Master Thesis Mathematics

# Natural Deduction Derived from Truth Tables

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# Introduction

Natural deduction originates from the work by the German mathematician Gerhard Gentzen (1909-1945): *Untersuchungen über das logische Schliessen*, 1935 [3]. Right after the Second World War he was arrested and died of hunger. During his short life, he studied mathematical foundations and proof theory. His most famous contributions are natural deduction and sequent calculus. He constructed both logical systems in his attempts to prove the consistency of number theory. He first defined natural deduction, for which he tried to prove consistency using the cut elimination theorem, which he called the Hauptsatz. However, he did not manage to prove the Hauptsatz for natural deduction. Therefore he introduced the sequent calculus for which he proved the Hauptsatz. Later, in 1956, the Swedish logician Dag Prawitz (born in 1936) gave a direct prove for the cut elimination theorem for natural deduction in *Natural deduction: a proof-theoretical study* [11].

The main idea of natural deduction is that propositions are deduced by applying derivation rules from a set of assumptions. Gentzen distinguished so-called introduction and elimination rules, that should be defined in a 'natural' way. The Hauptsatz states that elimination from introduced formulas can be avoided, which results in 'normal' derivations. Since the work of Gentzen and Prawitz, various other natural deduction systems have been introduced.

This thesis covers an analysis of the *truth table natural deduction system*, which is a natural deduction system recently defined by Herman Geuvers and Tonny Hurkens [4]. They developed a general method for deriving propositional natural deduction rules for an arbitrary connective from its truth table. It is remarkable that in this way not only classical rules can be derived, but also intuitionistic rules. The analysis in this thesis focuses on proof-theoretic properties and semantics. We devote an extensive part of the thesis to a study of cut elimination for intuitionistic logic, resulting in a number of normalization results. We have included a lot of examples to illustrate the properties of the system.

A main contribution of this thesis is the proof of strong normalization of the intuitionistic truth table natural deduction system. To establish this result, we have examined the work of Philippe de Groote who proved strong normalization for proof reduction in the natural deduction system from Gentzen and Prawitz [6]. In addition, we extend the simply typed lambda calculus to a parallel simply typed lambda calculus. This enables us to use the method of De Groote to prove strong normalization.

The thesis is structured in the following way. Chapter 1 is a preliminary chapter in which we present the basics of natural deduction studied by Gentzen and Prawitz. Those who are familiar with propositional logic can skip this chapter and consult it when necessary. Chapter 2 defines the truth table natural deduction system and gives a lot of examples. Many properties have already been established in [5], but some of them are new, such as Glivenko's theorem. Chapter 3 gives an elaborated analysis of normalization of the intuitionistic truth table natural deduction system, using some techniques from Chapter 1. We conclude with a chapter on related work and future research.

I would like to thank my supervisor Herman Geuvers who proposed to study this beautiful topic in the field of logic. During the process we had good discussions about the content. I would also like to thank Tonny Hurkens for his tremendous enthusiasm for mathematics. It is fantastic that he does mathematics in his free time.

# Chapter 1 | Natural Deduction

We start with basic results about natural deduction for propositional logic to set out general proof methods. This is useful in order to study the truth table natural deduction system in Chapter 2 and Chapter 3. Natural deduction is introduced by Gentzen in 1935 [3]. In 1965, Prawitz proved the *cut elimination theorem* for the natural deduction in his work called *Natural deduction: a proof-theoretical study* [11]. The cut elimination theorem states that every derivation can be transformed into a normal derivation. The theorem is also known under the name *weak normalization*. This chapter is the result of a close reading of Prawitz's work. Although Gentzen introduced the natural deduction system, we refer to it as the Prawitz system, because our focus is on weak normalization which is proved by Prawitz.

## 1.1 Propositional logic

The function of this section is to introduce the basics of natural deduction for propositional logic. This is useful in order to fix definitions and notation. We focus on propositional logic, because the truth table natural deduction system is only defined for propositional logic. The following definitions and notations are based on the work of Van Dalen [1], which is a modern reference book on logic.

**Definition 1.1.1.** The language of propositional logic contains the following symbols:

(1) *proposition symbols*: $p_0, p_1, p_2, \dots,$
(2) *connectives*: $\wedge, \vee, \rightarrow, \neg, \bot,$
(3) *auxiliary signs*: parentheses ( and ).

These connectives are the standard connectives used in propositional logic. The names of the connectives are, conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$), negation ($\neg$) and bottom or falsum ($\bot$). The proposition symbols and $\bot$ are indecomposable propositions, which are called *atomic propositions* or just *atoms*.

**Definition 1.1.2.** The set *PROP* of propositions is the smallest set $X$ such that

(1) for proposition symbols $p$ we have $p \in X$ and $\bot \in X$,
(2) if $A, B \in X$, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B) \in X$,
(3) if $A \in X$, then $(\neg A) \in X$.

We work in propositional logic, but we prefer to speak of *formulas* instead of propositions. We write formulas with capital letters $A$, $B$, $C$, etc.

The connectives $\wedge$, $\vee$ and $\rightarrow$ are 2-*ary connectives*, $\neg$ is a 1-*ary connective*, and $\bot$ is a 0-*ary connective*. The formulas $(A \wedge B)$ can also be denoted by $\wedge(A, B)$. The same holds for the other connectives, where $(\neg A)$ translates to $\neg(A)$. In this notation, it is possible to speak of an arbitrary connective $c$.

**Definition 1.1.3.** Formula $A'$ is a *subformula* of formula $A$ if one of the following holds.

- $A' = A$,
- $A = c(A_1, A_2)$ and $A'$ is subformula of $A_1$ or $A_2$,
- $A = \neg A_1$ and $A'$ is subformula of $A_1$.

**Definition 1.1.4.** The *rank* $r(A)$ of a formula $A$ is defined by

$$r(A) = \begin{cases} 0, & \text{for atomic } A \\ \max(r(A_1), r(A_2)) + 1, & \text{if } A = c(A_1, A_2) \text{ with } c \text{ a 2-ary connective,} \\ r(A_1) + 1, & \text{if } A = \neg(A_1). \end{cases}$$

The rank of a formula is an important concept, because it makes it possible to do induction on the structure of formulas.

Now we will examine natural deduction. Natural deduction is based on derivation rules that were first designed by Gentzen. These rules render an intuitive meaning of the connectives and make it possible to reason with logical formulas. It is possible to define an intuitionistic system and a classical set of rules.

**Definition 1.1.5.** For each connective we have the following *introduction* and *elimination rules* [11]. We distinguish between an intuitionistic and a classical rule for falsum ($\bot$). All the other rules are intuitionistic and classical.

$$(\wedge\text{-I}) \quad \frac{A \qquad B}{A \wedge B} \qquad\qquad (\wedge\text{-E}) \quad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$(\vee\text{-I}) \quad \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad (\vee\text{-E}) \quad \frac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

$$(\rightarrow\text{-I}) \quad \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \rightarrow B} \qquad\qquad (\rightarrow\text{-E}) \quad \frac{A \rightarrow B \qquad A}{B}$$

$$(\bot^{\mathrm{i}}) \quad \frac{\bot}{A} \qquad\qquad (\bot^{\mathrm{c}}, \text{RAA}) \quad \frac{\begin{array}{c}[\neg A]\\ \vdots \\ \bot\end{array}}{A}$$

Negation is defined as $\neg A \equiv (A \rightarrow \bot)$

We stick to the convention that $A$ is different from $\bot$ in the $\bot$-rules.

The top-formulas in the rules are called *assumptions.* Assumptions written between brackets $[\cdot]$ are *discharged assumptions* and assumptions without brackets are *open assumptions.* When writing $[A]$, we mean that 0, 1 or more instances of $A$ are discharged at the same time. Formulas immediately above the line of a rule are called *premises.* The formulas $A \wedge B$, $A \vee B$ and $A \rightarrow B$ in the elimination rules $\wedge$-E, $\vee$-E and $\rightarrow$-E are called *major premises.* The other formulas immediately above the line in those rules are called *minor premises.* The formula below the line of a derivation rule is the *conclusion* or *consequence* of the rule.

We are able to construct deductions of formulas, also known as derivations. Informally, a *derivation* is a tree built up from the rules from Definition 1.1.5. Derivations are inductively defined by the rules. See Definition 1.4.1. of [1] for a formal definition.

Derivations are often denoted by the Greek letters $\Pi$ or $\Sigma$. Top-formulas of a derivation are called *assumptions* and the bottom-formula is called the *conclusion* of the derivation. Discharged assumptions $[A]$ are often labelled with a natural number $k$, that is $[A]^k$, indicating at which rule the assumption is discharged. If $\Gamma$ is a set of open assumptions and $A$ is the conclusion of a deduction $\Pi$, then we say that $\Pi$ is a *deduction from* $\Gamma$ *to* $A$ or simply $A$ is *derivable from* $\Gamma$. If $\Pi$ does not contain any open assumptions, then $\Pi$ is a *proof* of $A$.

**Definition 1.1.6.** Let $\Gamma$ be a set of formulas and $A$ a formula. We write $\Gamma \vdash A$ to mean that there is a derivation with open assumption in $\Gamma$ and conclusion $A$. If $\Gamma$ is empty we write $\vdash A$.

**Example 1.1.7.** The tree below proves $\vdash \neg\neg\neg A \to \neg A$. This derivation only contains $\to$-I and $\to$-E rules, since $\neg A = A \to \bot$.

$$\dfrac{\dfrac{\dfrac{[\neg\neg\neg A]^4 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg A]^1 \qquad [A]^2}{\bot}\,\to\text{-E}}{\neg\neg A}\,\to\text{-I},\,1}{A \to \neg\neg A}\,\to\text{-I},\,2 \qquad [A]^3}{\neg\neg A}\,\to\text{-E}}{\bot}\,\to\text{-E}}{\neg A}\,\to\text{-I},\,3}{\neg\neg\neg A \to \neg A}\,\to\text{-I},\,4$$

We distinguish between intuitionistic propositional logic and classical logic.

**Definition 1.1.8.** *Intuitionistic propositional logic* (IPC) contains derivations with intuitionistic rules and *classical propositional logic* CPC is the set of derivations with classical rules, that is, IPC adopts the $\bot^{\mathrm{i}}$-rule and CPC uses the $\bot^{\mathrm{c}}$-rule. We will often refer to these systems as the *Prawitz natural deduction system* or just the *Prawitz system*.

Note that the $\bot^{\mathrm{i}}$-rule is weaker than the $\bot^{\mathrm{c}}$-rule, which means that IPC $\subsetneq$ CPC. The $\bot^{\mathrm{c}}$-rule is often called *Reductio ad absurdum* (RAA) in the literature. This rule makes it possible to reason with proof by contradiction. Typical classical statements are $\vdash_{\mathrm{CPC}} \neg\neg A \to A$ and Peirce's Law $\vdash_{\mathrm{CPC}} ((A \to B) \to A) \to A$.

We define some more notions concerning derivations. We say that formula $A$ stands *immediately above* formula $B$ in a derivation if $A$ is a premise of a rule of which $B$ is the conclusion. Formulas $A$ and $B$ are said to be *side-connected* if they stand on the same line. For example in $\wedge$-I, $A$ and $B$ are side-connected and $A$ stands immediately above $A \wedge B$.

Rule $\vee$-E is the only rule where some premises are the same as the conclusion. It may happen that you apply this rule without discharging any of the assumptions. Such an application of the rule is called a *redundant application* because it does not add any new information in the derivation tree. Therefore, we assume from now that no redundant applications of $\vee$-E are allowed in a derivation.

## 1.2 Normal derivations

This section is the result of a detailed examination of the well-known proof of weak normalization of natural deduction by Prawitz [11]. Prawitz proved weak normalization for both CPC and IPC. Here we only work in IPC, because in Chapter 3 we study normalization of the intuitionistic truth table natural deduction system. Important aspects in this section are the definitions of a normal deduction and the inspection of the induction value in the normalization proof of Prawitz, used in Theorem 1.2.10.

The most important motivation for defining a normal derivation in natural deduction is the fact that deductions satisfy the so-called *Inversion Principle*. That is the idea that an elimination rule of a connective is, in some sense, the inverse of the corresponding introduction rules. By an elimination rule one only conclude what had already been established if the major premise of the elimination was deduced from the corresponding introduction rules. The normalization consists of the removal of such introduction-elimination pairs. The inversion principle was also a great motivation for Gentzen, he wrote: 'the introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.' [3]

There are different definitions of a normal derivation. Here we focus on two of them. One definition comes from Prawitz [11] and the other is more commonly used, by for instance Troelstra [15] and indirectly by Van Dalen [1]. Before we state both definitions, we have to introduce the concept of a segment.

**Definition 1.2.1.** A *segment* in a deduction $\Pi$ is a sequence $A_1, \ldots, A_n$ of formulas such that $A_i$ stands immediately above $A_{i+1}$ and

(1) $A_1$ is not the conclusion of $\vee$-E,
(2) For $i < n$, $A_i$ is a minor premise of $\vee$-E,
(3) $A_n$ is not the minor premise of $\vee$-E.

Note that all formulas $A_i$ in a segment represent the same formula. The *rank* of a segment is the rank of the formula in that segment. Number $n$ is called the *length* of the segment.

**Definition 1.2.2.** A *maximum segment* is a segment that begins with the consequence of an I-rule or the $\bot^{\mathrm{i}}$-rule and ends with a major premise of an E-rule. A maximum segment of length 1 is called a *maximum formula.*

A maximum formula is introduced by an introduction or the $\bot^{\mathrm{i}}$-rule, immediately followed by the corresponding elimination rule.

**Definition 1.2.3.** A *cut formula* is a formula which is the conclusion of $\vee$-E, $\bot^{\mathrm{i}}$-rule or an I-rule and the major premise of an E-rule. A *cut segment* is a segment that contains a cut formula.

From the definitions, we see that each maximum segment is a cut segment, but the reverse is not true. Maximum segments play an important role in the definition of a normal derivation by Prawitz. Cut segments play an important role in the other definition of normal derivations.

**Example 1.2.4.** There are two segments of length 2 in this deduction, containing the formula $A \wedge B$. The right segment is a maximum segment. Both are cut segments.

$$\dfrac{\dfrac{(A \wedge B) \vee (B \wedge A) \qquad [A \wedge B]^1 \qquad \dfrac{\dfrac{[B \wedge A]^1}{A}\wedge\text{-E} \qquad \dfrac{[B \wedge A]^1}{B}\wedge\text{-E}}{A \wedge B}\wedge\text{-I}}{A \wedge B}\vee\text{-E}, 1}{A}\wedge\text{-E}$$

Now we state both definitions of a normal derivation. Since the definition of Prawitz is less used, we give it the name Prawitz normal.

**Definition 1.2.5** (Prawitz normal)**.** A deduction is *Prawitz normal* if it contains no maximum segment.

**Definition 1.2.6** (Normal)**.** A deduction is *normal* if every major premise of an E-rule is either an assumption or the conclusion of an elimination rule different from $\vee$-E.

Note that a deduction is normal if and only if it contains no cut segment. It is possible to distinguish between two types of cut segments. If $\sigma$ is a cut segment of length 1, then it is the conclusion of the $\perp^{\mathrm{i}}$-rule or an introduction rule and the major premise of an elimination rule. When $\mathrm{Length}(\sigma) > 1$, then the cut formula is derived from $\vee$-E. We get the following standard definitions.

**Definition 1.2.7.** An E-rule with a major premise derived from an I-rule is a *detour convertibility.*

**Definition 1.2.8.** An E-rule with a major premise derived from $\vee$-E is a *permutation convertibility.*

Now we see that a deduction is normal in the sense of Definition 1.2.6 if and only if it contains no detour and no permutation convertibilities and no cut formula derived from the $\perp^{\mathrm{i}}$-rule. When deleting such convertibilities, we speak about detour or permutation *conversions* or *reductions.* Example 1.2.4 contains a permutation convertibility. In the proof of weak normalization (Theorem 1.2.10), we see other examples of detour and permutation convertibilities.

We may ask ourselves whether the two definitions of normal derivations are equivalent. The answer is no, because a cut segment is a stronger concept than a maximum segment. This is illustrated by the following example.

**Example 1.2.9.** This is a derivation of formula $A$ from $(A \wedge B) \vee (A \wedge B)$.

$$\dfrac{\dfrac{(A \wedge B) \vee (A \wedge B) \qquad [A \wedge B]^1 \qquad [A \wedge B]^1}{A \wedge B\ (*)}\ \vee\text{-E, } 1}{A}\ \wedge\text{-E}$$

This derivation is Prawitz normal, since there is no introduction rule or $\perp^{\mathrm{i}}$-rule, hence no maximum segment. But it is *not* normal in the sense of Definition 1.2.6, because formula $A \wedge B$ marked by $(*)$ is neither an assumption nor the conclusion of an elimination rule different from $\vee$-E. One might want the following deduction:

$$\dfrac{(A \wedge B) \vee (A \wedge B) \qquad \dfrac{[A \wedge B]^1}{A}\ \wedge\text{-E} \qquad \dfrac{[A \wedge B]^1}{A}\ \wedge\text{-E}}{A}\ \wedge\text{-E, } 1$$

Here, the application of $\vee$-E is delayed as long as possible. This derivation is normal according to both definitions.

In general, if a deduction is normal, then it is also Prawitz normal. Example 1.2.9 shows that the other way around is not true, but a Prawitz normal deduction can be transformed into a normal deduction. This follows from the following theorem, which holds for both Definitions 1.2.5 and 1.2.6. The proof of the theorem is the result of a close reading of Prawitz [11].

**Theorem 1.2.10** (Weak normalization)**.** *There is an effective procedure for transforming a natural deduction of* $\Gamma \vdash A$ *in* IPC *into a (Prawitz) normal deduction of* $\Gamma \vdash A$*.*

*Proof.* Let $\Pi$ be a derivation of $\Gamma \vdash A$. This proof is done by induction. The induction value to be used depends on the chosen definition of a normal derivation.

- (Prawitz normal) The induction value is the pair $\langle d, l\rangle$ with $d$ the highest rank of a maximum segment in $\Pi$ and $l$ is the sum of the lengths of maximum segments of rank $d$. The induction is done on the lexicographic ordering of these pairs.
- (normal) The induction value is the pair $\langle d, l\rangle$ with $d$ the highest rank of a cut segment in $\Pi$ and $l$ is the sum of the lengths of cut segments of rank $d$. Again, the induction is done on the lexicographic ordering of these pairs.

