdash \mathsf{if}\ (\mathsf{if}\ A\ \mathsf{then}\ B\ \mathsf{else}\ C)\ \mathsf{then}\ D\ \mathsf{else}\ E$.

As a matter of fact, either one of the last two rules renders the connective $\mathsf{if\text{-}then\text{-}else}$ *classical.*

An important property is that (just as in classical logic), the constructive $\mathsf{if\text{-}then\text{-}else}$, together with $\top$ and $\bot$ is *functionally complete*: all other connectives can be defined in terms of it. We prove this for $\wedge$, $\vee$, $\to$ and $\neg$. A result from Schroeder-Heister [10] implies that all constructive connectives can be defined in terms of $\mathsf{if\text{-}then\text{-}else}$.

Definition 14 *We define the usual intuitionistic connectives in terms of* $\top$*,* $\bot$ *and* $\mathsf{if\text{-}then\text{-}else}$*, as follows:* $A \dot{\vee} B \ :=\ A{\to}A/B$*,* $A \dot{\wedge} B \ :=\ A{\to}B/A$*,* $A \dot{\to} B \ :=\ A{\to}B/\top$*,* $\dot{\neg}A \ :=\ A{\to}\bot/\top$*.*

Lemma 15 *The defined connectives in Definition 14 satisfy the derivation rules for these same connectives as given in Definition 9. As an immediate consequence, the intuitionistic connective* $\mathsf{if\text{-}then\text{-}else}$*, together with* $\top$ *and* $\bot$*, is functionally complete.*

Proof. Lemma 15 shows that the well-known intuitionistic connectives can all be defined in terms of $\mathsf{if\text{-}then\text{-}else}$, $\top$ and $\bot$. In [10], it is shown that all connectives can be defined in terms of $\vee$, $\wedge$, $\to$ and $\neg$. □

3 Kripke semantics

We now define a Kripke semantics for the intuitionistic rules and prove that it is complete. We follow standard methods, given e.g. in [11, 12], which we generalize to arbitrary connectives. Formulas are built from atoms using existing or defined connectives of any arity, so for each n-ary connective c, we assume a truth table $t_c : \{0,1\}^n \to \{0,1\}$ and we have inductively defined derivability $\vdash$ as a relation between a sets of formulas and a formula above.

Definition 16 *We define a* Kripke model *as a triple* $(W, \leq, \mathsf{at})$ *where* W *is a set of worlds with a reflexive, transitive relation* $\leq$ *on it and a function* $\mathsf{at} : W \to \wp(\mathsf{At})$ *satisfying* $w \leq w' \Rightarrow \mathsf{at}(w) \subseteq \mathsf{at}(w')$*.*

In a Kripke model we want to define the relation $w \Vdash \varphi$ between worlds and formulas (φ *is true in world* w). We do this by defining $[\![\varphi]\!]_w \in \{0,1\}$, with the meaning that $[\![\varphi]\!]_w = 1$ if $w \Vdash \varphi$ and $[\![\varphi]\!]_w = 0$ if $w \nVdash \varphi$.

Definition 17 *Given a Kripke model* $(W, \leq, \mathsf{at})$ *we define* $[\![\varphi]\!]_w \in \{0, 1\}$*, by induction on* φ *as follows.*

– *If* φ *is atomic, we define* $[\![\varphi]\!]_w := 1$ *if* $\varphi \in \mathsf{at}(w)$*.*
– *If* $\varphi = c(\varphi_1, \ldots, \varphi_n)$*, we define* $[\![\varphi]\!]_w := 1$ *if* $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 1$ *for each* $w' \geq w$*, where* t_c *is the truth table of* c*.*

We define $\Gamma \models \psi$ *(*ψ *is a consequence of* Γ*) as: for each Kripke model and each world* w*, if for each* φ *in* Γ*,* $[\![\varphi]\!]_w = 1$*, then* $[\![\psi]\!]_w = 1$*.*

An immediate consequence of this definition is that for all worlds w, w', if $[\![\varphi]\!]_w = 1$ and $w' \geq w$, then $[\![\varphi]\!]_{w'} = 1$.