Now we can prove the assertion. Here it is proved for Definition 1.2.6, so we are going to transform $\Pi$ in a normal derivation of $\Gamma \vdash A$. Let $\sigma$ be a cut segment of highest rank $d$ such that:

(a) there is no cut segment of rank $d$ above $\sigma$,
(b) no cut segment of rank $d$ stands above a formula side-connected with the last formula in $\sigma$,
(c) no cut segment of rank $d$ contains a formula side-connected with the last formula in $\sigma$.

It is important to mention that such a cut segment $\sigma$ exists if derivation $\Pi$ is not normal, see [11] for a proof. Let $F$ be the cut formula in $\sigma$. We treat different cases for $F$.

(1) $F$ is derived from the $\bot^{\mathrm{i}}$-rule,
(2) $F$ is in a detour convertibility,
(3) $F$ is in a permutation convertibility.

In case 1, cut formula $F$ in $\sigma$ is the consequence of the $\bot^{\mathrm{i}}$-rule and the reduction goes as follows.

$$\dfrac{\dfrac{\begin{matrix}\Sigma_1\\ \bot\end{matrix}}{F}\,\bot^{\mathrm{i}}\text{-rule} \qquad \Sigma_2}{\begin{matrix}C\\ \Pi\end{matrix}}\,\text{E-rule} \qquad \rightsquigarrow \qquad \dfrac{\begin{matrix}\Sigma_1\\ \bot\end{matrix}}{\begin{matrix}C\\ \Pi\end{matrix}}\,\bot^{\mathrm{i}}\text{-rule}$$

For case 2, $F$ is in a detour convertibility. This means that $F$ is the conclusion of an I-rule which is followed by its corresponding E-rule. Such reductions are not so difficult, therefore we only show the detour reductions for $\vee$ and $\rightarrow$.

$$\dfrac{\dfrac{\begin{matrix}\Sigma_1\\ A\end{matrix}}{A \vee B} \qquad \begin{matrix}[A]^1 \dots [A]^1\\ \Sigma_2\\ C\end{matrix} \qquad \begin{matrix}[B]^1 \dots [B]^1\\ \Sigma_3\\ C\end{matrix}}{\begin{matrix}C\\ \Pi\end{matrix}}\,1 \qquad \rightsquigarrow \qquad \begin{matrix}\Sigma_1 & & \Sigma_1\\ A & \dots & A\\ & \Sigma_2 & \\ & C & \\ & \Pi & \end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[A]^1 \dots [A]^1\\ \Sigma_1\\ B\end{matrix}}{A \rightarrow B}\,1 \qquad \begin{matrix}\Sigma_2\\ A\end{matrix}}{\begin{matrix}B\\ \Pi\end{matrix}} \qquad \rightsquigarrow \qquad \begin{matrix}\Sigma_2 & & \Sigma_2\\ A & \dots & A\\ & \Sigma_1 & \\ & B & \\ & \Pi & \end{matrix}$$

In the reduction of $\vee$ there can be multiple subderivations containing $\Sigma_1$. This is not a problem for the induction value, because of clause (a) in the assumption on $\sigma$. So either $d$ is lowered or $d$ remains the same and $l$ is lowered. The same holds for the reduction for $\rightarrow$ by clause (b).

In case 3, cut formula $F$ is in a permutation convertibility, that is, $F$ is the consequence of the $\vee$-E rule. The reduction is as follows:

$$\dfrac{\dfrac{\begin{matrix}\Sigma_1\\ A\vee B\end{matrix} \quad \begin{matrix}[A]^1\ldots[A]^1\\ \Sigma_2\\ F\end{matrix} \quad \begin{matrix}[B]^1\ldots[B]^1\\ \Sigma_3\\ F\end{matrix}}{F}\,1 \qquad \Sigma_4}{C}\,\mathrm{E} \qquad \rightsquigarrow \qquad \dfrac{\begin{matrix}\Sigma_1\\ A\vee B\end{matrix} \quad \dfrac{\begin{matrix}[A]^1\ldots[A]^1\\ \Sigma_2\\ F\end{matrix} \quad \Sigma_4}{C}\,\mathrm{E} \quad \dfrac{\begin{matrix}[B]^1\ldots[B]^1\\ \Sigma_3\\ F\end{matrix} \quad \Sigma_4}{C}\,\mathrm{E}}{C}\,1$$

$$\Pi \qquad\qquad\qquad\qquad \Pi$$

Again we have to check whether the induction value $\langle d, l\rangle$ is indeed decreased. There can be two problems. First, subderivation $\Sigma_4$ is duplicated. But because of clause (b), there will be no extra cut segment of rank $d$. Secondly, there could be more cut segments containing formula occurrence $C$. We have to check that such a $C$ has a lower rank than $F$ in order to conclude that the induction value decreases. If the E-rule indicated in the left derivation is not $\vee$-E, then $C$ has a lower rank than $F$ by definition of the rules. Suppose $C$ has also rank $d$ and that it is derived from $\vee$-E. Then $\Sigma_4$ consists of two derivations, both ending in $C$. This means that there is a cut segment (containing $C$) of rank $d$ containing a formula side-connected with the last formula $F$ in $\sigma$. This contradicts clause (c) above. So we have shown that $C$ has a lower rank than $F$, by the assumptions on $\sigma$. So we can conclude that the induction value is lowered.

We have checked all possibilities for cut formula $F$ in chosen cut segment $\sigma$. In all cases, the induction value of the reduced derivation is less than the original one. This makes sure that after repeated applications of these reductions we obtain a normal derivation. □

Weak normalization states that for every deduction, there can be found a normal one. In 2002, De Groote published a paper where he proved strong normalization for the union of detour and permutation convertibilities [6]. This means that he does not treat a cut formula derived from $\perp^{\mathrm{i}}$-rule as a convertibility. Strong normalization means that it does not matter in what order you reduce the detour or permutation convertibilities, you always end up with a 'normal' derivation. Here we will not explain his proof, this is done in Chapter 3, where we prove strong normalization for the intuitionistic truth table natural deduction system.

## 1.3 Subformula property, consistency and decidability

Normal derivations have several convenient properties. In this section we consider the subformula property, consistency and decidability of IPC. In order to formulate and prove these properties, we have to look at the form of a normal derivation. We introduce some more terminology based on Prawitz [11] and Van Dalen [1].

**Definition 1.3.1.** A *path* in a deduction $\Pi$ is a sequence of formulas $A_1, \ldots, A_n$ such that

(1) $A_1$ is an assumption not discharged by $\vee$-E,
(2) $A_i$ is not a minor premise of $\to$-E and either
  - $A_i$ is not a major premise of $\vee$-E and $A_{i+1}$ stands immediately below $A_i$ or
  - $A_i$ is the major premise of $\vee$-E and $A_{i+1}$ is a discharged assumption in this rule,
(3) $A_n$ is either
  - a minor premise of $\to$-E,
  - the conclusion of $\Pi$.

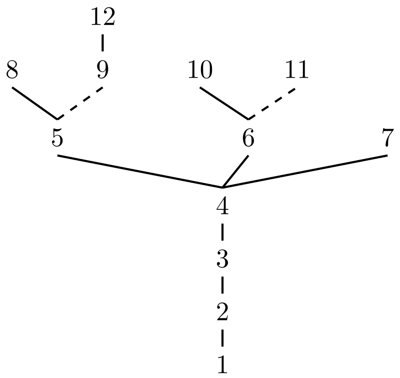
Note that if $A_i$ is the major premise of $\vee$-E, there are two possibilities to continue the path, because there are two discharged assumption, since we assume that there are no redundant applications of the $\vee$-E rule. Also note that segments are totally included in a path.

**Definition 1.3.2.** A path containing the conclusion of the derivation is called a *main path.*

**Example 1.3.3.** To denote the different paths in a deduction, it is useful to look at the underlying tree structure. Derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(C \lor D) \to (A \lor B)]^3 \qquad \dfrac{[C]^4}{C \lor D}\lor\text{-I}}{A \lor B}\to\text{-E} \qquad \dfrac{[A \to B]^2 \qquad [A]^1}{B}\to\text{-E} \qquad [B]^1}{B}\lor\text{-E, 1}}{(A \to B) \to B}\to\text{-I, 2}}{((C \lor D) \to (B \lor A)) \to (A \to B) \to B}\to\text{-I, 3}}{C \to ((C \lor D) \to (B \lor A)) \to (A \to B) \to B}\to\text{-I, 4}$$

adopts the tree form




A dashed line indicates the minor premise of rule $\to$-E, which is the end of a path. The derivation contains the following paths: (8,5,11), (8,5,7,4,3,2,1), (12,9) and (10,6,4,3,2,1). This is a normal derivation.

Now it is possible to prove statements about the form of a (Prawitz) normal derivation. Note that each formula in a deduction tree belongs to at least one path.

**Proposition 1.3.4.** *A path $\pi$ in a (Prawitz) normal derivation is divided into at most three parts: an E-part, followed by a $\perp$-part, followed by an I-part, such that*

*(1) for each segment $\sigma$ in the E-part, it holds that the last formula in $\sigma$ is the major premise of an E-rule or the end-formula of $\pi$,*
*(2) the last formula in segment $\sigma$ of the $\perp$-part is the premise of the $\perp^{\mathrm{i}}$-rule,*
*(3) for each segment in the I-part, the last formula is a premise of an I-rule or the end-formula of $\pi$.*

*Each of the parts may be empty.*

*Proof.* Since normal derivations are also Prawitz normal, it suffices to prove the assertion for Prawitz normal derivations. Let $\Pi$ be a Prawitz normal derivation and $\pi$ a path of $\Pi$. Then there is no maximum segment in $\Pi$, that is, there is no segment that starts with an I-rule or $\perp^{\mathrm{i}}$-rule and ends with a major premise of an E-rule. In addition, if a segment starting with an I-rule or $\perp^{\mathrm{i}}$-rule ends with a minor premise of an E-rule, then it must be the minor premise of $\to$-E, which is the end of path $\pi$. Therefore, if the first rule in $\pi$ is an E-rule, then all segments that end in an E-rule come first. This forms the E-part. Look at the last segment that is a consequence of an E-rule. This results in either

- the end-formula of $\pi$.
- $\bot$, in which case only the $\bot^{\mathrm{i}}$-rule may be applied. Consider now the segment $\sigma'$ that starts with the $\bot^{\mathrm{i}}$-rule. Note that formula $A$ in $\sigma'$ is not equal to $\bot$ by convention. Segment $\sigma'$ cannot end in a major premise of an E-rule, since $\Pi$ is normal. It can end in the minor premise of $\to$-E, the conclusion of $\Pi$ or an I-rule. This means that $\sigma'$ belongs to the I-part.
- An I-rule. Consider now the segment that starts with an I-rule. This segment cannot end in the $\bot^{\mathrm{i}}$-rule or a major premise of an E-rule. Again, it may end in the minor premise of $\to$-E, the conclusion of $\Pi$ or an I-rule, which yields the I-part.

□

Note that if we look at normal derivations in the sense of Definition 1.2.6, then the segments in the E-part have length 1, except for the possible last segment ending in the conclusion. This is because major premises of elimination rules can not be derived from $\vee$-E, so segments ending in a major premise of an elimination rule have length 1.

**Example 1.3.5.** Here we identify the parts in the paths from Example 1.3.3. We write | to switch to the next part, that is (E-part | $\bot$-part | I-part). Some parts may be empty. We write ; between segments. We have (8; 5; 11 | _ | _), (8; 5 | _ | 7,4; 3; 2; 1), (_ | _ |12,9) and (10| _ | 6,4; 3; 2; 1).

**Definition 1.3.6.** Let $\pi$ be a path in a (Prawitz) normal deduction. Define the order $o$ of $\pi$ as follows.

- $o(\pi) = 0$ for a main path
- If $\pi$ ends in a minor premise of $\to$-E, then

$$o(\pi) = 1 + \min\{o(\pi') \mid \pi' \text{ contains corresponding major premise}\}.$$

Informally, the order of a path determines 'the shortest way' to a main path. There is at least one main path in a derivation, since each formula is contained in at least one path. The order of paths is an important concept to prove the subformula property.

**Example 1.3.7.** Consider again the derivation of Example 1.3.3. We determine the orders of the paths.

$$o(8,5,7,4,3,2,1) = 0$$
$$o(10,6,4,3,2,1) = 0$$
$$o(8,5,11) = o(10,6,4,3,2,1) + 1 = 1$$
$$o(12,9) = \min\{o(8,5,7,4,3,2,1), o(8,5,11)\} + 1 = \min\{1,2\} = 1$$

**Lemma 1.3.8.** *Let $\pi = (A_1, \ldots, A_n)$ be a path. Each formula in the E-part or $\bot$-part is subformula of $A_1$. Each formula in the I-part is subformula of $A_n$.*

*Proof.* If $B$ occurs in the E-part, then $B$ is contained in a segment $\sigma$, such that the first formula in $\sigma$ is either an assumption or it is derived from an E-rule different from $\vee$-E. Denote this formula by $A_j$, note that $A_j = B$ as formulas. If $A_j$ is derived from an E-rule, then $B$ is subformula of the major premise $A_{j-1}$ of that E-rule. When $A_j$ is an assumption, then $A_j = A_1$ or $A_j$ is a discharged assumption of $\vee$-E. In the first case we are done. In the second case, $A_j$ is the subformula of the corresponding major premise $A_{j-1} = A_j \vee C$ for some $C$. When repeating the process for $A_{j-1}$, which is also in the E-part, we see that $B$ is a subformula of $A_1$.

If $B$ occurs in the $\bot$-part, then $B$ is contained in a segment $\sigma$, such that the first formula $A_j = \bot$ in $\sigma$ is an assumption or the conclusion of an E-rule different from $\vee$-E. This means that $B$ is subformula of an open assumption or subformula of some formula in the E-part, hence $B$ is subformula of $A_1$.

If $B$ is a formula in the I-part, then $B$ is subformula of the end-formula $A_n$, because of the form of the I-rules. □

If we look closer at the proof of Lemma 1.3.8, we see that for a path $(A_1, \dots, A_n)$ we have the following: For E-part and $\perp$-part $(A_1, \dots, A_j)$ we have that $A_{i+1}$ is subformula of $A_i$ and for I-part $(A_{j+1}, \dots, A_n)$ we have that $A_i$ is subformula of $A_{i+1}$.

**Theorem 1.3.9** (Subformula property)**.** *Let* $\Pi$ *be a (Prawitz) normal derivation of* $\Gamma \vdash A$*. Then each formula* $B$ *in* $\Pi$ *is a subformula of* $A$ *or of a formula in* $\Gamma$*.*

*Proof.* Let $\Pi$ be a Prawitz normal deduction of $\Gamma \vdash A$. We proceed by induction on the order of paths. Consider a formula $B$ in $\Pi$ in a path $\pi = (A_1, \dots, A_n)$. By Proposition 1.3.4, $B$ can occur in the E-part, $\perp$-part or I-part of $\pi$.

If $B = A_n$, then $B$ is either the conclusion of $\Pi$ or the minor premise of the $\to$-E rule. If $B$ is the conclusion, then $B = A$. If $B$ is the minor premise of $\to$-E, then the major premise is of the form $B \to C$ for some $C$, hence $B$ is subformula of the major premise which is contained in a path $\pi'$ with $o(\pi') < o(\pi)$. Applying the induction hypothesis we find that $B$ is a subformula of an open assumption or the conclusion.

If $B$ is contained in the I-part of $\pi$, then $B$ is a subformula of the end-formula $A_n$, by Lemma 1.3.8. So $B$ is a subformula of $A$ or subformula of a formula in $\Gamma$.

If $B = A_1$, then either $B \in \Gamma$ or $B$ is discharged by the $\to$-I rule. If $B$ is a cancelled assumption of $\to$-I, then $B$ is a subformula of the corresponding introduced formula $B \to C$ for some $C$. Formula $B \to C$ is contained in the I-part of $\pi$, or in a path $\pi'$ with $o(\pi') < o(\pi)$. So we can conclude that $B$ is subformula of $A$ or subformula of a formula in $\Gamma$.

If $B$ occurs in the E-part or $\perp$-part, we derive with Lemma 1.3.8 that $B$ is a subformula of $A_1$, which case we treated above. This finishes the proof. $\square$

**Corollary 1.3.10.** IPC *is consistent, that is,* $\nvdash_{\text{IPC}} \perp$*.*

*Proof.* Suppose $\vdash \perp$, then there is a normal deduction ending in $\perp$ without any open assumptions. There is a main path which contains the conclusion $\perp$. This means that there are no I-rules in this path. But this means that the first formula of the path is not discharged. This is a contradiction. $\square$

**Corollary 1.3.11.** IPC *is decidable, that is, there is an effective way to determine whether* $\Gamma \vdash_{\text{IPC}} A$ *or not.*

*Proof.* By the weak normalization theorem (Theorem 1.2.10) we can limit our search to a normal derivation of $\Gamma \vdash A$. This means that we avoid cut segments. We also search in such a way that the conclusion of the $\perp^{\text{i}}$-rule is a proposition letter.

(1) First try whether $A \in \Gamma$, otherwise
(2) try an I-rule if $A$ is composite and try the $\perp^{\text{i}}$-rule if $A$ is a proposition letter,
(3) and try an E-rule for all $B \in \Gamma$ which are composite.

In the recursive case, this gives finitely many possibilities to check and each try creates a new search of the form $\Gamma, C \vdash D$ or $\Gamma \vdash C$, where $C$ and $D$ are subformulas of $\Gamma$ or $A$ by the subformula property. The number of subformulas in the context increases and otherwise the size of the goal-formula decreases. Since the number of all subformulas in $\Gamma$ and $D$ is finite, this search terminates. $\square$

# Chapter 2 | The Truth Table Natural Deduction System

In a recent paper [4], Herman Geuvers and Tonny Hurkens developed a method for deriving propositional natural deduction rules from the truth table for an arbitrary connective. It leads to a method for defining rules in a 'standard format'. This makes it possible to define propositional logic for arbitrary sets of connectives. It is remarkable that in this way not only classical rules can be defined, but also intuitionistic deduction rules. We call this natural deduction system the *truth table natural deduction system*.

In this chapter, we examine the truth table natural deduction system on proof-theoretic properties and semantics of the system. We give the definition and we look at examples and the difference between the classical and intuitionistic rules. Many results are based on the work of Geuvers and Hurkens, but we will add some new results here and there. In Chapter 3, we elaborate more on the system by looking at cut elimination and normalization.

## 2.1 Definition

Here we will introduce the method of deriving elimination and introduction rules for an arbitrary connective $c$ from its truth table [4]. In contrast to [4], we first give the basis of the concerned propositional language. This section can be compared to Section 1.1, where we treated the Prawitz system.

We consider arbitrary connectives, where each connective $c$ is defined via its truth table. The truth table of a connective $c$ is denoted by $t_c$. We write $t_c(a_1, \ldots, a_n)$ with $a_i \in \{0, 1\}$ to mean the value of $c(a_1, \ldots, a_n)$ for those $a_i$'s. This corresponds to a value of a row in $t_c$. If the truth table has $n + 1$ columns, $c$ is called an $n$-ary connective. So $\bot$ and $\top$ are considered as 0-ary connectives. In this thesis we further look at well-known connective $\vee, \wedge, \rightarrow$ and $\neg$. Here, $\neg$ is a defined connective and not an abbreviation $\neg A \equiv A \rightarrow \bot$. We also consider 3-ary connectives if-then-else and most, see Appendix A for the truth tables.

**Definition 2.1.1.** The language of the truth table system contains the following symbols:

(1) *propositional letters:* $p_0, p_1, p_2, \ldots,$
(2) *connectives:* $c_0, c_1, c_2, \ldots,$
(3) *auxiliary signs:* comma , and parentheses ( and ).

**Definition 2.1.2.** Let $\mathcal{C}$ be a set of connectives. The set $PROP_{\mathcal{C}}$ of propositions is the smallest set $X$ such that

(1) for propositional letter $p$ we have $p \in X$,
(2) for each connective $c \in \mathcal{C}$, if $A_1, \ldots, A_n \in X$, then $c(A_1, \ldots, A_n) \in X$.

We work in propositional logic, but we prefer to speak of *formulas* instead of propositions. We write formulas with capital letters $A, B, C$, etc. Sometimes we use Greek letters $\Phi, \Psi$ for formulas. For connectives $\bot$ and $\top$, $\bot()$ and $\top()$ are formulas. Since the parentheses are superfluous, we write $\bot$ and $\top$ for formulas as well.

**Definition 2.1.3.** Formula $A'$ is a *subformula* of formula $A$ if one of the following holds.

- $A' = A$,
- $A = c(A_1, \dots A_n)$ for some connective $c$ and formulas $A_1, \dots, A_n$ such that $A'$ is subformula of $A_i$ for some $i$.

**Definition 2.1.4.** The *rank* $r(A)$ of a formula $A$ is defined by

$$r(A) = \begin{cases} 0, & \text{for proposition letter } A \text{ or 0-ary connective } c() = A, \\ \max(r(A_1), \dots, r(A_n)) + 1, & \text{if } A = c(A_1, \dots, A_n) \text{ for } n\text{-ary connective c, } n \geq 1. \end{cases}$$

The definition of the rank of a formula makes it possible to do induction on the structure of formulas.

Now we turn to the most important definition of this chapter. It defines the natural deduction rules derived from truth tables [4].

**Definition 2.1.5.** Let $c$ be an $n$-ary connective with a truth table $t_c$. We write $\varphi = c(p_1, \dots, p_n)$ where $p_1, \dots, p_n$ are proposition letters and we write $\Phi = c(A_1, \dots A_n)$ where $A_1, \dots, A_n$ are formulas. If the row in truth table $t_c$ gives a 0, then we obtain an *elimination rule* (el). If the row gives a 1, then it gives rise to a classical and intuitionistic *introduction rule* ($\text{in}^{\text{c}}$ and $\text{in}^{\text{i}}$). The rules are defined in the following way, where the truth tables are stated on the left-hand side.

$$\begin{array}{c|c} p_1 \dots p_n & \varphi \\ \hline a_1 \dots a_n & 0 \end{array} \quad \mapsto \quad \frac{\vdash \Phi \qquad \dots \vdash A_j \ (a_j = 1) \dots \qquad \dots A_i \vdash D \ (a_i = 0) \dots}{\vdash D} \text{el}$$

$$\begin{array}{c|c} p_1 \dots p_n & \varphi \\ \hline b_1 \dots b_n & 1 \end{array} \quad \mapsto \quad \frac{\dots \vdash A_j \ (b_j = 1) \dots \qquad \dots A_i \vdash \Phi \ (b_i = 0) \dots}{\vdash \Phi} \text{in}^{\text{i}}$$

$$\begin{array}{c|c} p_1 \dots p_n & \varphi \\ \hline c_1 \dots c_n & 1 \end{array} \quad \mapsto \quad \frac{\Phi \vdash D \qquad \dots \vdash A_j \ (c_j = 1) \dots \qquad \dots A_i \vdash D \ (c_i = 0) \dots}{\vdash D} \text{in}^{\text{c}}$$

For an $n$-ary connective $c$, there will be $2^n$ classical and $2^n$ intuitionistic rules, since there are $2^n$ rows in the truth table. From the definition, it is clear that the elimination rules for connective $c$ are the same for classical and intuitionistic logic.

The rules in Definition 2.1.5 are given in sequent notation in abbreviated form, that is without a possible set containing extended hypotheses. Let $\Gamma$ be a set of formulas, then the elimination rule is as follows:

$$\frac{\Gamma \vdash \Phi \qquad \dots \Gamma \vdash A_j \ (a_j = 1) \dots \qquad \dots \Gamma, A_i \vdash D \ (a_i = 0) \dots}{\Gamma \vdash D} \text{el}$$

The same holds for the introduction rules. We only specify $\Gamma$ when necessary.

We call the sequents above the line in a rule the *premises.* $\Gamma \vdash \Phi$ in the elimination rule is called the *major premise*, all others above the line are called *minor premises.* The sequent below the line is the *consequence* of the rule. In general, a derivation rule has the form

$$\frac{\Gamma \vdash \Phi_1 \ \dots \ \Gamma \vdash \Phi_n \qquad \Gamma, \Psi_1 \vdash D \ \dots \ \Gamma, \Psi_n \vdash D}{\Gamma \vdash D}$$

We call the $\Phi$'s a *lemma* and the $\Psi$'s a *case*. If we look at the rules in Definition 2.1.5, we see that $A_j$ occurs as a lemma, if $a_j = 1$ in truth table $t_c$, and $A_i$ occurs as a case, if $a_i = 0$ in $t_c$.

The rules in Definition 2.1.5 give rise to the classical and intuitionistic natural deduction systems based on truth tables.

**Definition 2.1.6.** Let $\mathcal{C}$ be a set of connectives. We define the *intuitionistic* and *classical* natural deduction systems for $\mathcal{C}$, $\mathrm{IPC}_\mathcal{C}$ and $\mathrm{CPC}_\mathcal{C}$ respectively, as follows:

- Both systems have an *axiom rule*

$$\frac{}{\Gamma \vdash A}\ \text{axiom, if } A \in \Gamma$$

- $\mathrm{IPC}_\mathcal{C}$ contains the elimination rules and intuitionistic introduction rules for all connectives in $\mathcal{C}$.
- $\mathrm{CPC}_\mathcal{C}$ contains the elimination rules and classical introduction rules for all connectives in $\mathcal{C}$.

We will often refer to these systems as the *truth table natural deduction system system* or just as the *truth table system.*

We write $\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} A$ if $\Gamma \vdash A$ is derivable using the rules of $\mathrm{IPC}_\mathcal{C}$, idem for $\mathrm{CPC}_\mathcal{C}$ with $\mathrm{IPC}_\mathcal{C}$ replaced by $\mathrm{CPC}_\mathcal{C}$. In these systems it is possible to make derivations of formulas. A *derivation* in $\mathrm{IPC}_\mathcal{C}$ or $\mathrm{CPC}_\mathcal{C}$ is a tree built up from the rules in $\mathrm{IPC}_\mathcal{C}$ or $\mathrm{CPC}_\mathcal{C}$ respectively, where its leaves are derived from the axiom rule. We use Greek capital letters $\Pi$ or $\Sigma$ to denote derivations. The end-sequent is called the *conclusion* or *consequence.* If the end-sequent in a derivation $\Pi$ is of the form $\Gamma \vdash D$, then $\Gamma$ is the set of *assumptions* and we say that $D$ is *derived from* $\Gamma$.

For the Prawitz system holds IPC $\subsetneq$ CPC. This property holds also in the truth table system of deduction, that is, $\mathrm{IPC}_\mathcal{C} \subsetneq \mathrm{CPC}_\mathcal{C}$, for every set of connectives $\mathcal{C}$. This can be shown by examining the introduction rules. If $\Phi$ is derived from the intuitionistic introduction rule

$$\frac{\ldots \vdash A_j\ (b_j = 1) \ldots \qquad \ldots A_i \vdash \Phi\ (b_i = 0) \ldots}{\vdash \Phi}\ \mathrm{in}^{\mathrm{i}},$$

then this deduction can be transformed into a classical deduction as follows.

$$\frac{\Phi \vdash \Phi \qquad \ldots \vdash A_j\ (c_j = 1) \ldots \qquad \ldots A_i \vdash \Phi\ (c_i = 0) \ldots}{\vdash \Phi}\ \mathrm{in}^{\mathrm{c}}.$$

The terminology of 'intuitionistic' rules and 'classical' rules will be justified when we define a Kripke semantics and a 1-point Kripke semantics respectively in Sections 2.4 and 2.5.

**Example 2.1.7.** Consider the truth table of $\wedge$.

| $A$ | $B$ | $A \wedge B$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

We derive the following intuitionistic rules for $\wedge$ labeled by their corresponding entries in the rows.

$$\frac{\vdash A \wedge B \qquad A \vdash D \qquad B \vdash D}{\vdash D}\ \wedge\text{-el}_{00} \qquad \frac{\vdash A \wedge B \qquad \vdash A \qquad B \vdash D}{\vdash D}\ \wedge\text{-el}_{10}$$

$$\frac{\vdash A \wedge B \qquad A \vdash D \qquad \vdash B}{\vdash D}\ \wedge\text{-el}_{01} \qquad \frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\ \wedge\text{-in}^{\mathrm{i}}$$

See also Appendix A for the rules of the usual connectives $\vee$, $\rightarrow$, $\neg$, $\bot$ and $\top$.

Natural deduction derived from truth tables can be defined for each connective via its truth table. This means that every possible truth table of arity $n$ gives rise to deduction rules. We can for instance examine the rules for if-then-else and $\mathsf{most}(A, B, C)$ which have both arity 3. We write $A \to B/C$, to mean 'if $A$, then $B$, else $C$'. See Appendix A for the truth tables and rules for $A \to B/C$ and $\mathsf{most}(A, B, C)$.

An important aspect of the rules derived from truth tables is that for each connective, the rules are completely 'self-contained'. This means that we do not need to explain one connective in terms of another. It can be illustrated by Peirce's Law, $((A \to B) \to A) \to A$. This statement holds in classical logic. In the Prawitz system, we have to use the RAA rule, while $\perp$ does not occur in Peirce's law. In Example 2.5.1, we will see that we can prove $\vdash_{\mathrm{CPC}_{\{\to\}}} ((A \to B) \to A) \to A$, so we only need the rules for $\to$.

We end this section with some more terminology, which we also used in the Prawitz system, see Section 1.1. We adopt the same notions of formula $A$ standing *immediately above* formula $B$ and $A$ and $B$ being *side-connected*. In the truth table system we also exclude redundant application of rules. An application of a rule is *redundant*, if a case of that rule is not used in an axiom rule. This means that this case is not discharged as an assumption. In contrast to the Prawitz system, introduction rules can also be redundant.

**Example 2.1.8.** Here we consider a short example where both case $B$ and case $C$ are not used in an axiom rule. This means that the $\vee$-el is redundant. $\Sigma$ is a subderivation with conclusion $\vdash B \vee C$. Sequent $\vdash A \to A$ is two times proved using intuitionistic introduction rules for $\to$, see Appendix A for the rules.

$$\dfrac{\begin{array}{c}\Sigma\\ \vdash B \vee C\end{array} \qquad \dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A \vdash A \to A} \qquad \dfrac{A \vdash A \qquad A \vdash A}{A \vdash A \to A}}{\vdash A \to A} \qquad \dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A \vdash A \to A} \qquad \dfrac{A \vdash A \qquad A \vdash A}{A \vdash A \to A}}{\vdash A \to A}}{\vdash A \to A}\ \vee\text{-el}$$

## 2.2 Optimizing the rules

Each connective gives rise to many deduction rules, namely $2^n$ for an $n$-ary connective. However, it is possible to reduce the number of rules in some cases. In this section we will see the reduced formats [4].

We first look at two standard lemmas. The first lemma is the weakening property. The second lemma shows how to put derivations together in one derivation.

**Lemma 2.2.1** (Weakening)**.** *If $\Pi$ is a derivation of $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$, then $\Pi$ is also a derivation of $\Delta \vdash A$.*

*Proof.* Proof by simple induction on the derivation of $\Gamma \vdash A$. □

**Lemma 2.2.2.** *If $\Gamma \vdash A$ and $\Delta, A \vdash B$, then $\Gamma, \Delta \vdash B$.*

*Proof.* This is proved by induction on the derivation of $\Delta, A \vdash B$. Suppose $\Gamma \vdash A$.

(Axiom) First suppose that $\Delta, A \vdash B$ is derived from the axiom rule. Then $B \in \Delta$ or $A = B$. If $B \in \Delta$ then $\Gamma, \Delta \vdash B$. If $A = B$ then $\Gamma \vdash B$ and so $\Gamma, \Delta \vdash B$, by the weakening property.

(Elimination) Let $\Delta, A \vdash B$ be the conclusion of an elimination rule of some connective $c$ with major premise $\Phi = c(C_1, \ldots, C_n)$, for some formulas $C_1, \ldots, C_n$:

$$\frac{\Delta, A \vdash \Phi \qquad \ldots \Delta, A \vdash C_j \ldots \qquad \ldots \Delta, A, C_i \vdash B \ldots}{\Delta, A \vdash B}\ \text{el}$$

From the induction hypothesis we know that $\Gamma, \Delta \vdash \Phi$ and $\Gamma, \Delta \vdash C_j$ for 1-entries in the truth table $t_c$ and $\Gamma, \Delta, C_i \vdash B$ for 0-entries. With the elimination rule for $c$, we conclude $\Gamma, \Delta \vdash B$.

(Introduction) The proof for classical and intuitionistic introduction is analogous to the elimination case. □

This principle can be written in a more suggestive way using proof trees. If $\Sigma$ is a derivation of $\Gamma \vdash A$ and $\Pi$ of $\Delta, A \vdash B$, then the derivation of $\Gamma, \Delta \vdash B$ becomes:

$$\begin{array}{ccc} \Sigma & & \Sigma \\ \Gamma \vdash A & \ldots & \Gamma \vdash A \\ & \Pi & \\ & \Gamma, \Delta \vdash B & \end{array}$$

This is possible, because the only place in $\Pi$ where the hypothesis $A$ can be used is at an instance of the axiom rule of the shape $\Delta', A \vdash A$ for some $\Delta' \supseteq \Delta$.

The following two lemmas state the reductions to the optimized rules. For proofs see [4]. Note that Lemma 2.2.2 is needed for Lemma 2.2.4.

**Lemma 2.2.3.** *Two deriving rules of the form*

$$\frac{\vdash \Phi_1 \ldots \vdash \Phi_n \qquad \Psi_1 \vdash D \ldots \Psi_m \vdash D \qquad A \vdash D}{\vdash D}\ ,\quad \frac{\vdash \Phi_1 \ldots \vdash \Phi_n \qquad \vdash A \qquad \Psi_1 \vdash D \ldots \Psi_m \vdash D}{\vdash D}$$

*are equivalent to the rule*

$$\frac{\vdash \Phi_1 \ldots \vdash \Phi_n \qquad \Psi_1 \vdash D \ldots \Psi_m \vdash D}{\vdash D}\ .$$

**Lemma 2.2.4.** *The following rules are equivalent:*

$$\frac{\vdash \Phi_1\ \ldots\ \vdash \Phi_n \qquad \Psi \vdash D}{\vdash D} \qquad \textit{and} \qquad \frac{\vdash \Phi_1\ \ldots\ \vdash \Phi_n}{\vdash \Psi}.$$

In Appendix A, all optimized rules are written down for $\vee, \wedge, \rightarrow, \neg$, $\bot$ and $\top$. We also present there the intuitionistic optimized rules for if-then-else and most.

**Example 2.2.5.** The classical optimized rules are equivalent to the intuitionistic optimized rules for $\wedge$. There are two elimination rules and one introduction.

$$\frac{\vdash A \wedge B}{\vdash A}\ \wedge\text{-el}_1 \qquad \frac{\vdash A \wedge B}{\vdash B}\ \wedge\text{-el}_2 \qquad \frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\ \wedge\text{-in}$$

Note that these rules correspond to the Prawitz rules.

Reductions with Lemma 2.2.3 and Lemma 2.2.4 do not always result in unique optimized rules. The following example gives the usual intuitionistic optimized rules for if-then-else.

**Example 2.2.6.** The connective if-then-else is defined by the following truth table, which is also presented in Appendix A.

| $A$ | $B$ | $C$ | $A \to B/C$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

We consider the intuitionistic rules. There are four elimination rules and four introduction rules from the definition. These can be optimized to two elimination and two introduction rules. The first two elimination rules optimize to else-el and the last two elimination rules are equivalent to then-el. The first two introduction rules optimize to else-in and the last two to then-in.

$$\frac{\vdash A \to B/C \quad A \vdash D \quad C \vdash D}{\vdash D}\ \text{else-el} \qquad \frac{\vdash A \to B/C \qquad \vdash A}{\vdash B}\ \text{then-el}$$

$$\frac{A \vdash A \to B/C \qquad \vdash C}{\vdash A \to B/C}\ \text{else-in} \qquad \frac{\vdash A \qquad \vdash B}{\vdash A \to B/C}\ \text{then-in}$$

These are the usual intuitionistic optimized rules for if-then-else. But it is possible to optimize in another way, such as the following.

$$\frac{\vdash A \to B/C \quad B \vdash D \quad C \vdash D}{\vdash D}\ \text{if-el} \qquad \frac{\vdash B \qquad \vdash C}{\vdash A \to B/C}\ \text{if-in}$$

Just as we saw in Example 2.2.5 for $\wedge$, it is possible that the classical rules and the intuitionistic rules are the same in optimized form. This is also the case for $\vee$ and most. In general, we establish a new result stating that IPC and CPC are equivalent for monotone connectives.