Lemma 18 (Soundness) *If* $\Gamma \vdash \psi$*, then* $\Gamma \models \psi$

Proof. By induction on the derivation of $\Gamma \vdash \psi$. We treat the case for the last rule being an introduction: $\psi = c(\psi_1, \ldots, \psi_n)$ and we have a line $p_1, \ldots, p_n|1$ in the truth table for c. The introduction rule then is as follows.

$$\frac{\Gamma \vdash \psi_j \text{ (for } \psi_j \text{ with } p_j = 1) \ldots \quad \ldots \Gamma, \psi_i \vdash \psi \text{ (for } \psi_i \text{ with } p_i = 0) \ldots}{\Gamma \vdash \psi} \text{ in}$$

Given a Kripke model and a world w in this model with $[\![\varphi]\!]_w = 1$ for all $\varphi \in \Gamma$, we need to prove that $[\![\psi]\!]_w = 1$. The induction hypothesis says that $[\![\psi_j]\!]_w = 1$ for all j with $p_j = 1$. Let $w' \geq w$. There are two cases: (1) $[\![\psi_i]\!]_{w'} = 1$ for some i with $p_i = 0$. Then by induction hypothesis: $[\![\psi]\!]_{w'} = 1$, so $t_c([\![\psi_1]\!]_{w'}, \ldots, [\![\psi_n]\!]_{w'}) = 1$. (2) $[\![\psi_i]\!]_{w'} = 0$ for all i with $p_i = 0$. Then $t_c([\![\psi_1]\!]_{w'}, \ldots, [\![\psi_n]\!]_{w'}) = 1$. So, for all $w' \geq w$: $t_c([\![\psi_1]\!]_{w'}, \ldots, [\![\psi_n]\!]_{w'}) = 1$. So $[\![\psi]\!]_w = 1$. □

Now we prove completeness: if $\Gamma \models \psi$, then $\Gamma \vdash \psi$. We prove this by constructing a special, universal Kripke model.

Definition 19 *For* ψ *a formula and* Γ *a set of formulas, we say that* Γ is ψ-maximal *if* $\Gamma \nvdash \psi$ *and for every formula* $\varphi \notin \Gamma$ *we have:* $\Gamma, \varphi \vdash \psi$*.*

If $\Gamma \nvdash \psi$, we can extend Γ to a ψ-maximal set Γ' that contains Γ as follows. Take an enumeration of the formulas as $\varphi_1, \varphi_2, \ldots$ and define recursively $\Gamma_0 := \Gamma$ and $\Gamma_{i+1} := \Gamma_i$ if $\Gamma_i, \varphi_{i+1} \vdash \psi$ and $\Gamma_{i+1} := \Gamma_i, \varphi_{i+1}$ if $\Gamma_i, \varphi_{i+1} \nvdash \psi$. Then take $\Gamma' := \bigcup_{i \in \mathbf{N}} \Gamma_i$. (NB. as always, Γ_i, φ_{i+1} denotes $\Gamma_i \cup \{\varphi_{i+1}\}$.)

Fact 20 *We list a couple of simple important facts about* ψ*-maximal sets* Γ*.*

1. *For every* φ*, we have* $\varphi \in \Gamma$ *or* $\Gamma, \varphi \vdash \psi$*.*
2. *So, for every* φ*, if* $\varphi \notin \Gamma$ *then* $\Gamma, \varphi \vdash \psi$*.*
3. *For every* φ*, if* $\Gamma \vdash \varphi$*, then* $\varphi \in \Gamma$*.*

Definition 21 *We define the Kripke model* $U = (W, \leq, \mathsf{at})$ *as follows:*

– *A world* $w \in W$ *is a pair* (Γ, ψ) *where* Γ *is a* ψ*-maximal set of formulas.*
– $(\Gamma, \psi) \leq (\Gamma', \psi') := \Gamma \subseteq \Gamma'$*.*

– $\mathsf{at}(\Gamma, \psi) := \Gamma \cap \mathsf{At}$.