**Definition 2.2.7.** An $n$-ary connective $c$ is *monotone* if for each row in the truth table $t_c$ with $t_c(p_1, \ldots, p_n) = 1$, we have that $t_c(p_1, \ldots, p_{i-1}, 1, p_{i+1}, \ldots, p_n) = 1$, for each $i$.

Indeed, $\vee, \wedge$ and most are monotone connectives. We can also phrase this definition in a slightly different way, where we use a preorder on rows of propositional letters $p \in \{0, 1\}$.

**Definition 2.2.8.** For sequences of propositional letters of length $n$, we define ordering $\sqsubseteq$ as follows. If $p_i \leq q_i$ for all $i$, then $(p_1, \ldots, p_n) \sqsubseteq (q_1, \ldots, q_n)$.

So $n$-ary connective $c$ is monotone if: if $t_c(p_1, \ldots, p_n) = 1$ and $(p_1, \ldots, p_n) \sqsubseteq (q_1, \ldots, q_n)$, then $t_c(q_1, \ldots, q_n) = 1$.

It is useful to identify optimized rules of $c$ by certain sequences of 0's and 1's including gaps as follows. Let $r$ be a rule for $n$-ary connective $c$ with lemmas indexed by set $L \subseteq \{1, \ldots, n\}$ and cases indexed by $C \subseteq \{1, \ldots, n\} \setminus L$. Recall that lemmas correspond to certain 1-entries in the truth table and cases to 0-entries. Let $I \subseteq \{1, \ldots, n\} \setminus (L \cup C)$ be a set of gaps. Then we identify rule $r$ by sequence $(p_1, \ldots, p_n)$ with $p_i = 1$ if $i \in L$, $p_i = 0$ if $i \in C$ and $i = \_$ for $i \in I$. In this way, for both CPC and IPC, each rule has a sequence, but not all sequences correspond to a derivation rule of $c$.

**Example 2.2.9.** The two optimized introduction rules of disjunction

$$\frac{\vdash A}{\vdash A \vee B}\ \vee\text{-in}_1 \quad \text{and} \quad \frac{\vdash B}{\vdash A \vee B}\ \vee\text{-in}_2$$

can be identified by $(1, \_)$ and $(\_, 1)$. Sequences $(0, \_)$ and $(\_, 0)$ do not correspond to rules of $\vee$.

We can now adopt the ordering on optimized sequences as follows.

**Definition 2.2.10.** Let $I \subseteq \{1, \dots, n\}$ be the index set of gaps. We define $\sqsubseteq_I$ on sequences with gaps in $I$ as follows. Define $(p_1, \dots, p_n) \sqsubseteq_I (q_1, \dots, q_n)$, if $p_i = q_i = \_$ for all $i \in I$ and $p_j \leq q_j$ for all $j \notin I$.

Note that sequences can only be related if they have gaps on the same places. This means that sequences $(1, \_)$, $(\_, 1)$ and $(1, 1)$ are not comparable. We have $(\_, 0) \sqsubseteq_2 (\_, 1)$. If $I = \{1, \dots, n\}$ then $\sqsubseteq_I$ is the same as $\sqsubseteq$. Now we consider crucial definitions.

**Definition 2.2.11.** Let $r$ be an introduction rule for $n$-ary connective $c$ identified by sequence $(p_1, \dots, p_n)$ with index set $I$ of gaps. We call $r$ *minimal with regard to* $I$ if $(q_1, \dots, q_n)$ is not an introduction rule for all $(q_1, \dots, q_n) \sqsubseteq_I (p_1, \dots, p_n)$ and $(q_1, \dots, q_n) \neq (p_1, \dots, p_n)$.

Note that $(q_1, \dots, q_n)$ does not have to be any rule at all. Since set of gaps $I$ belongs to $r$ we will just write *minimal*. Because of the link from optimized rules to sequences, we can also write $r \sqsubseteq_I r'$ for rules $r, r'$.

**Definition 2.2.12.** Let $r$ be a minimal introduction rule of connective $c$ with index set $I$ of gaps. For both IPC and CPC, we define the *upper set* of $r$ as

$$U_r = \{r' \mid r \sqsubseteq_I r'\}.$$

**Example 2.2.13.** Consider the truth table of disjunction $\vee$.

| $A$ | $B$ | $A \vee B$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

Both introduction rules defined by rows $(0, 1)$ and $(1, 0)$ are minimal rules. The introduction rule defined by $(1, 1)$ is not minimal. The two optimized rules identify by $(1, \_)$ and $(\_, 1)$ are both minimal. We have $U_{(0,1)} = \{(0, 1), (1, 1)\}$, $U_{(1,0)} = \{(1, 0), (1, 1)\}$, $U_{(1,\_)} = \{(1, \_)\}$ and $U_{(\_,1)} = \{(\_, 1)\}$.

We have the following observation which is evident from the definition of monotonicity.

**Lemma 2.2.14.** *Let $c$ be a monotone connective with non-optimized minimal introduction rules $r_1, \dots, r_t$ for* IPC *or* CPC*. Then $U_{r_1} \cup \dots \cup U_{r_t}$ contains all introduction rules of $c$ for* IPC *or* CPC *respectively.*

*Proof.* From monotonicity it is clear that $U_{r_1} \cup \dots \cup U_{r_t}$ only contains introduction rules. It contains all introduction rules since for each non-optimized introduction rule $r$ we have $r_i \sqsubseteq r$ for some non-optimized minimal rule $r_i$. ☐

To prove the next proposition, we need the following definition and lemma.

**Definition 2.2.15.** Connective $c$ is *monotone in index set $I$* if the following holds. If $(p_1, \ldots, p_n)$ with gaps in $I$ correspond to an introduction rule, and $(p_1, \ldots, p_n) \sqsubseteq_I (q_1, \ldots, q_n)$, then $(q_1, \ldots, q_n)$ with the same gaps is also an introduction rule.

**Lemma 2.2.16.** *If connective $c$ is monotone in index set $I$, $c$ is monotone in $I \cup \{j\}$ for every $j$.*

*Proof.* Suppose $(p_1, \ldots, p_n)$ is an introduction rules with gaps in $I \cup \{j\}$ with $j \notin I$ such that $(p_1, \ldots, p_n) \sqsubseteq_{I\cup\{j\}} (q_1, \ldots, q_n)$. By Lemma 2.2.3, sequence $(p_1, \ldots, p_n)$ is equivalent to the two rules

$$(p_1, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n) \text{ and } (p_1, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n).$$

These are introduction rules, because only introduction rules can reduce to an introduction rule. Connective $c$ is monotone in index set $I$, so $(p_1, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n) \sqsubseteq_I (q_1, \ldots, q_{j-1}, 0, q_{j+1}, \ldots, q_n)$ and $(p_1, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n) \sqsubseteq_I (q_1, \ldots, q_{j-1}, 1, q_{j+1}, \ldots, q_n)$ are also introduction rules. By Lemma 2.2.3, $(q_1, \ldots, q_n)$ is an introduction rule. ☐

**Proposition 2.2.17.** *Let $c$ be a monotone in index set $I$. Let $r$ be an optimized minimal rule with gaps in $I$ and lemmas on 1-entries $L = \{i_1, \ldots, i_k\}$. Then the introduction rules in $U_r$ are equivalent to the following introduction rule*

$$\frac{\vdash A_{i_1} \qquad \ldots \qquad \vdash A_{i_k}}{\vdash c(A_1, \ldots, A_n)}\ .$$

*This is true for both* IPC *and* CPC.

*Proof.* Let $r_{\min}$ be a minimal rule with lemmas indexed by $L = \{i_1, \ldots, i_k\}$ and gaps $I$. We have cases $C = \{1, \ldots, n\} \setminus (L \cup I)$. Note that $L$, $I$ and $C$ are disjoint sets. We proceed by induction on the number of elements in $I$ until $\#I = n - \#L$. Write $\Phi = c(A_1, \ldots, A_n)$.

If $\#I = n - \#L$, then in the intuitionistic rule, there are only lemmas $A_{i_1}, \ldots, A_{i_k}$, so it is already in the right form. For the classical case, we apply Lemma 2.2.4 such that we have

$$\frac{\Phi \vdash D \qquad \vdash A_{i_1} \ldots \vdash A_{i_k}}{\vdash D} \quad \text{is equivalent to} \quad \frac{\vdash A_{i_1} \qquad \ldots \qquad \vdash A_{i_k}}{\vdash \Phi}\ .$$

For $\#I < n - \#L$, the first part of the proof is the same for IPC and CPC where we use Lemma 2.2.3. Consider pairs $[r_1, r_2]$ of rules in $U_{r_{\min}}$ such that the sequences of the rules differ on one entry. This means that there exists a $j \in C$ such that $r_1 = (p_1, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n)$ and $r_2 = (p_1, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n)$. Note that $p_i = 1$ for all $i \in L$. Such a pair $[r_1, r_2]$ is equivalent to rule $(p_1, \ldots, p_{j-1}, \_, p_{j+1}, \ldots, p_n)$ by applying Lemma 2.2.3. In addition, for each $j \in C$ there is an $r$ such that $[r_{\min}, r]$ amounts to a minimal introduction rule $r'$ with regard to index set $I' = I \cup \{j\}$. Moreover, since $c$ is monotone in $I$, $c$ is also monotone in $I \cup \{j\}$ for every $j \in C$. Note that all rules in $U_{r_{\min}}$ are equivalent to all rules in $\bigcup U_{r'}$. For each optimized minimal rule $r'$ we can apply the induction hypothesis to $U_{r'}$. For the intuitionistic case, we now have that the rules in each $U_{r'}$ are equivalent to the rule

$$\frac{\vdash A_{i_1} \qquad \ldots \qquad \vdash A_{i_k}}{\vdash \Phi}$$

So for IPC, we conclude that rules in $U_{r_{\min}}$ are equivalent to that derivation. For the classical case we know that the rules in each $U_{r'}$ are equivalent to

$$\frac{\Phi \vdash D \qquad \vdash A_{i_1} \ldots \vdash A_{i_k}}{\vdash D}$$

So rules in $U_{r_{\min}}$ are equivalent to that rule. Now apply Lemma 2.2.4 to get the desired result for CPC. ☐

The following example illustrates the strategy of the proof.

**Example 2.2.18.** Consider an arbitrary monotone 4-ary connective $c$ with non-optimized minimal introduction rule indicated by $r_1 = (1,1,0,0)$. Since $c$ is monotone we have also non-optimized introduction rules $r_2 = (1,1,1,0)$, $r_3 = (1,1,0,1)$ and $r_4 = (1,1,1,1)$. Pair $[r_1, r_2]$ optimizes to $r'_1 = (1,1,\_,0)$. We get also optimized rules $r'_2 = (1,1,0,\_)$, $r'_3 = (1,1,1,\_)$, $r'_4 = (1,1,\_,1)$. Rules $r'_1$ and $r'_2$ are minimal with $U_{r'_1} = \{r'_1, r'_4\}$ and $U_{r'_2} = \{r'_2, r'_3\}$. Then both $U_{r'_1}$ and $U_{r'_2}$ optimize to $(1,1,\_,\_)$. So we conclude that rules of $U_{r_1}$ are equivalent to $(1,1,\_,\_)$.

Elimination rules are the same for IPC and CPC, so for a monotone connective, IPC and CPC are equivalent. This is also true for a set of monotone connectives, since rules can only be optimized when they are derived from the same connective. This yields the following corollary.

**Corollary 2.2.19.** *Let $\mathcal{C}$ be a set of monotone connectives. Then* $\mathrm{IPC}_\mathcal{C}$ *and* $\mathrm{CPC}_\mathcal{C}$ *are equivalent.*

*Proof.* Follows from Proposition 2.2.17 with $I = \emptyset$ and Lemma 2.2.14 together with the remark of above. □

## 2.3 A few examples

This section gives insights in how to use the truth table system by presenting examples of derivations. The goal of presenting these examples is to get a feeling of how to use the (optimized) derivation rules. It is in particular not our aim to explain difficult examples, because big examples do not contribute to the understanding of the rules.

First we look at examples with $\to$. The two optimized introduction rules

$$\frac{\vdash B}{\vdash A \to B}\,\to\text{-in}^{\mathrm{i}}{}_1 \qquad \text{and} \qquad \frac{A \vdash A \to B}{\vdash A \to B}\,\to\text{-in}^{\mathrm{i}}{}_2$$

are equivalent to the more common introduction rule of Prawitz, which is

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,.$$

This is shown in the following lemma.

**Lemma 2.3.1** (Deduction Theorem). *Let $\mathcal{C}$ be a set of connectives containing $\to$. For* $\mathrm{IPC}_\mathcal{C}$ *and* $\mathrm{CPC}_\mathcal{C}$ *we have*

$$\Gamma \vdash A \to B \quad \textit{if and only if} \quad \Gamma, A \vdash B.$$

*Proof.* First suppose $\Gamma \vdash A \to B$, then

$$\cfrac{\cfrac{\Gamma \vdash A \to B}{\Gamma, A \vdash A \to B} \qquad \cfrac{}{\Gamma, A \vdash A}\,\text{axiom}}{\Gamma, A \vdash B}\,\to\text{-el}$$

Now suppose $\Gamma, A \vdash B$. It suffices to show the intuitionistic case:

$$\cfrac{\cfrac{\Gamma, A \vdash B}{\Gamma, A \vdash A \to B}\,\to\text{-in}^{\mathrm{i}}{}_1}{\Gamma \vdash A \to B}\,\to\text{-in}^{\mathrm{i}}{}_2$$

□

The following example states more common properties of $\to$.

**Example 2.3.2.** Intuitionistically, we give derivations of $A \to B, B \to C \vdash A \to B$ and also of $A \to (B \to C), B \vdash A \to C$. Let $\Gamma := \{A \to B, B \to C\}$ and $\Delta := \{A \to (B \to C), B\}$. See Appendix A for the labels of the rules.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash B \to C}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, A \vdash A \to B}\,\text{ax} \quad \dfrac{}{\Gamma, A \vdash A}\,\text{ax}}{\Gamma, A \vdash B}\,\to\text{-el}}{\Gamma, A \vdash C}\,\to\text{-el}}{\Gamma \vdash A \to C}\,\text{Lemma 2.3.1}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta, A \vdash A \to (B \to C)}\,\text{ax} \quad \dfrac{}{\Delta, A \vdash A}\,\text{ax}}{\Delta, A \vdash B \to C}\,\to\text{-el} \qquad \dfrac{\dfrac{}{\Delta, A \vdash A \to B}\,\text{ax} \quad \dfrac{}{\Delta, A \vdash A}\,\text{ax}}{\Delta, A \vdash B}\,\to\text{-el}}{\Delta, A \vdash C}\,\to\text{-el}}{\Delta \vdash A \to C}\,\text{Lemma 2.3.1}$$

We have seen that the optimized truth tables rules for $\to$ are equivalent to the Prawitz rules. This is also the case for $\vee$ and $\wedge$. There is one big difference between both systems. In the Prawitz system, negation is abbreviated as $\neg A \equiv A \to \bot$, whereas in the truth table system, $\neg$ is considered as an apart connective. The following example presents this difference.

**Example 2.3.3.** Here we give a proof of $\neg\neg\neg A \vdash_{\text{IPC}} \neg A$ in the truth table system. Compare this proof with the proof in the Prawitz system in Example 1.1.7.

$$\dfrac{\dfrac{\dfrac{}{\neg\neg\neg A, A \vdash \neg\neg\neg A}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg\neg A, A, \neg A \vdash \neg A}\,\text{ax} \quad \dfrac{}{\neg\neg\neg A, A, \neg A \vdash A}\,\text{ax}}{\neg\neg\neg A, A, \neg A \vdash \neg\neg A}\,\neg\text{-el}}{\neg\neg\neg A, A \vdash \neg\neg A}\,\neg\text{-in}^{\text{i}}}{\neg\neg\neg A, A \vdash \neg A}\,\neg\text{-el}}{\neg\neg\neg A \vdash A}\,\neg\text{-in}$$

So far, we have considered a few examples of the well-known connectives. The following example focuses on the connective if-then-else.

**Example 2.3.4.** We show $A \to B/C \vdash_{\text{IPC}} (A \to B) \wedge (\neg A \to C)$. Later, in Example 2.4.7, we will see that the reverse is not true. Note that the reverse is true in classical logic. We only show necessary formulas of contexts for readability.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A \to B/C}\,\text{ax} \quad \dfrac{}{\vdash A}\,\text{ax}}{A \to B/C, A \vdash B}\,\text{then-el}}{A \to B/C \vdash (A \to B)}\,\text{Lemma 2.3.1} \qquad \dfrac{\dfrac{\dfrac{}{\vdash A \to B/C}\,\text{ax} \quad \dfrac{\dfrac{}{\vdash \neg A}\,\text{ax} \quad \dfrac{}{\vdash A}\,\text{ax}}{\neg A, A \vdash C}\,\neg\text{-el} \quad \dfrac{}{C \vdash C}\,\text{ax}}{A \to B/C, \neg A \vdash C}\,\text{else-el}}{A \to B/C \vdash \neg A \to C}\,\text{Lemma 2.3.1}}{A \to B/C \vdash (A \to B) \wedge (\neg A \to C)}\,\wedge\text{-in}^{\text{i}}$$

It is also possible to prove $(A \wedge B) \vee (\neg A \wedge C) \vdash_{\text{IPC}} A \to B/C$, where again the reverse is only true in classical logic.

Lastly, we give two examples of classical derivations. The first is rather short, the second involves rules of most. More examples in classical logic can be found in Section 2.5, such as Peirce's law.

**Example 2.3.5.** This is the derivation of $\vdash_{\mathrm{CPC}} A \vee \neg A$.

$$\dfrac{\dfrac{\dfrac{}{\neg A \vdash \neg A}\,\text{ax}}{\neg A \vdash A \vee \neg A}\,\vee\text{-in}_2 \qquad \dfrac{\dfrac{}{A \vdash A}\,\text{ax}}{A \vdash A \vee \neg A}\,\vee\text{-in}_1}{\vdash_{\mathrm{CPC}} A \vee \neg A}\,\neg\text{-in}^{\mathrm{c}}$$

**Example 2.3.6.** Here we prove $\vdash_{\mathrm{CPC}} \mathsf{most}(A \to B, B \to C, C \to A)$. See the rules in Appendix A. In the derivation below we write $M = \mathsf{most}(A \to B, B \to C, C \to A)$ and abbreviate $\mathsf{most}$-in to $\mathsf{m}$-in. Due to lack of space we omit the premises of the introduction rules of $\mathsf{most}$, where we sometimes need the rule

$$\dfrac{\vdash P}{\vdash Q \to P}\,\to\text{-in}_1 \ .$$

$$\dfrac{\dfrac{\dfrac{}{A \to B, C \to A \vdash M}\,\mathsf{m}\text{-in}_2 \quad \dfrac{}{A \to B, C \vdash M}\,\mathsf{m}\text{-in}_1}{A \to B \vdash M}\,\to\text{-in}^{\mathrm{c}}{}_2 \qquad \dfrac{\dfrac{}{A, B \to C \vdash M}\,\mathsf{m}\text{-in}_3 \quad \dfrac{}{A, B \vdash M}\,\mathsf{m}\text{-in}_2}{A \vdash M}\,\to\text{-in}^{\mathrm{c}}{}_2}{\vdash M}\,\to\text{-in}^{\mathrm{c}}{}_2$$

## 2.4 Intuitionistic semantics

The difference between the classical and intuitionistic rules is established in the introduction rules. But why do we call them *classical* or *intuitionistic*? At this point, it is not clear why the rules correspond to intuitionistic and classical logic. In this and the following section we will justify the terminology by defining a Kripke semantics. In the next section we will examine classical logic, here we will focus on intuitionistic logic.

In [4], a general Kripke semantics is defined and proved that it is complete for the intuitionistic rules. We will shortly state these results and present some examples. After that, we will look at a generalization of the disjunction property, which is also an important property of intuitionistic logic.

**Definition 2.4.1.** A *Kripke model* is a triple $(W, \leq, \mathsf{at})$, where $W$ is a set of worlds with a reflexive, transitive relation $\leq$ and a function $\mathsf{at} : \mathsf{W} \to \mathcal{P}(\mathsf{At})$ from the set of worlds to the powerset of atoms such that $w \leq w' \Rightarrow \mathsf{at(w)} \subseteq \mathsf{at(w')}$.

When defining the semantics, we want to define a forcing relation $w \Vdash A$ saying that formula $A$ is true in world $w$. We adopt the notation as in [4], where $[\![A]\!]_w = 1$ means $w \Vdash A$ and $[\![A]\!]_w = 0$ means $w \nVdash A$.

**Definition 2.4.2.** For a Kripke model $(W, \leq, \mathsf{at})$ we define $[\![A]\!]_w \in \{0, 1\}$, by induction on $A$ as follows.

- (Atom) If $A$ is a proposition letter, then $[\![A]\!]_w = 1$ iff $A \in \mathsf{at(w)}$.
- (Connective) For $A = c(A_1, \dots, A_n)$, then $[\![A]\!]_w = 1$ iff for each $w' \geq w$, we have that $t_c([\![A_1]\!]_{w'}, \dots, [\![A_n]\!]_{w'}) = 1$, with $t_c$ the truth table for $c$.

This forcing definition is the general notion of the forcing relation on well-known Kripke models.

**Definition 2.4.3.** We define $\Gamma \models_{\mathrm{IPC}} B$ as: for each Kripke model and each world $w$, if for each $A \in \Gamma$ we have $[\![A]\!]_w = 1$, then $[\![B]\!]_w = 1$.

Soundness and completeness are proved in [4] using $B$-maximal sets. Here we only state both statements. In the next section, we will see how $B$-maximal sets are defined and how they can also be used in order to show soundness and completeness of 1-point Kripke models for the classical rules.

**Lemma 2.4.4** (Soundness). *If* $\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} B$, *then* $\Gamma \models_{\mathrm{IPC}_\mathcal{C}} B$.

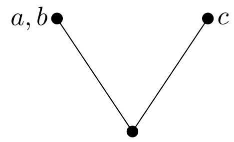
**Theorem 2.4.5** (Completeness). *If* $\Gamma \models_{\mathrm{IPC}_\mathcal{C}} B$, *then* $\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} B$.

Kripke models can be used to prove non-derivability of certain formulas. We give three examples, of which the first one is a well-known example. The latter two are examples of if-then-else.

**Example 2.4.6.** This is the well-known example that $p \vee \neg p$ is not derivable for intuitionistic logic. This example also applies to the truth table system.

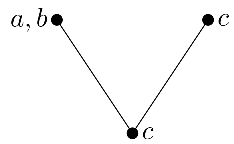

Let $w_0$ be the world at the bottom. We have $[\![p \vee \neg p]\!]_{w_0} = 0$.

**Example 2.4.7.** In Example 2.3.4, we saw that $A \to B/C \vdash_{\mathrm{IPC}} (A \to B) \wedge (\neg A \to C)$. Now we show that the reverse is not true by giving a Kripke model such that for some world $w$ we have $[\![(a \to b) \wedge (\neg a \to c)]\!]_w = 1$ and $[\![a \to b/c]\!]_w = 0$, where $a, b, c$ are proposition letters.




Let $w_0$ be the world at the bottom. It can be verified that $[\![(a \to b) \wedge (\neg a \to c)]\!]_{w_0} = 1$. But $[\![a \to b/c]\!]_{w_0} = 0$, since $t([\![a]\!]_{w_0}, [\![b]\!]_{w_0}, [\![c]\!]_{w_0}) = t(0,0,0) = 0$, where $t$ is the truth table for if-then-else (see Appendix A). Therefore $(a \to b) \wedge (\neg a \to c) \not\models a \to b/c$, so $(a \to b) \wedge (\neg a \to c) \nvdash a \to b/c$.

**Example 2.4.8.** In Example 2.3.4, we said that $(A \wedge B) \vee (\neg A \wedge C) \vdash_{\mathrm{IPC}} A \to B/C$. The reverse is not true. Consider the following Kripke model, where $w_0$ is the world at the bottom.




See Appendix A for the truth tables. We have $[\![a \to b/c]\!]_{w_0} = 1$. But $[\![(a \wedge b) \vee (\neg a \wedge c)]\!]_{w_0} = 0$, because $[\![\neg a]\!]_{w_0} = 0$. So we can conclude that $a \to b/c \not\models (a \wedge b) \vee (\neg a \wedge c)$. Therefore $a \to b/c \nvdash (a \wedge b) \vee (\neg a \wedge c)$.

The Prawitz intuitionistic logic is characterized by the *disjunction property*, that is, if $\Gamma \vdash A \vee B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$. This suggests that disjunction is a strong concept in intuitionistic logic. This is not the case for classical logic, since $\vdash_{\mathrm{CPC}} A \vee \neg A$. It is possible to generalize the notion for arbitrary connectives in the truth table system by introducing the concept of splitting connectives [4].

**Definition 2.4.9.** Let $c$ be an $n$-ary connective and $1 \leq i \lneq j \leq n$. We say that $c$ is an *$i, j$-splitting* connective if the truth table for $c$ has the following form, with $\varphi = c(p_1, \ldots, p_n)$.

| ... | $p_i$ | ... | $p_j$ | ... | $\varphi$ |
|---|---|---|---|---|---|
| ... | 0 | ... | 0 | ... | 0 |
| ... | 0 | ... | 0 | ... | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| ... | 0 | ... | 0 | ... | 0 |

In short, for each row in $t_c$ where $p_i = p_j = 0$ we have $t_c(p_1, \ldots, p_n) = 0$.

Connective $\vee$ is $1,2$-splitting. Connective if-then-else is 1,3-splitting and 2,3 splitting. Connective most is $i,j$-splitting for each $1 \leq i \lneq j \leq 3$.

The following propostion gives the generalization of the disjunction property. It is proved in [4] using the completeness of the Kripke semantics.

**Proposition 2.4.10.** *Let $c$ be an $i,j$-splitting connective. If $\Gamma \vdash_{\mathrm{IPC}} c(A_1,\ldots,A_n)$, then $\Gamma \vdash_{\mathrm{IPC}} A_i$ or $\Gamma \vdash_{\mathrm{IPC}} A_j$.*

## 2.5 Classical semantics

Analogous to the previous section, we define a semantics for the classical rules in order to justify the terminology of 'classical' rules. But before we give this semantics, we first examine the syntax of the classical rules. We compare it with the classical natural deduction of Prawitz. In the natural deduction of Prawitz, the classical property is determined by the proof of contradiction, also known as *Reductio ad absurdum* (RAA). That is the following rule:

$$\begin{array}{c} \neg A \\ \vdots \\ \dfrac{\bot}{A}\ \bot^{\mathrm{c}},\ \mathrm{RAA} \end{array}$$

If this rule is added to the other (intuitionistic) rules of the Prawitz system, one gets classical logic.

A big difference between the Prawitz system and the truth table system is that the classical aspect of the latter system lies in the rules themselves. We say that, for each connective, the rules are *self-contained*. We do not need to explain one classical rule in terms of another. This means that it is not necessary to add a deduction rule, such as RAA. Therefore, whereas one need RAA in the Prawitz system in the proof of the classical statement Peirce's Law, $((A \to B) \to A) \to A$ , in the truth table system it suffices to only use rules of $\to$.

**Example 2.5.1.** We give a proof of Peirce's Law in system $\mathrm{CPC}_{\to}$. Write $P = ((A \to B) \to A) \to A$. With the optimized rules of $\to$ we have the following derivation, see Appendix A for the rules.

$$\dfrac{\dfrac{\dfrac{}{A \to B, P \vdash P}\,\mathrm{ax} \qquad \dfrac{\dfrac{\dfrac{}{(A \to B) \to A \vdash (A \to B) \to A}\,\mathrm{ax} \qquad \dfrac{}{A \to B \vdash A \to B}\,\mathrm{ax}}{A \to B, (A \to B) \to A \vdash A}\,\to\text{-el}}{A \to B, (A \to B) \to A \vdash P}\,\to\text{-in}^{\mathrm{c}}{}_1}{A \to B \vdash P}\,\to\text{-in}^{\mathrm{c}}{}_2 \qquad \dfrac{\dfrac{}{A \vdash A}\,\mathrm{ax}}{A \vdash P}\,\to\text{-in}^{\mathrm{c}}{}_1}{\vdash P}\,\to\text{-in}^{\mathrm{c}}{}_2$$

Other typical classical statements are $\vdash_{\mathrm{CPC}} A \vee \neg A$ and $\neg\neg A \vdash_{\mathrm{CPC}} A$. We proved the first one in Example 2.3.5. Below we prove the second.

**Example 2.5.2.** For every formula $A$ we have $\neg\neg A \vdash_{\mathrm{CPC_C}} A$.

$$\dfrac{\dfrac{\dfrac{}{\neg\neg A \vdash \neg\neg A}\,\mathrm{ax} \qquad \dfrac{}{\neg A \vdash \neg A}\,\mathrm{ax}}{\neg\neg A, \neg A \vdash A}\,\neg\text{-el} \qquad \dfrac{}{\neg\neg A, A \vdash A}\,\mathrm{ax}}{\neg\neg A \vdash A}\,\neg\text{-in}^{\mathrm{c}}$$

From the examples above we see that we can prove typical classical statements using the classical deduction rules. In the following we justify the name of 'classical' by defining a 1-point Kripke semantics for the classical rules and prove that it is complete. Note that 1-point Kripke models correspond to truth tables. In [4], only a semantics was given for the intuitionistic rules. Here we present how it works in the classical case.

Recall that $PROP_{\mathcal{C}}$ is the set of all formulas containing connectives from set $\mathcal{C}$.

**Definition 2.5.3.** A mapping $v : PROP_{\mathcal{C}} \to \{0,1\}$ is a *valuation* if for each $c \in \mathcal{C}$ we have $v(c(A_1,\dots,A_n)) = t_c(v(A_1),\dots,v(A_n))$, with $t_c$ the truth table of $c$.

We see that each row of the truth table of $c$ corresponds to a valuation. Note that a valuation $v$ is uniquely determined by its values on the atoms. Using a similar notation as in [4], we write $[\![A]\!]_v$ to mean $v(A)$.

**Definition 2.5.4.** We define $\Gamma \models_{\mathrm{CPC}} B$ as: for each valuation $v$, if for each $A \in \Gamma$ we have $[\![A]\!]_v = 1$, then $[\![B]\!]_v = 1$.

We can prove soundness and completeness.

**Lemma 2.5.5** (Soundness)**.** *If* $\Gamma \vdash_{\mathrm{CPC}_{\mathcal{C}}} B$*, then* $\Gamma \models_{\mathrm{CPC}_{\mathcal{C}}} B$.

*Proof.* Proof by induction on the derivation of $\Gamma \vdash_{\mathrm{CPC}_{\mathcal{C}}} B$. Here we write $\Gamma \vdash B$ instead of $\Gamma \vdash_{\mathrm{CPC}_{\mathcal{C}}} B$ for simplicity. Since the axiom rule and the elimination rule are the same for IPC and CPC and soundness of all Kripke models for IPC is shown, we only have to check the classical introduction rule. However, we present all cases to see how the proof works for all cases.

(Axiom) First suppose that $\Gamma \vdash B$ is derived from the axiom rule. Then if $[\![A]\!]_v = 1$ for each $A \in \Gamma$, then always $[\![B]\!]_v = 1$ since $B \in \Gamma$ by the axiom rule.

(Elimination) Let $\Gamma \vdash B$ be the conclusion of an elimination rule of some connective $c$ such that $t_c(p_1,\dots,p_m) = 0$, write $A = c(A_1,\dots,A_n)$:

$$\frac{\Gamma \vdash A \qquad \dots\Gamma \vdash A_j \ (p_j = 1)\dots \qquad \dots\Gamma, A_i \ \vdash B \ (p_i = 0)\dots}{\Gamma \vdash B}\ \mathrm{el}$$

From the induction hypothesis we know $\Gamma \models A$, $\Gamma \models A_j$ $(p_j = 1)$ and $\Gamma, A_i \models B$ $(p_i = 0)$. Let $v$ be a valuation such that $[\![C]\!]_v = 1$ for all $C \in \Gamma$. We distinguish two cases:

- $[\![A_i]\!]_v = 1$ for some $i$ with $p_i = 0$. Then with $\Gamma, A_i \models B$, we know $[\![B]\!]_v = 1$.
- $[\![A_i]\!]_v = 0$ for all $i$ with $p_i = 0$. From $\Gamma \models A_j$ for all $j$ with $p_j = 1$ it follows that $[\![A_j]\!]_v = 1$. Hence $[\![A]\!]_v = t_c([\![A_1]\!]_v,\dots,[\![A_n]\!]_v) = 0$. But $\Gamma \models A$ gives us $[\![A]\!]_v = 1$. This leads to a contradiction, so this case is not possible.

Therefore $[\![B]\!]_v = 1$.

(Introduction) Now let $\Gamma \vdash B$ be the conclusion of an introduction rule of some connective $c$ such that $t_c(p_1,\dots,p_m) = 1$, write $A = c(A_1,\dots,A_n)$:

$$\frac{\Gamma, A \vdash B \qquad \dots\Gamma \vdash A_j \ (p_j = 1)\dots \qquad \dots\Gamma, A_i \vdash B \ (p_i = 0)\dots}{\Gamma \vdash B}\ \mathrm{in}^{\mathrm{c}}$$

From the induction hypothesis we know $\Gamma, A \models B$, $\Gamma \models A_j$ $(p_j = 1)$ and $\Gamma, A_i \models B$ $(p_i = 0)$. Let $v$ be a valuation such that $[\![C]\!]_v = 1$ for all $C \in \Gamma$. Again we have two cases:

- $[\![A_i]\!]_v = 1$ for some $i$ with $p_i = 0$. Then with $\Gamma, A_i \models B$, we know $[\![B]\!]_v = 1$.
- $[\![A_i]\!]_v = 0$ for all $i$ with $p_i = 0$. By induction hypothesis: $[\![A_j]\!]_v = 1$ for all $j$. So from truth table $t_c$ we know $[\![A]\!]_v = t_c([\![A_1]\!]_v,\dots,[\![A_n]\!]_v) = 1$. Then by $\Gamma, A \models B$ we have $[\![B]\!]_v = 1$.

This completes the proof of soundness. ∎

In order to show completeness, the following definition and facts are useful. These are also used in the proof of completeness for the Kripke semantics for intuitionistic logic in [4].

**Definition 2.5.6.** A set of formulas $\Gamma$ is called *$B$-maximal* if $\Gamma \nvdash B$ and for each formula $A \notin \Gamma$ we have $\Gamma, A \vdash B$.

For a formula $B$ and a set of formulas $\Gamma$ such that $\Gamma \nvdash B$, we can construct a $B$-maximal set $\Gamma'$ such that $\Gamma' \supseteq \Gamma$. This can be done in the following way. Take an enumeration of all formulas $A_1, A_2, \ldots$. Define recursively $\Gamma_0 := \Gamma$ and $\Gamma_{i+1} := \Gamma_i$ if $\Gamma_i, A_{i+1} \vdash B$ and $\Gamma_{i+1} := \Gamma_i \cup \{A_{i+1}\}$ if $\Gamma_i, A_{i+1} \nvdash B$. Then take $\Gamma' := \bigcup_{i \in \mathbb{N}} \Gamma_i$. Note that this $B$-maximal set $\Gamma'$ depends on the enumeration of the formulas.

**Lemma 2.5.7.** *Let $\Gamma$ be a $B$-maximal set of formulas. Then*

*(i) for every $A$ we have $A \in \Gamma$ or $\Gamma, A \vdash B$*
*(ii) and for each $A$, if $\Gamma \vdash A$, then $A \in \Gamma$. (Thus if $A \notin \Gamma$, then $\Gamma \nvdash A$.)*

*Proof.* Statement (i) is an elementary verification from the definition of $B$-maximal sets. For statement (ii) suppose that $\Gamma \vdash A$. For a contradiction assume that $A \notin \Gamma$, then $\Gamma, A \vdash B$. By Lemma 2.2.2 we conclude that $\Gamma \vdash B$, which contradicts the fact that $\Gamma$ is a $B$-maximal set. □

**Theorem 2.5.8** (Completeness)**.** *If $\Gamma \models_{\mathrm{CPC}_\mathcal{C}} B$, then $\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} B$.*

*Proof.* The proof follows the method of Van Dalen [1] and Milne [7]. Here we write $\Gamma \models B$ instead of $\Gamma \models_{\mathrm{CPC}_\mathcal{C}} B$ for simplicity. Suppose $\Gamma \models B$. Assume for a contradiction that $\Gamma \nvdash B$. Take a $B$-maximal superset $\Gamma' \supseteq \Gamma$, that is, $\Gamma' \nvdash B$ and for each formula $A \notin \Gamma'$ we have $\Gamma', A \vdash B$.