Lemma 22 *In the model U we have, for all worlds $(\Gamma, \psi) \in W$:*

$$\forall \varphi, \varphi \in \Gamma \Leftrightarrow [\![\varphi]\!]_{(\Gamma,\psi)} = 1.$$

Proof. The proof is by induction on φ. If $\varphi \in \mathsf{At}$, the result is immediate, so suppose that $\varphi = c(\varphi_1, \ldots, \varphi_n)$ where c has truth table t_c. We prove the two directions separately.

($\Rightarrow$): Assume $\varphi \in \Gamma$.

We have $[\![\varphi]\!]_{(\Gamma,\psi)} = 1$ iff for all $\Gamma' \supseteq \Gamma$ and for all ψ', writing $w' = (\Gamma', \psi')$, we have $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 1$.

So let $\Gamma' \supseteq \Gamma$ and let ψ' be a formula such that Γ' is ψ'-maximal. For the sub-formulas of φ we have the following possibilities

– $[\![\varphi_j]\!]_{w'} = 1$, and then by induction hypothesis: $\varphi_j \in \Gamma'$ and so $\Gamma' \vdash \varphi_j$.
– $[\![\varphi_i]\!]_{w'} = 0$, and then by induction hypothesis: $\varphi_i \notin \Gamma'$ and so $\Gamma', \varphi_i \vdash \psi'$.

This corresponds to an entry in the truth table t_c for the connective c.

Suppose $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 0$. Then this row in the truth table yields an elimination rule that allows us to prove ψ':

$$\frac{\Gamma' \vdash \varphi \quad \ldots \Gamma' \vdash \varphi_j \text{ (for } \varphi_j \text{ with } [\![\varphi_j]\!]_{w'} = 1) \ldots \quad \ldots \Gamma', \varphi_i \vdash \psi' \text{ (for } \varphi_i \text{ with } [\![\varphi_i]\!]_{w'} = 0) \ldots}{\Gamma' \vdash \psi'}\,\text{el}$$

Note that all hypotheses of the rule are derivable, because $\varphi \in \Gamma'$ and the other hypotheses are derivable by induction. So we have $\Gamma' \vdash \psi'$. Contradiction! So: $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 1$, what we needed to prove.

($\Leftarrow$): Assume $[\![\varphi]\!]_{(\Gamma,\psi)} = 1$ and suppose (towards a contradiction) $\varphi \notin \Gamma$.

Then $\Gamma \nvdash \varphi$ (because if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$ by the facts we remarked about Kripke model U.) So there is a φ-maximal theory $\Gamma' \supseteq \Gamma$ with $\Gamma' \nvdash \varphi$. So $(\Gamma,' \varphi)$ is a world in U with $(\Gamma, \psi) \leq (\Gamma', \varphi)$. We write $w' := (\Gamma', \varphi)$ and we have

$$t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 1.$$

We consider the different sub-formulas of φ:

– the φ_j with $[\![\varphi_j]\!]_{w'} = 1$, and so (by induction hypothesis) $\varphi_j \in \Gamma'$ and so $\Gamma' \vdash \varphi_j$;
– the φ_i with $[\![\varphi_i]\!]_{w'} = 0$, and so (by induction hypothesis) $\varphi_i \notin \Gamma'$ and so $\Gamma', \varphi_i \vdash \varphi$.