Consider the valuation $v$ such that $v(p) = 1$ for all atoms $p \in \Gamma'$ and $v(p) = 0$ otherwise. Note that this valuation is unique. Now we claim:

$$[\![A]\!]_v = 1 \text{ if and only if } A \in \Gamma'.$$

The claim is proven below. First assume this claim holds. From $\Gamma' \nvdash B$ we have $B \notin \Gamma'$. So by the claim we have $[\![B]\!]_v = 0$. But since $\Gamma \subseteq \Gamma'$ we have $[\![A]\!]_v = 1$ for all $A \in \Gamma$ by the claim. This means that $\Gamma \not\models B$, which contradicts the assumption. Therefore we conclude that $\Gamma \vdash B$.

Now we prove the claim by induction on $A$. If $A$ is atomic, the claim follows immediately by the assumption on $v$. So suppose $A = c(A_1, \ldots, A_n)$ for some connective $c$. We split the proof in two directions.

(⇒) Suppose $[\![A]\!]_v = 1$ and assume $A \notin \Gamma'$. Consider the subformulas of $A = c(A_1, \ldots, A_n)$:
- $A_j$ with $[\![A_j]\!]_v = 1$. Then by induction hypothesis: $A_j \in \Gamma'$, so $\Gamma' \vdash A_j$.
- $A_i$ with $[\![A_i]\!]_v = 0$. Then by induction hypothesis: $A_i \notin \Gamma'$, so $\Gamma', A_i \vdash B$, since $\Gamma'$ is $B$-maximal.

The assumption $[\![A]\!]_v = 1$ is the same as $t_c([\![A_1]\!]_v, \ldots, [\![A_n]\!]_v) = 1$. Then this row in the truth table yields an introduction rule which allows us to prove $A$. Note that $\Gamma', A \vdash B$, since $A \notin \Gamma'$.

$$\frac{\Gamma', A \vdash B \qquad \ldots \Gamma' \vdash A_j \ ([\![A_j]\!]_v = 1) \ldots \qquad \ldots \Gamma', A_i \vdash B \ ([\![A_i]\!]_v = 0) \ldots}{\Gamma' \vdash B}\ \mathrm{in}^c.$$

But we already had $\Gamma' \nvdash B$, so we have a contradiction. Hence $A \in \Gamma'$.

(⇐) Suppose $A \in \Gamma'$. Keep in mind that $\Gamma'$ is a $B$-maximal set. Consider subformulas of $A$:
- $A_j$ with $[\![A_j]\!]_v = 1$. Then by induction hypothesis: $A_j \in \Gamma'$, so $\Gamma' \vdash A_j$.
- $A_i$ with $[\![A_i]\!]_v = 0$. Then by induction hypothesis: $A_j \notin \Gamma'$, so $\Gamma', A_i \vdash B$.

We also have $\Gamma' \vdash A$, since $A \in \Gamma'$.
Suppose $[\![A]\!]_v = 0$, or equally, $t_c([\![A_1]\!]_v, \ldots, [\![A_n]\!]_v = 0$. Then the elimination rule concludes:

$$\frac{\Gamma' \vdash A \qquad \ldots \Gamma' \vdash A_j \; ([\![A_j]\!]_v = 1) \ldots \qquad \ldots \Gamma', A_i \vdash B \; ([\![A_i]\!]_v = 0) \ldots}{\Gamma' \vdash B} \text{ el.}$$

This cannot be the case, because $\Gamma'$ is a $B$-maximal set. So we have $[\![A]\!]_v = 1$.

This finishes the proof of the claim. □

**Corollary 2.5.9.** *For any set of connectives $\mathcal{C}$, $\mathrm{CPC}_\mathcal{C}$ is decidable, that is, there is an effective way to determine whether $\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} A$ or not.*

*Proof.* In order to check whether $\Gamma \vdash A$ or not, it is sufficient to determine whether $\Gamma \models A$ or not. Here we will give an effective procedure to determine the latter. For each $B \in \Gamma$, it is possible to make a truth table. This can be done in finitely many steps, since there is a finite number of subformulas of $B$. It is also possible to make a truth table for $A$. Now we have to check for each valuation $v$, if $[\![B]\!]_v = 1$ for each $B \in \Gamma$ whether $[\![A]\!]_v = 1$. If this is the case for each $v$, then $\Gamma \vdash A$. We have $\Gamma \nvdash A$ otherwise. □

## 2.6 Glivenko's translation

We have seen that $\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} A$ implies $\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} A$. An interesting question is how to transfer from classical logic to intuitionistic logic. Well-known methods are the translations of Gödel and Glivenko. These translations are applied to the Prawitz natural deduction system containing formulas with negation $\neg$. In the truth table systems $\mathrm{CPC}_\mathcal{C}$ and $\mathrm{IPC}_\mathcal{C}$ it is not guaranteed that $\mathcal{C}$ contains $\neg$. Therefore a new kind of translation has to be found. Unfortunately, this goal is not reached yet.

Here we consider Glivenko's translation applied to $\mathrm{CPC}_\mathcal{C}$ and $\mathrm{IPC}_\mathcal{C}$ for a set of connectives $\mathcal{C}$ which contains negation $\neg$.

**Proposition 2.6.1** (Glivenko's Theorem)**.** *Let $\mathcal{C}$ be a set of connectives such that $\neg \in \mathcal{C}$. Then*

$$\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} A \textit{ if and only if } \neg\neg\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg A,$$

*where $\neg\neg\Gamma$ is the set containing $\neg\neg B$ for all $B \in \Gamma$.*

The proof is based on the following two lemmas.

**Lemma 2.6.2.** *In $\mathrm{CPC}_\mathcal{C}$ and $\mathrm{IPC}_\mathcal{C}$ we have the following facts.*

*(i)* $A \vdash \neg\neg A$,

*(ii)* $\neg\neg\neg A \dashv\vdash \neg A$,

*(iii)* $\dfrac{\Gamma, A \vdash B}{\Gamma, \neg B \vdash \neg A}$ .

*Proof.* See for the rules Appendix A. All statements follow from $\neg$-el and $\neg$-in$^\mathrm{i}$. Case (i) is used in the proof of (ii). □

**Lemma 2.6.3.** *In $\mathrm{CPC}_\mathcal{C}$ and $\mathrm{IPC}_\mathcal{C}$ we have* $\dfrac{\Gamma \vdash \neg\neg D}{\neg\neg\Gamma \vdash \neg\neg D}$ .

*Proof.* Write $\Gamma = \{B_1, \ldots, B_n\}$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash \neg\neg D}{B_1,\dots,B_n,\neg\neg\Gamma \vdash \neg\neg D}}{B_2,\dots,B_n,\neg\neg\Gamma,\neg D \vdash \neg B_1}\text{Lemma 2.6.2 (ii) and (iii)} \qquad \dfrac{}{B_2,\dots,B_n,\neg\neg\Gamma,\neg D \vdash \neg\neg B_1}\text{axiom}}{B_2,\dots,B_n,\neg\neg\Gamma,\neg D \vdash \neg\neg D}\neg\text{-el}}{B_2,\dots,B_n,\neg\neg\Gamma \vdash \neg\neg D}\neg\text{-in}^{\text{i}}$$

$$\vdots$$

$$\text{etc}$$

$$\vdots$$

$$\neg\neg\Gamma \vdash \neg\neg D$$

□

The combination of this lemma with Lemma 2.2.2 makes it possible to have deductions that look like

$$\dfrac{\Delta \vdash \neg\neg\Gamma \qquad \Gamma \vdash \neg\neg D}{\Delta \vdash \neg\neg D}\ ,$$

where we write $\Delta \vdash \neg\neg\Gamma$ to mean $\Delta \vdash \neg\neg A$ for every $A \in \Gamma$.

Now we are able to prove Proposition 2.6.1 (Glivenko's Theorem), that is, $\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} A$ if and only if $\neg\neg\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg A$, for $\mathcal{C}$ containing $\neg$.

*Proof of Proposition 2.6.1.* ($\Leftarrow$) $\neg\neg\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg A$ implies $\neg\neg\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} \neg\neg A$, which implies $\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} A$ since $\neg\neg A \vdash_{\mathrm{CPC}_\mathcal{C}} A$ for all formulas $A$.

($\Rightarrow$) By induction on the derivation of $\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} A$. The atomic case is clear.

(Elimination) Suppose

$$\dfrac{\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} B \qquad \dots\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} B_j\ (p_j = 1)\dots \qquad \dots\Gamma, B_i \vdash_{\mathrm{CPC}_\mathcal{C}} A\ (p_i = 0)\dots}{\Gamma \vdash_{\mathrm{CPC}_\mathcal{C}} A}\text{el}$$

for some $B = c(B_1,\dots,B_n)$ with $t_c(p_1,\dots p_n) = 0$. The induction hypothesis gives us

(1) $\neg\neg\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg B$,
(2) $\neg\neg\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg B_j\ (p_j = 1)$,
(3) $\neg\neg\Gamma, \neg\neg B_i \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg A\ (p_i = 0)$.

Writing $\Delta = \{B_j \mid p_j = 1\}$ we obtain the following derivation. For readability we omit $\neg\neg\Gamma$ in this tree.

$$\dfrac{\dfrac{(1)}{\vdash \neg\neg B} \qquad \dfrac{(2)}{\vdash \neg\neg\Delta} \qquad \dfrac{\dfrac{}{B,\Delta \vdash B}\text{ax} \qquad \dfrac{}{B,\Delta \vdash B_j}\text{ax} \qquad \dfrac{\dfrac{\text{Lemma 2.6.2}}{B_i \vdash \neg\neg B_i} \qquad \dfrac{(3)}{\neg\neg B_i \vdash \neg\neg A}}{B_i \vdash \neg\neg A}2.2.2}{B,\Delta \vdash \neg\neg A}\text{el}}{\vdash \neg\neg A}\text{Lemmas 2.2.2 and 2.6.3}$$

This gives the desired result $\neg\neg\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg A$.

(Introduction) Suppose

$$\dfrac{\Gamma, B \vdash A \qquad \dots\Gamma \vdash B_j\ (p_j = 1)\dots \qquad \dots\Gamma, B_i \vdash A\ (p_i = 0)\dots}{\Gamma \vdash A}\text{in}^{\text{c}}$$

for some $B = c(B_1,\dots,B_n)$ with $t_c(p_1,\dots p_n) = 1$. The induction hypothesis gives us

(1) $\neg\neg\Gamma, \neg\neg B \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg A$,
(2) $\neg\neg\Gamma \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg B_j\ (p_j = 1)$,
(3) $\neg\neg\Gamma, \neg\neg B_i \vdash_{\mathrm{IPC}_\mathcal{C}} \neg\neg A\ (p_i = 0)$.

This leads to the proof in similar notation as above.

$$\dfrac{\dfrac{(2)}{\vdash \neg\neg\Delta} \qquad \dfrac{\dfrac{\dfrac{(1) + \text{Lemma 2.6.2}}{\neg A \vdash \neg B} \qquad \dfrac{\dfrac{}{\Delta \vdash B_j}\,\text{ax} \qquad \dfrac{\dfrac{(3) + \text{Lemma 2.6.2}}{\neg A \vdash \neg B_i} \qquad \dfrac{}{B_i \vdash B_i}\,\text{ax}}{\neg A, B_i \vdash B}\,\neg\text{-el}}{\Delta, \neg A \vdash B}\,\text{in}^{\text{i}}}{\Delta, \neg A \vdash \neg\neg A}\,\neg\text{-el}}{\Delta \vdash \neg\neg A}\,\neg\text{-in}^{\text{i}}}{\vdash \neg\neg A}\,\text{Lemmas 2.2.2 and 2.6.3}$$

Hence $\neg\neg\Gamma \vdash_{\mathrm{IPC}_c} \neg\neg A$. This concludes the proof of Glivenko's Theorem.

□

# Chapter 3 | Normalization

This chapter covers an elaborated study of normalization of the intuitionistic truth table system. We look at so-called detour and permutation conversions of intuitionistic derivations which have already been defined in [5]. In [5], normalization properties have been established, such as weak normalization with regard to detour and permutation conversions. In this section we also prove weak normalization, but now using the method of Prawitz, which we have presented in Chapter 1. This leads to a study of the form of normal derivations in Section 3.3.

A main contribution of this chapter is the strong normalization of the intuitionistic truth table system. The proof is based on the work of De Groote who proved strong normalization for proof reduction in the Prawitz system. His idea is based on a translation from the Prawitz system to simply typed lambda calculus, which is strongly normalizing. For the truth table system, the simply typed lambda calculus is not sufficient enough. Therefore we introduce in Section 3.4 an extended variant, which we call *parallel simply typed lambda calculus*. This makes it possible to apply proof techniques of the De Groote in order to prove strong normalization for the intuitionistic truth table system in Section 3.6.

## 3.1 Cut elimination

In this section we look at cut elimination of $\text{IPC}_{\mathcal{C}}$. This forms the preliminary work for the weak and strong normalization of intuitionistic logic which will be examined in Section 3.2 and Section 3.6. In a normal derivation all major premises of elimination rules are assumptions. In short, cuts are patterns in a derivation for which a major premise of an elimination rule is not an assumption. There are two different cuts, which we call *detour convertibilities* and *permutation convertibilities*. In [4], they are called *direct cuts* and *indirect cuts* respectively, but we prefer to stick to a more common terminology.

A detour convertibility is an introduction rule of a formula $\Phi$ immediately followed by an elimination rule of $\Phi$. We will see that there are two possibilities circumventing such patterns.

**Definition 3.1.1.** Let $c$ be an $n$-ary connective with an elimination rule and an intuitionistic introduction rule derived from truth table $t_c$. So we have the following truth table.

| $p_1$ | $\dots$ | $p_n$ | $c(p_1,\dots,p_n)$ |
|---|---|---|---|
| $a_1$ | $\dots$ | $a_n$ | 0 |
| $b_1$ | $\dots$ | $b_n$ | 1 |

A *detour convertibility* is an intuitionistic introduction rule followed by an elimination rule, where $\Phi = c(A_1,\dots,A_n)$. This has the following form with subtrees $\Sigma_j$, $\Sigma_i$, $\Pi_k$ and $\Pi_l$.

$$\dfrac{\dfrac{\begin{matrix}\Sigma_j \\ \dots\Gamma\vdash A_j\dots\end{matrix} \qquad \begin{matrix}\Sigma_i \\ \dots\Gamma, A_i\vdash \Phi\dots\end{matrix}}{\Gamma\vdash\Phi}\text{in} \qquad \begin{matrix}\Pi_k \\ \dots\Gamma\vdash A_k\dots\end{matrix} \qquad \begin{matrix}\Pi_l \\ \dots\Gamma, A_l\vdash D\dots\end{matrix}}{\Gamma\vdash D}\text{el}$$

Here, $A_j$ ranges over all formulas where $b_j = 1$ and $A_i$ ranges over all formulas where $b_i = 0$ in the truth table. Similarly, $A_k$ ranges over all formulas where $a_k = 1$ and $A_l$ ranges over all formulas where $a_l = 0$.

**Example 3.1.2.** Consider the following detour convertibility for most. For simplicity, we use the optimized rules (see Appendix A).

$$\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ \Gamma\vdash A\end{array}\qquad \begin{array}{c}\Sigma_2\\ \Gamma\vdash B\end{array}}{\Gamma\vdash \mathsf{most}(A,B,C)}\,\textsf{most-in}_1 \qquad \begin{array}{c}\Pi_1\\ \Gamma, A\vdash D\end{array}\qquad \begin{array}{c}\Pi_2\\ \Gamma,B\vdash D\end{array}}{\Gamma\vdash D}\,\textsf{most-el}_3$$

**Definition 3.1.3.** A *detour conversion* is defined by replacing a detour convertibility from Definition 3.1.1 in a derivation by one of the following derivations. As we will see, there are several ways to eliminate a detour convertibility. This depends on the columns from the truth table in which the entries of the introduction and elimination rule differ. The two general possibilities result in the following detour conversions.

(1) $l' = j'$ for some $l', j'$ which means $A_{l'} = A_{j'}$:

$$\begin{array}{c}\begin{array}{ccc}\Sigma_{j'} & & \Sigma_{j'}\\ \Gamma\vdash A_{j'} & \dots & \Gamma\vdash A_{j'}\end{array}\\ \Pi_{l'}\\ \Gamma\vdash D\end{array}$$

(2) $k' = i'$ for some $k', i'$ which means $A_{k'} = A_{i'}$:

$$\dfrac{\begin{array}{c}\begin{array}{ccc}\Pi_{k'} & & \Pi_{k'}\\ \Gamma\vdash A_{i'} & \dots & \Gamma\vdash A_{i'}\end{array}\\ \Sigma_{i'}\\ \Gamma\vdash \Phi\end{array}\qquad \begin{array}{c}\Pi_k\\ \dots\Gamma\vdash A_k\dots\end{array}\qquad \begin{array}{c}\Pi_l\\ \dots\Gamma, A_l\vdash D\dots\end{array}}{\Gamma\vdash D}$$

Both diagrams yield correct derivations because of Lemma 2.2.2. In a detour convertibility, there could be several columns in the truth table in which the entries of the elimination and introduction rule differ. So there could be several 'matching cases' $l' = j'$ or $k' = i'$. This means that detour conversion is non-deterministic. We will illustrate this with the following example of most.

**Example 3.1.4.** This example continues from Example 3.1.2. The detour convertibility represented in that example can be reduced to one of the following derivations, using matching case $l' = j'$ of Definition 3.1.3.

$$\begin{array}{c}\begin{array}{ccc}\Sigma_1 & & \Sigma_1\\ \Gamma\vdash A & \dots & \Gamma\vdash A\end{array}\\ \Pi_1\\ \Gamma\vdash D\end{array}\qquad\qquad \begin{array}{c}\begin{array}{ccc}\Sigma_2 & & \Sigma_2\\ \Gamma\vdash B & \dots & \Gamma\vdash B\end{array}\\ \Pi_2\\ \Gamma\vdash D\end{array}$$

We see that in this example there are two possibilities to eliminate the detour convertibility of Example 3.1.2. This means that detour conversion is non-deterministic.

Now we can turn to the permutation convertibilities. A permutation convertibility is a pattern in which an elimination rule is applied to the major premise of an elimination rule. It is interesting to look at such patterns, because indirect cuts may block a detour convertibility in a way that an introduction of a formula $\Phi = c(A_1, \dots, A_n)$ is not directly followed by an elimination rule for $c$, but first by other elimination rules where $\Phi$ is a minor premise of those elimination rules. To solve this blockage, we can permute one elimination rule over another. This justifies the name of permutation convertibility.

**Definition 3.1.5.** Let $c$ be an $n$-ary connective and let $c'$ be an $n'$-ary connective with elimination rules $r$ and $r'$ respectively. So we have the following rows in truth tables $t_c$ and $t_{c'}$:

| $p_1$ | $\dots$ | $p_n$ | $c(p_1,\dots,p_n)$ |
|---|---|---|---|
| $b_1$ | $\dots$ | $b_n$ | 0 |

| $p_1$ | $\dots$ | $p_{n'}$ | $c'(p_1,\dots,p_{n'})$ |
|---|---|---|---|
| $a_1$ | $\dots$ | $a_{n'}$ | 0 |

A *permutation convertibility* is a pattern of the following form, in which $\Phi = c(B_1,\dots B_n)$ and $\Psi = c'(A_1,\dots,A_{n'})$.

$$\dfrac{\dfrac{\Gamma\vdash\Psi \qquad \overset{\Sigma_j}{\dots\Gamma\vdash A_j\dots} \qquad \overset{\Sigma_i}{\dots\Gamma,A_i\vdash\Phi\dots}}{\Gamma\vdash\Phi}\,\text{el}_{r'} \qquad \overset{\Pi_k}{\dots\Gamma\vdash B_k\dots} \qquad \overset{\Pi_l}{\dots\Gamma,B_l\vdash D\dots}}{\Gamma\vdash D}\,\text{el}_r$$

Here, $A_j$ ranges over all formulas where $a_j = 1$ and $A_i$ ranges over all formulas where $a_i = 0$. Similarly, $B_k$ ranges over all formulas where $b_k = 1$ and $B_l$ ranges over all formulas where $b_l = 0$.

**Example 3.1.6.** This example with optimized rules is also stated in [5]. For optimized rules, permutation convertibilities are defined in a similar way.

$$\dfrac{\dfrac{\Gamma\vdash A\vee B \qquad \dfrac{\Gamma,A,C\vdash C\to D}{\Gamma,A\vdash C\to D}\,\to\text{-in}^{\mathrm{i}}{}_2 \qquad \Gamma,B\vdash C\to D}{\Gamma\vdash C\to D}\,\vee\text{-el} \qquad \Gamma\vdash C}{\Gamma\vdash D}\,\to\text{-el}$$

In this example we have a permutation convertibility where the conclusion of $\vee$-el is the major premise of $\to$-el. In this derivation, this permutation convertibility blocks the detour convertibility of the combination of $\to$-in$^{\mathrm{i}}{}_2$ and $\to$-el.

**Definition 3.1.7.** A *permutation conversion* is defined by replacing a permutation convertibility from Definition 3.1.5 in a derivation by the following derivation, where the two elimination rules are permuted.

$$\dfrac{\Gamma\vdash\Psi \qquad \overset{\Sigma_j}{\dots\Gamma\vdash A_j\dots} \qquad \dots\dfrac{\overset{\Sigma_i}{\Gamma,A_i\vdash\Phi} \qquad \overset{\Pi_k}{\dots\Gamma,A_i\vdash B_k\dots} \qquad \overset{\Pi_l}{\dots\Gamma,A_i,B_l\vdash D\dots}}{\Gamma,A_i\vdash D}\,\text{el}_r\dots}{\Gamma\vdash D}\,\text{el}_{r'}$$

This is a correct derivation, because of the weakening property, that is, $\Pi_k$ is a derivation of $\Gamma, A_i \vdash B_k$ since $\Gamma \vdash B_k$. Similarly, $\Pi_l$ derivation for $\Gamma, A_i, B_l \vdash D$.

**Example 3.1.8.** Now we look at the permutation conversion of the derivation of Example 3.1.6. We get the following derivation.

$$\dfrac{\Gamma\vdash A\vee B \qquad \dfrac{\dfrac{\Gamma,A,C\vdash C\to D}{\Gamma,A\vdash C\to D}\,\to\text{-in}^{\mathrm{i}}{}_2 \qquad \Gamma,A\vdash C}{\Gamma,A\vdash D}\,\to\text{-el} \qquad \dfrac{\Gamma,B\vdash C\to D \qquad \Gamma,B\vdash C}{\Gamma,B\vdash D}\,\to\text{-el}}{\Gamma\vdash D}\,\vee\text{-el}$$

We observe that now we have created a detour convertibility of the combination of $\to$-in$^{\mathrm{i}}{}_2$ and $\to$-el. Consequently, this can be reduced with a detour conversion of case $k' = i'$ of Definition 3.1.1.

Detour conversion and permutation conversion are also known under the term *cut elimination*. Such as in the Prawitz system, detour and permutation conversions are related to normal derivations.

**Definition 3.1.9.** A derivation in $\mathrm{IPC}_{\mathcal{C}}$ is *normal* if every major premise of an elimination rule is an assumption.

Note that a derivation is normal iff it does not contain any detour or permutation convertibilities. The definitions of detour and permutation convertibilities can be compared with the definitions of those in the Prawitz system, Definition 1.2.7 and Definition 1.2.8. There is one difference when we look at normal forms. In the Prawitz system, a derivation is normal iff it contains no detour and no permutation convertibilities and no cut formula derived from the $\bot^{\mathrm{i}}$-rule. The difference is that in the truth table system the $\bot$-rule is considered as an elimination rule, whereas in the Prawitz system the $\bot^{\mathrm{i}}$-rule is taken as an apart rule. This means that in the truth table system a reduction with $\bot$ is considered as a permutation conversion.

**Example 3.1.10.** We consider permutation conversion with $\bot$-el and $\to$-el ánd $\neg$-el and $\bot$-el. We see that some subderivations are deleted when permuting the rules. This is due to the fact that $\bot$-el contains no case. In this example we also see that $\bot$-el is rarely necessary in a derivation, where it would be essential in a proof in the Prawitz system. This is because the rules in the truth table system are self-contained. Note that $\neg$-el also contains no case. A two step reduction is shown below, first permuting the $\bot$-el – $\to$-el pattern and then the $\neg$-el – $\bot$-el pattern. We write $\longrightarrow_{\mathrm{P}}$ for a permutation conversion.

$$\dfrac{\dfrac{\dfrac{\dfrac{A\wedge\neg A\vdash A\wedge\neg A}{A\wedge\neg A\vdash\neg A}\wedge\text{-el}_2 \qquad \dfrac{A\wedge\neg A\vdash A\wedge\neg A}{A\wedge\neg A\vdash A}\wedge\text{-el}_1}{A\wedge\neg A\vdash\bot}\neg\text{-el}}{A\wedge\neg A\vdash A\to D}\bot\text{-el} \qquad \dfrac{A\wedge\neg A\vdash A\wedge\neg A}{A\wedge\neg A\vdash A}\wedge\text{-el}_1}{A\wedge\neg A\vdash D}\to\text{-el}$$

$$\longrightarrow_{\mathrm{P}} \qquad \dfrac{\dfrac{\dfrac{A\wedge\neg A\vdash A\wedge\neg A}{A\wedge\neg A\vdash\neg A}\wedge\text{-el}_2 \qquad \dfrac{A\wedge\neg A\vdash A\wedge\neg A}{A\wedge\neg A\vdash A}\wedge\text{-el}_1}{A\wedge\neg A\vdash\bot}\neg\text{-el}}{A\wedge\neg A\vdash D}\bot\text{-el}$$

$$\longrightarrow_{\mathrm{P}} \qquad \dfrac{\dfrac{A\wedge\neg A\vdash A\wedge\neg A}{A\wedge\neg A\vdash\neg A}\wedge\text{-el}_2 \qquad \dfrac{A\wedge\neg A\vdash A\wedge\neg A}{A\wedge\neg A\vdash A}\wedge\text{-el}_1}{A\wedge\neg A\vdash D}\neg\text{-el}$$

Note that the two step reduction of permuting the $\neg$-el – $\bot$-el pattern before the $\bot$-el – $\to$-el pattern yields the same result.

We have defined the detour and permutation conversions for non-optimized rules. It is also possible to define those conversions for the optimized rules, which is done in [5].

In Example 3.1.4, we saw that detour conversion is non-deterministic. This means that $\mathrm{IPC}_{\mathcal{C}}$ cannot satisfy the Church-Rosser property. The Church-Rosser property says that if derivation $\Sigma$ reduces to $\Pi_1$ and $\Sigma$ also reduces to $\Pi_2$, then there is a $\Pi_3$, such that there are conversions from both $\Pi_1$ and $\Pi_2$ resulting in $\Pi_3$. Informally speaking, the Church-Rosser property says that the order of the conversions does not make a difference to the eventual result. This is also known under the name confluence.

**Proposition 3.1.11.** *The intuitionistic truth table system does not satisfy the Church-Rosser property.*

*Proof.* We give an example in which different conversions result in different normal derivations. This means that conversion does not always lead to a unique normal form. Let $\Gamma = \{A, B, B \to A\}$. Derivation

$$\dfrac{\dfrac{\overline{\Gamma\vdash A}\qquad \overline{\Gamma\vdash B}}{\Gamma\vdash A\wedge B}\wedge\text{-in} \qquad \dfrac{}{\Gamma, A\vdash A} \qquad \dfrac{\overline{\Gamma,B\vdash B\to A}\qquad\overline{\Gamma,B\vdash B}}{\Gamma\vdash A}\to\text{-el}}{\Gamma\vdash A}\wedge\text{-el}_{00}$$

contains a detour convertibility, which reduces to two different normal derivations using matching case $l' = j'$ of Definition 3.1.3.

$$\overline{\Gamma\vdash A} \qquad \text{and} \qquad \dfrac{\overline{\Gamma\vdash B\to A}\qquad\overline{\Gamma\vdash B}}{\Gamma,B\vdash A}\to\text{-el}$$

Both derivations are normal, so the intuitionistic truth table system cannot satisfy the Church-Rosser property. □

## 3.2 Weak normalization

In this section, we give a proof for the weak normalization of the non-optimized intuitionistic natural deduction rules derived from truth tables. Weak normalization is already proven with the use of proof terms by Geuvers and Hurkens [5]. Here we follow a same strategy as Prawitz used in his proof of weak normalization mentioned in Section 1.2.

**Definition 3.2.1.** Let $R = \{\longrightarrow_1, \dots, \longrightarrow_n\}$ be a set of reduction relations. A derivation $\Pi$ is *weakly normalizing* with regard to $R$ if there is a finite reduction sequence of reductions from $R$ starting from $\Pi$. We say that a logical system has *the weak normalization property with regard to* $R$ if every derivation is weakly normalizing with regard to $R$.

In the truth table system we set $R = \{\ \longrightarrow_{\text{D}}\ ,\ \longrightarrow_{\text{P}}\ \}$, where $\longrightarrow_{\text{D}}$ stands for the detour conversions and $\longrightarrow_{\text{P}}$ for the permutation conversions.

In order to prove weak normalization, it is more practical to use another notation for the deduction rules. The new notation is also used in Prawitz [11] and Von Plato [9]. This notation shows more explicitly at which rules in a tree the discharges of assumptions take place. In the case of the elimination rule the new notation would be:

$$\dfrac{\Phi \qquad \dots A_j \dots \qquad \begin{matrix}[A_i]^1\\ \vdots\\ \dots D \dots\end{matrix}}{D}\,\text{el},1$$

In this notation we call $\Phi$ the *major premise* of the elimination rule, $A_j$ a *lemma* and we say that the $D$ above the line is *derived from case* $A_i$. If a rule uses such a $D$, then we say that the rule *contains a case* or that it is a rule *with a case*. The same holds for lemmas. Assumptions written between brackets $[\cdot]$ are discharged assumptions. Discharged assumptions [A] are labelled with a natural number $k$, that is $[A]^k$, indicating at which rule the assumption is discharged. Open assumptions are those which are not discharged. Context $\Gamma$ is hidden in this notation, where all open assumptions of the derivation form the context $\Gamma$. In the rest of this section we use this notation.

Now we introduce the definition of a segment. In natural deduction derived from truth tables, a segment is slightly different in comparison to a segment defined by Prawitz (Definition 1.2.1). This is due to the form of the introduction rules.

**Definition 3.2.2.** A *segment* in a derivation is a sequence $D_1, \ldots, D_n$ of formulas such that $D_i$ stand immediately above $D_{i+1}$ and

(1) $D_1$ is not the conclusion of a deduction rule that contains a case,
(2) for $i < n$, $D_i$ is derived from a case of a deduction rule (elimination or introduction),
(3) $D_n$ is not a minor premise of a deduction rule where $D_n$ is derived from a case.

A segment ends in a lemma or in a major premise. Note that in a segment $D_1, \ldots, D_n$ it holds that $D_i = D_j$ for all $i$ and $j$. Therefore, we define the *rank* of a segment by the *rank* of the formula in that segment. Recall Definition 2.1.4 of the rank of a formula. Number $n$ represents the *length* of the segment.

**Example 3.2.3.** In this example we can identify (at least) two segments with formula $C \wedge D$ and (at least) two segments containing formula $E$. Numbers $1, 2, 3$ indicate the places where corresponding assumptions are discharged.

$$\dfrac{\dfrac{A \wedge B \qquad \begin{array}{c}[A]^1\\ \vdots \\ C \wedge D\end{array} \qquad \dfrac{B \to (C \wedge D) \qquad [B]^1 \qquad [C \wedge D]^2}{C \wedge D}\,\to\text{-el},\,2}{C \wedge D}\,\wedge\text{-el}_{00},\,1 \qquad \begin{array}{c}[C]^3\\ \vdots \\ E\end{array} \qquad \begin{array}{c}[D]^3\\ \vdots \\ E\end{array}}{E}\,\wedge\text{-el}_{00},\,3$$

In contrast to the Prawitz system, segments can also occur in the introduction rules. This is necessary, because of the detour reductions of kind $k' = i'$. This is illustrated by the following example.

**Example 3.2.4.** This example has two segments with the formula $A \to B$ and (at least) one segment with formula $D$. One segment with $A \to B$ has length 2 and the other has length 3.

$$\dfrac{\dfrac{\dfrac{[A]^1 \qquad B}{A \to B}\,\to\text{-in}_{11} \qquad \dfrac{\dfrac{[A]^2 \qquad [B]^1}{A \to B}\,\to\text{-in}_{11} \qquad [B]^1}{A \to B}\,\to\text{-in}_{01},\,2}{A \to B}\,\to\text{-in}_{00},\,1 \qquad A \qquad \begin{array}{c}[B]^3\\ \vdots \\ D\end{array}}{D}\,\to\text{-el},\,3$$

We reintroduce the definition of a cut formula, now for the intuitionistic truth table deduction system.

**Definition 3.2.5.** A *cut formula* is a formula which

(1) is the major premise of an elimination rule,
(2) and the conclusion of a derivation rule, that is, the formula is not an axiom.

A segment that contains a cut formula is called a *cut segment.*

A cut formula appears only in a detour convertibility or a permutation convertibility. This means that a normal derivation is free of cut segments.

**Example 3.2.6.** In Example 3.2.3 the segments containing $C \wedge D$ are cut segments, but the segments containing $E$ are not. In Example 3.2.4 the segments containing $A \to B$ are cut segments, but the segment containing $D$ is not.

Before we turn to the weak normalization of detour and permutation conversion, we consider an illustrative example of how the lengths of segments reduce. These segments can occur both in introduction rules and in elimination rules.

**Example 3.2.7.** Consider the following reductions of a proof of $A, \neg A \vdash D$.

$$\dfrac{\dfrac{\dfrac{[\neg A]^1 \qquad A}{\neg\neg A}\,\neg\text{-el}}{\neg\neg A}\,\neg\text{-in},1 \qquad \neg A}{D}\,\neg\text{-el} \quad \longrightarrow_{\mathrm{D}(k'=i')} \quad \dfrac{\dfrac{\neg A \qquad A}{\neg\neg A}\,\neg\text{-el} \qquad \neg A}{D}\,\neg\text{-el} \quad \longrightarrow_{\mathrm{P}} \quad \dfrac{\neg A \qquad A}{D}\,\neg\text{-el}$$

In the first derivation there is one cut segment of length 2 containing formula $\neg\neg A$. After detour reduction, the length is decreased to 1. After permutation there are no cut segments anymore.

The proof of the next theorem follows the same method as the proof of Prawitz (Theorem 1.2.10).

**Theorem 3.2.8** (Weak normalization)**.** *There is an effective procedure for transforming a deduction $\Gamma \vdash A$ in $IPC_{\mathcal{C}}$ into a normal derivation of $\Gamma \vdash A$.*

*Proof.* This is proved by induction. Let $\Pi$ be a derivation of $\Gamma \vdash A$. A deduction is normal if it contains no cut segments. Let pair $\langle d, l \rangle$ be the induction value with

- $d$ the highest rank of a cut formula in $\Pi$,
- $l$ the sum of the lengths of segments containing a cut formula of rank $d$.

Let $\sigma$ be a cut segment of highest rank $d$ such that:

(a) there is no cut segment of rank $d$ above $\sigma$,
(b) no cut segment of rank $d$ stands above a formula side-connected with the last formula in $\sigma$,
(c) no cut segment of rank $d$ contains a formula side-connected with the last formula in $\sigma$.

It is important to mention that such a cut segment $\sigma$ exists if derivation $\Pi$ is not normal. This is proved in the following way, such as in [11]. Consider the finite set of cut segments of rank $d$ that satisfy clause (a). If $\sigma_1$ from this set does not satisfy (b) or (c), then there is a segment $\sigma_2$ that makes clause (b) or (c) fail for $\sigma_2$. If (b) or (c) also fail for $\sigma_2$, then there is a segment $\sigma_3 \neq \sigma_1$ in the set that makes (b) or (c) fail for $\sigma_2$. Now we get a sequence of segments $\sigma_1, \sigma_2, \sigma_3, \ldots$ with $\sigma_i \neq \sigma_j$ if $i \neq j$. Since the set is finite, we must find a segment that satisfies clauses (b) and (c). Therefore we can find segment $\sigma$ which satisfies (a), (b) and (c).

Now let $\Phi$ be the cut formula in $\sigma$. We treat different cases for $\sigma$.

(1) Length$(\sigma) = 1$
(2) Length$(\sigma) > 1$

Both cases can be divided into two subcases: either $\Phi$ is in a detour convertibility or $\Phi$ is in a permutation convertibility.

In case 1, $\sigma$ consists of one formula, which is $\Phi$. This formula is then the consequence of a rule without any case. This can be an introduction rule or an elimination rule. This correspond to a detour convertibility and a permutation convertibility.