Now, using an introduction rule for connective c, we can derive φ:

$$\frac{\Gamma' \vdash \varphi_j \text{ (for } \varphi_j \text{ with } [\![\varphi_j]\!]_{w'} = 1) \ldots \quad \ldots \Gamma', \varphi_i \vdash \varphi \text{ (for } \varphi_i \text{ with } [\![\varphi_i]\!]_{w'} = 0) \ldots}{\Gamma' \vdash \varphi}\,\text{in}$$

So we have $\Gamma' \vdash \varphi$, because the hypotheses of the rule are all derivable. Contradiction! So $\varphi \in \Gamma'$. □

Theorem 23 *If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.*

Proof. Suppose $\Gamma \models \psi$ and $\Gamma \nvdash \psi$. We can find a ψ-maximal superset Γ' of Γ such that $\Gamma' \nvdash \psi$. In particular: $\psi \notin \Gamma'$. So (Γ', ψ) is a world in the Kripke model U in which each member of Γ is true: $[\![\varphi]\!]_{(\Gamma',\psi)} = 1$ for $\varphi \in \Gamma$, by Lemma 22. However, ψ is not true in (Γ', ψ): $[\![\psi]\!]_{(\Gamma',\psi)} = 0$. So $\Gamma \not\models \psi$. Contradiction, so $\Gamma \vdash \psi$. $\square$

In intuitionistic logic, the connective $\vee$ has a special property that does not hold for classical logic, called the *disjunction property*: If $\vdash A \vee B$, then $\vdash A$ or $\vdash B$. This implies that the disjunction is "strong": if one has a proof of a disjunction, one has a proof of one of the disjoints. (Which is classically not the case, viz. $\vdash A \vee \neg A$.) The disjunction property can easily be proved using Kripke semantics, relying on the completeness theorem. We want to generalize this to other connectives and we introduce the notion of a *splitting connective*.

Definition 24 *Let c be an n-ary connective, $1 \leq i, j \leq n$. We say that c is* i, j-splitting *in case the truth table for c has the following shape*

A_1	$\dots$	A_i	$\dots$	A_j	$\dots$	A_n	$c(A_1, \dots, A_n)$
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0

So, in all rows where $p_i = p_j = 0$ we have $c(p_1, \dots, p_n) = 0$. Phrased purely in terms of t_c, that is: $t_c(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_{j-1}, 0, p_{j+1}, \dots, p_n) = 0$ for all $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n \in \{0, 1\}$.

A connective can be i, j-splitting for more than one i, j-pair. Examples are the ternary connectives most and if-then-else. We now state and prove our generalization of the disjunction property.

Lemma 25 *Let c be an i, j-splitting connective and suppose $\vdash c(A_1, \dots, A_n)$. Then $\vdash A_i$ or $\vdash A_j$.*

Proof. Let c be an i, j-splitting connective and let $\varphi = c(A_1, \dots, A_n)$ be a formula with $\vdash \varphi$.

Suppose $\nvdash A_i$ and $\nvdash A_j$. Then there are Kripke models K_1 and K_2 such that $K_1 \nVdash A_i$ and $K_2 \nVdash A_j$. We may assume that the sets of worlds of K_1 and K_2 are disjoint so we can construct a Kripke model K as the union of K_1 and K_2 where we add a special "root world" w_0 that is below all worlds of K_1 and K_2, with $\mathsf{at}(w_0) = \emptyset$. It is easily verified that K is a Kripke model and we have $w_0 \nVdash A_i$, because w_0 is below some world w in K_1 with $w \nVdash A_i$; similarly $w_0 \nVdash A_j$. So, $[\![A_i]\!]_{w_0} = [\![A_j]\!]_{w_0} = 0$. But then $w_0 \nVdash \varphi$, because $[\![\varphi]\!]_{w_0} = [\![c(A_1, \dots, A_n)]\!]_{w_0} = 1$ iff for all $w \geq w_0$: $t_c([\![A_1]\!]_w, \dots, [\![A_n]\!]_w) = 1$. However, for $w := w_0$, whatever the values of $[\![A_k]\!]_w$ are for $k \neq i, j$, $t_c([\![A_1]\!]_w, \dots, [\![A_n]\!]_w) = 0$. On the other hand, $w_0 \Vdash \varphi$, because $\vdash \varphi$, so we have a contradiction. We conclude that $\vdash A_i$ or $\vdash A_j$. $\square$

Example 26 *Looking at the truth tables in Section 2.1, we see that* most *is i, j-splitting for every i, j. Indeed, if $\vdash$ most(A, B, C), we can derive $\vdash A$ or $\vdash B$ but also $\vdash A$ or $\vdash C$ and also $\vdash B$ or $\vdash C$.*

The connective if-then-else *is not $1, 2$-splitting but it is $1, 3$-splitting and $2, 3$-splitting: if $\vdash A \to B/C$, then we have $\vdash A$ or $\vdash C$ and also $\vdash B$ or $\vdash C$.*

4 Cuts and cut-elimination

The idea of a cut in intuitionistic logic is an introduction of a formula φ immediately followed by an elimination of φ. We will call this a *direct intuitionistic cut*. In general in between the intro rule for φ and the elim rule for φ, there may be other auxiliary rules, so occasionally we may have to first permute the elim rule with these auxiliary rules to obtain a direct cut that can be contracted. We leave that for future research and now just define the notion of direct cut and contraction of a direct cut.

Definition 27 *Let c be a connective of arity n, with an elim rule and an intuitionistic intro rule derived from the truth table, as in Definition 1. So suppose we have the following rules in the truth table t_c.*

$A_1 \ldots A_n$	$c(A_1, \ldots, A_n)$
$p_1 \ldots p_n$	0
$q_1 \ldots q_n$	1

An intuitionistic direct cut *in a derivation is a pattern of the following form, where $\varphi = c(A_1, \ldots, A_n)$ and: (1) A_j ranges over all formulas where $q_j = 1$, A_i ranges over all formulas where $q_i = 0$; (2) A_k ranges over all formulas where $p_k = 1$, A_ℓ over all formulas where $p_\ell = 0$,*

$$\dfrac{\dfrac{\ldots\; \begin{array}{c}\vdots\,\Sigma_j\\ \Gamma\vdash A_j\end{array}\;\ldots\quad\ldots\; \begin{array}{c}\vdots\,\Sigma_i\\ \Gamma, A_i\vdash\varphi\end{array}\;\ldots}{\Gamma\vdash\varphi}\qquad \ldots\; \begin{array}{c}\vdots\,\Pi_k\\ \Gamma\vdash A_k\end{array}\;\ldots\quad\ldots\;\begin{array}{c}\vdots\,\Pi_\ell\\ \Gamma, A_\ell\vdash D\end{array}\;\ldots}{\Gamma\vdash D}$$

The elimination of a direct cut *is defined by replacing the derivation pattern above by*

1. *If $\ell = j$ (for some ℓ, j):*

$$\begin{array}{c} \begin{array}{c}\vdots\,\Sigma_j\\ \Gamma\vdash A_j\end{array}\;\ldots\;\begin{array}{c}\vdots\,\Sigma_j\\ \Gamma\vdash A_j\end{array}\\ \vdots\,\Pi_\ell \\ \Gamma\vdash D\end{array}$$

2. If $k = i$ (for some k, i):

$$
\frac{
\begin{array}{c}
\begin{array}{ccc}
\vdots\,\Pi_k & & \vdots\,\Pi_k \\
\Gamma \vdash A_i & \ldots & \Gamma \vdash A_i
\end{array} \\
\vdots\,\Sigma_i \\
\Gamma \vdash \varphi
\end{array}
\qquad \ldots\
\begin{array}{c} \vdots\,\Pi_k \\ \Gamma \vdash A_i \end{array}
\ \ldots \qquad \ldots\
\begin{array}{c} \vdots\,\Pi_\ell \\ \Gamma, A_\ell \vdash D \end{array}
\ \ldots
}{\Gamma \vdash D}
$$

There may be several choices for the i and j in the previous definition, so cut-elimination is non-deterministic in general. As an example, we give the cut-elimination rules for if-then-else with optimized deduction rules.

Example 28 *The intuitionistic cut-elimination rules for* if-then-else *are the following.*

(then-then)

$$
\frac{\dfrac{\Gamma \vdash A \qquad \begin{array}{c} \vdots\,\Sigma \\ \Gamma \vdash B \end{array}}{\Gamma \vdash A \to B / C} \qquad \Gamma \vdash A}{\Gamma \vdash B}
\quad \mapsto \quad
\begin{array}{c} \vdots\,\Sigma \\ \Gamma \vdash B \end{array}
$$

(else-then)

$$
\frac{\dfrac{\begin{array}{c} \vdots\,\Sigma \\ \Gamma, A \vdash A \to B / C \end{array} \qquad \Gamma \vdash C}{\Gamma \vdash A \to B / C} \qquad \begin{array}{c} \vdots\,\Pi \\ \Gamma \vdash A \end{array}}{\Gamma \vdash B}
\quad \mapsto \quad
\frac{\begin{array}{c}
\begin{array}{ccc}
\vdots\,\Pi & & \vdots\,\Pi \\
\Gamma \vdash A & \ldots & \Gamma \vdash A
\end{array} \\
\vdots\,\Sigma \\
\Gamma \vdash A \to B / C
\end{array} \qquad \begin{array}{c} \vdots\,\Pi \\ \Gamma \vdash A \end{array}}{\Gamma \vdash B}
$$

(then-else)

$$
\frac{\dfrac{\begin{array}{c} \vdots\,\Sigma \\ \Gamma \vdash A \end{array} \qquad \Gamma \vdash B}{\Gamma \vdash A \to B / C} \qquad \begin{array}{c} \vdots\,\Pi \\ \Gamma, A \vdash D \end{array} \qquad \Gamma, C \vdash D}{\Gamma \vdash D}
\quad \mapsto \quad
\begin{array}{c}
\begin{array}{ccc}
\vdots\,\Sigma & & \vdots\,\Sigma \\
\Gamma \vdash A & \ldots & \Gamma \vdash A
\end{array} \\
\vdots\,\Pi \\
\Gamma \vdash D
\end{array}
$$

(else-else)

$$
\frac{\dfrac{\Gamma, A \vdash A \to B / C \qquad \begin{array}{c} \vdots\,\Sigma \\ \Gamma \vdash C \end{array}}{\Gamma \vdash A \to B / C} \qquad \Gamma, A \vdash D \qquad \begin{array}{c} \vdots\,\Pi \\ \Gamma, C \vdash D \end{array}}{\Gamma \vdash D}
\quad \mapsto \quad
\begin{array}{c}
\begin{array}{ccc}
\vdots\,\Sigma & & \vdots\,\Sigma \\
\Gamma \vdash C & \ldots & \Gamma \vdash C
\end{array} \\
\vdots\,\Pi \\
\Gamma \vdash D
\end{array}
$$

5 Conclusion and Further work

We have introduced a general procedure for deriving natural deduction rules from truth tables that applies both to classical and intuitionistic logic. Our deduction rules obey a specific format, making it easier to study. To show that the intuitionistic rules are truly constructive we have defined a complete Kripke semantics for the intuitionistic rules. We have defined cut-elimination for intuitionistic logic in general. In an extended version of the paper [5] we have described a Curry-Howard proofs-as-terms isomorphism for the deriavtions in constructive logic. We have studied it in more detail for if-then-else.

The work described here raises many new research questions that we will pursue further: Is cut-elimination normalizing in general for an arbitrary set of connectives? How to define cut-elimination for the classical case, and what is its connection with a term calculus for classical logic as studied e.g. in [8, 1, 2]?

Another issue is the possibility of "hidden cuts" that need to be made explicit via a permuting conversion operation on the derivation (or on the proof-term). These already occur in the fragment with just if-then-else and we describe these permuting conversions in [5]. The question is if we can describe and study these permuting conversions in general.

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