- *Detour*: Normally, there are two possibilities for a detour conversion, but since in this case we do not have any case for the introduction rule, we only have to examine the case that $l' = j'$ for some $l'$ and $j'$. The reduction goes as follows, with cut formula $\Phi$:

$$\dfrac{\dfrac{\begin{matrix}\Sigma_j\\ \ldots A_j \ldots\end{matrix}}{\Phi}\,\text{in} \qquad \begin{matrix}\Pi_k\\ \ldots A_k\ldots\end{matrix} \qquad \begin{matrix}[A_l]^2\\ \Pi_l\\ \ldots D \ldots\end{matrix}}{\begin{matrix}D\\ \Pi\end{matrix}}\,\text{el,2} \qquad \longrightarrow_{\text{D}} \qquad \begin{matrix}\Sigma_{j'} & & \Sigma_{j'}\\ A_{j'} & \ldots & A_{j'}\\ & \Pi_{l'} & \\ & D & \\ & \Pi & \end{matrix}$$

In this reduction there can be multiple subderivations containing $\Sigma_{j'}$. This is not a problem for the induction value, because of clause (a) in the assumption on $\sigma$. So either $d$ is lowered or $d$ remains the same and $l$ is lowered.

- *Permutation*: Cut formula $\Phi$ is the conclusion of an elimination rule without a case, so the reduction is as follows:

$$\dfrac{\dfrac{\begin{matrix}\Sigma\\ \Psi\end{matrix} \qquad \begin{matrix}\Sigma_j\\ \ldots A_j \ldots\end{matrix}}{\Phi}\,\text{el} \qquad \begin{matrix}\Pi_k\\ \ldots B_k\ldots\end{matrix} \qquad \begin{matrix}[B_l]^2\\ \Pi_l\\ \ldots D \ldots\end{matrix}}{\begin{matrix}D\\ \Pi\end{matrix}}\,\text{el,2} \qquad \longrightarrow_{\text{P}} \qquad \dfrac{\begin{matrix}\Sigma\\ \Psi\end{matrix} \qquad \begin{matrix}\Sigma_j\\ \ldots A_j \ldots\end{matrix}}{\begin{matrix}D\\ \Pi\end{matrix}}\,\text{el}$$

In this case it is clear that the induction value is decreased.

For case 2 we have Length($\sigma$) $> 1$. This means that cut formula $\Phi$ in $\sigma$ is derived from a rule including a case. Again, this can be a detour convertibility or a permutation convertibility.

- *Detour*: We have to consider two cases. In the case that $l' = j'$ for some $l'$ and $j'$ the reduction is the same as above, so we only do the case that $k' = i'$ for some $k'$ and $i'$. The reduction is as follows, with cut formula $\Phi$:

$$\dfrac{\dfrac{\begin{matrix}\Sigma_j\\ \ldots A_j \ldots\end{matrix} \qquad \begin{matrix}[A_i]^1\\ \Sigma_i\\ \ldots \Phi \ldots\end{matrix}}{\Phi}\,\text{in,1} \qquad \begin{matrix}\Pi_k\\ \ldots A_k\ldots\end{matrix} \qquad \begin{matrix}[A_l]^2\\ \Pi_l\\ \ldots D \ldots\end{matrix}}{\begin{matrix}D\\ \Pi\end{matrix}}\,\text{el,2}$$

$$\longrightarrow_{\text{D}} \qquad \dfrac{\begin{matrix}\Pi_{k'} & & \Pi_{k'}\\ A_{i'} & \ldots & A_{i'}\\ & \Sigma_{i'} & \\ & \Phi & \end{matrix} \qquad \begin{matrix}\Pi_k\\ \ldots A_k\ldots\end{matrix} \qquad \begin{matrix}[A_l]^1\\ \Pi_l\\ \ldots D \ldots\end{matrix}}{\begin{matrix}D\\ \Pi\end{matrix}}\,\text{el,1}$$

In this reduction there could be multiple subderivations containing $\Pi_{k'}$. This is not a problem for the induction value, because of clause (b) in the assumption on $\sigma$. So again, either $d$ is decreased or $d$ remains the same and $l$ is lowered.

- *permutation*: The reduction with cut formula $\Phi$ in $\sigma$:

$$\dfrac{\dfrac{\begin{matrix}\Sigma\\ \Psi\end{matrix} \qquad \begin{matrix}\Sigma_j\\ \ldots A_j \ldots\end{matrix} \qquad \begin{matrix}[A_i]^1\\ \Sigma_i\\ \ldots \Phi \ldots\end{matrix}}{\Phi}\,\text{el,1} \qquad \begin{matrix}\Pi_k\\ \ldots B_k\ldots\end{matrix} \qquad \begin{matrix}[B_l]^2\\ \Pi_l\\ \ldots D \ldots\end{matrix}}{\begin{matrix}D\\ \Pi\end{matrix}}\,\text{el,2}$$

$$\longrightarrow_{\text{P}} \quad \dfrac{\begin{array}{c}\Sigma\\ \Psi\end{array} \qquad \begin{array}{c}\Sigma_j\\ \ldots A_j \ldots\end{array} \qquad \ldots \quad \dfrac{\begin{array}{c}[A_i]^1\\ \Sigma_i\\ \Phi\end{array} \qquad \begin{array}{c}\Pi_k\\ \ldots B_k \ldots\end{array} \qquad \begin{array}{c}[B_l]^2\\ \Pi_l\\ \ldots D \ldots\end{array}}{D}\,\text{el,2} \ldots}{\begin{array}{c}D\\ \Pi\end{array}}\,\text{el,1}$$

Now there can be two problems. First, subderivations $\Pi_k$ and $\Pi_l$ could be duplicated. But because of clause (c) and (b), there will be no extra cut segment of rank $d$. Secondly, there could be more cut segments containing formula $D$. In order to conclude that the induction value decreases, we have to check that, in the case $D$ would be in a cut segment, $D$ has a lower rank than $\Phi$. This is precisely because of clause (c) in the assumption on $\sigma$. Hence the induction value $\langle d, l\rangle$ decreases.

We have shown for all different possibilities for $\sigma$ that the induction value becomes lower after reduction. This makes sure that after repeated applications of those reductions we end up with a normal derivation. □

**Example 3.2.9.** In this example we consider detour conversions. This is the reduction of the derivation of Example 3.2.4. We see that the lengths of cut segments with $A \to B$ decreases.

$$\dfrac{\dfrac{\dfrac{[A]^1 \quad B}{A \to B}\,\to\text{-in}_{11} \qquad \dfrac{\dfrac{[A]^2 \quad [B]^1}{A \to B}\,\to\text{-in}_{11} \quad [B]^1}{A \to B}\,\to\text{-in}_{01},\,2}{A \to B}\,\to\text{-in}_{00},\,1 \qquad A \qquad \begin{array}{c}[B]^3\\ \vdots\\ D\end{array}}{D}\,\to\text{-el},\,3$$

$$\longrightarrow_{D\ (k'=i')} \quad \dfrac{\dfrac{A \quad B}{A \to B}\,\to\text{-in}_{11} \qquad A \qquad \begin{array}{c}[B]^3\\ \vdots\\ D\end{array}}{D}\,\to\text{-el},\,3 \qquad \longrightarrow_{D\ (l'=j')} \quad \begin{array}{c}B \ \ldots \ B\\ \vdots\\ D\end{array}$$

## 3.3 Subformula property, consistency and decidability

In this section we prove the subformula property using the same ideas of Prawitz, which we have examined in Section 1.3. In [5], the subformula property, consistency and decidability are already proven for the truth table system using proof terms. Here we will focus on proof trees using terminology of Prawitz [11] and Van Dalen [1]. First we will look at the form of normal deductions in the truth table system. This differs slightly from the form of a normal proof in the Prawitz system.

**Definition 3.3.1.** A *path* in a deduction $\Pi$ is a sequence of formulas $D_1, \ldots, D_n$ such that

(1) $D_1$ is an assumption not discharged by an elimination rule,
(2) $D_i$ with $i < n$ is not a lemma of an elimination rule and either
   - $D_i$ is not a major premise of an elimination rule with a case and $D_{i+1}$ stands immediately below $D_i$ or
   - $D_i$ is the major premise of an elimination rule with a case and $D_{i+1}$ is a discharged assumption in this elimination rule,
(3) $D_n$ is either
   - a lemma of an elimination rule, or
   - the conclusion of $\Pi$.

Recall that we do not allow redundant applications of the rules (see Example 2.1.8). Consequently, if $D_i$ is the major premise of an elimination rule with $k$ cases, then there are $k$ possibilities to continue the path.

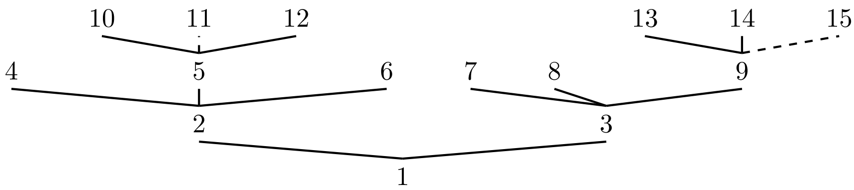
Also note that segments are included as a whole in paths. So a segment cannot be 'cut' by a path.

**Definition 3.3.2.** A path containing the conclusion is called a *main path.*

**Example 3.3.3.** In this example we give a normal derivation of $A \wedge B \vdash B \wedge A$. We use the notation introduced in previous section.

$$\dfrac{\dfrac{A \wedge B \qquad \dfrac{A \wedge B \quad [A]^1 \quad [B]^2}{B}\ \wedge\text{-el}_{10},2 \qquad [B]^1}{B}\ \wedge\text{-el}_{00},1 \qquad \dfrac{A \wedge B \quad [A]^3 \qquad \dfrac{A \wedge B \quad [A]^4 \quad [B]^3}{A}\ \wedge\text{-el}_{01},4}{A}\ \wedge\text{-el}_{00},\ 3}{B \wedge A}\ \wedge\text{-in}$$

This derivation adopts the following tree form




The derivation contains the following paths: (4,11) (4,6,2,1), (10,12,5,2,1), (7,8,3,1), (7,15) and (13,14,9,3,1). The dashed edges indicate the lemmas of elimination rules, which are the end-formulas of paths.

Now we can look at the form of a normal derivation in the truth table system. Note that each formula in a deduction tree belongs to at least one path. Compare the following proposition with Proposition 1.3.4. In this system we also have three parts, where the middle part does not has to contain the $\perp$-rule. For example, the $\neg$-el does also belong the the middle part, which is illustrated in Example 3.3.6.

**Proposition 3.3.4.** *A path $\pi$ in a normal derivation in the truth table system is divided into at most three parts: an E-part, followed by an M-part, followed by an I-part, such that*

*(1) the E-part consists of formulas that are a major premise of an elimination rule with a case except for the possible last formulas which form a segment ending in the end-formula of $\pi$,*
*(2) the M-part (middle part) contains at most one formula which is the major premise of an elimination rule without a case,*
*(3) for each segment $\sigma$ in the I-part, the last formula is a premise of an I-rule or the end-formula of path $\pi$.*

*Each of the parts may be empty.*

*Proof.* Let $\Pi$ be a normal deduction and $\pi = (D_1, \ldots, D_n)$ a path in $\Pi$. Since $\Pi$ is normal, every major premise of an elimination rule is an assumption. For the E-part, suppose that the first formula $D_1$ in $\pi$ is a premise of an elimination rule which contains a case. Note that $D_1$ is an assumption of $\Pi$. There are three cases: $D_1$ is the major premise, a lemma or a case of that rule. It cannot be a case, since that case would be directly discharged by the elimination rule and paths cannot start in discharged assumption by an elimination rule. If $D_1$ is a lemma, then $D_1$ is the end-formula of path $\pi$. If $D_1$ is the major premise, then $D_2$ is a case of that elimination rule. Let $\sigma$ be the segment containing $D_2$, then one of the following five is true.

(i) $D_2$ is a major premise of an elimination rule which contains a case, which repeats the process for the E-part.
(ii) $D_2$ is a major premise of an elimination rule without a case.
(iii) The last formula of $\sigma$ is a lemma of an elimination rule. This is the end of path $\pi$.
(iv) The last formula of $\sigma$ is the conclusion of $\Pi$. This is the end of path $\pi$.
(v) The last formula of $\sigma$ is a premise of an introduction rule.

Number (ii) is the M-part and we show that (v) begins the I-part. For the M-part, we have to show that in $\pi$ there is one formula that ends in a major premise of an elimination rule without a case. Consider the segment $\sigma'$ that starts with the elimination rule without a case. The last formula in $\sigma'$ cannot be a major premise of an elimination rule, since $\Pi$ is normal. Now we can show that $\sigma'$ belongs to the I-part, because the last formula in $\sigma'$ is either a lemma of an elimination rule, the conclusion of $\Pi$ or a premise of an introduction rule.

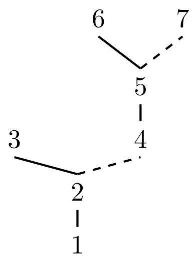
If $\sigma$ ends in the premise of an introduction rule we have to prove that the next formulas in the path belong to the I-part. Consider the next segment $\sigma'$ which starts from an introduction rule. $\sigma'$ cannot end in a major premise of an elimination rule, since $\Pi$ is normal. Therefore, $\sigma'$ ends in the end-formula of $\pi$ or an introduction rule. This yields the I-part. □

Note that the E-part consists of segments of length 1 except for the possible last segment, and that the possible M-part also has 1 formula. This is due to the fact that major premises of elimination rules are assumption in normal proofs.

**Example 3.3.5.** Here we identify the parts in the paths of Example 3.3.3. We write | to switch to the next part, that is (E-part | M-part | I-part). Some parts may be empty. We write ; between segments. We have (4; 11 | _ | _), (4 | _ | 6,2; 1), (10 | _ | 12,5,2; 1), (7 | _ | 8,3; 1), (7; 15 | _ | _) and (13| _ | 14,9,3; 1).

**Example 3.3.6.** We consider the normal derivation of $\neg\neg\neg A \vdash_{\mathrm{IPC}} \neg A$ and its tree structure, which is an example of paths including the M-part.

$$\dfrac{\neg\neg\neg A \qquad \dfrac{\dfrac{[\neg A]^1 \quad [A]^2}{\neg\neg A}\,\neg\text{-el}}{\neg\neg A}\,\neg\text{-in},1}{\dfrac{\neg A}{\neg A}\,\neg\text{-in},2}\,\neg\text{-el}$$




The deduction has the following paths: (3,2,1), (6,5,4), (7). These can be divided as follows, (_ | 3 | 2,1), (_ | 6 | 5,4) and (7 | _ | _).

We define the order of a path in the same way as in Definition 1.3.6. This makes it possible to prove the subformula property.

**Definition 3.3.7.** Let $\pi$ be a path in a normal derivation. Define the order $o$ of $\pi$ inductively as follows.

- $o(\pi) = 0$ for a main path.
- If $\pi$ ends in a lemma of an elimination rule, then

$$o(\pi) = 1 + \min\{o(\pi') \mid \pi' \text{ contains corresponding major premise}\}.$$

**Example 3.3.8.** Consider again the derivation of Example 3.3.6. We have the following orders: $o(3,2,1) = 0$, $o(6,5,4) = 1$ and $o(7) = 2$.

Now let us prove the subformula property, consistency and decidability. The proof of the subformula property which we present here is by induction on the order of paths, just as in the proof of Theorem 1.3.9. The subformula property is also proved in [5] using induction on the typing derivation rules. The following results can be compared to results in the Prawitz system of Section 1.3.

**Lemma 3.3.9.** *Let $\pi = (A_1, \dots, A_n)$ be a path. Each formula in the E-part or M-part is a subformula of $A_1$. Each formula in the I-part is subformula of $A_n$.*

*Proof.* For formula $B$ in the E-part, we prove that $A_j = B$ is a subformula of $A_{j-1}$. There are two possibilities, $B$ is the major premise of an elimination rule with a case or it is contained in a segment which starts with an assumption. In the first case, $B$ is an assumption, since $\Pi$ is a normal derivation. In the second case, $B$ is the same formula as the assumption. For both possibilities we have either $B \equiv A_1$ or $B = A_j$ for some case $A_j$ where $A_{j-1} = c(B_1, \dots, B_m)$ is the corresponding major premise with $A_j = B_i$ for some $i$. This means that $B = A_j$ is subformula of $A_{j-1}$. Repeating this process in the E-part, we see that $B = A_j$ is subformula of $A_1$.

If $B$ is the formula of the M-part, then $B$ is an assumption, since $\Pi$ is normal. $B$ can be an open assumption, that is $B \in \Gamma$, or a case of an elimination rule. For the second we have that $B$ is a subformula of a formula in the E-part. Hence, $B$ is subformula of $A_1$.

If $B$ is contained in the I-part of $\pi$, then $B = A_j$ is a subformula of $A_{j+1}$, because of the form of the intuitionistic introduction rules. Therefore, $B$ is a subformula of $A_n$. □

If we look closer at the proof of Lemma 3.3.9, we see that for a path $(A_1, \dots, A_n)$ we have the following: for E-part and M-part $(A_1, \dots, A_j)$ we have that $A_{i+1}$ is subformula of $A_i$ and for I-part $(A_{j+1}, \dots, A_n)$ we have that $A_i$ is subformula of $A_{i+1}$.

**Theorem 3.3.10** (Subformula property)**.** *Let $\Pi$ be a normal derivation of $\Gamma \vdash A$. Then each formula $B$ in $\Pi$ is a subformula of the conclusion $A$ or of a formula in $\Gamma$.*

*Proof.* Let $\Pi$ be a normal deduction of $\Gamma \vdash A$. We proceed by induction on the order of paths. Consider a formula $B$ in $\Pi$ in a path $\pi = (A_1, \dots, A_n)$. By Proposition 3.3.4, $B$ can occur in the E-part, M-part or I-part of $\pi$.

If $B = A_n$, then $B$ is either the conclusion or a lemma of an elimination rule. If $B$ is the conclusion, then $B = A$. If $B$ is a lemma of an elimination rule, then the major premise of that rule is of the form $c(B_1, \dots, B_m)$ where $B_i = B$ for some $i$. This means that $B$ is a subformula of the major premise which is contained in a path $\pi'$ with $o(\pi') < o(\pi)$. Applying the induction hypothesis we find that $B$ is a subformula of a formula in $\Gamma$ or the conclusion.

Now, if $B$ is contained in the I-part of $\pi$, then $B$ is a subformula of the end-formula $A_n$, by Lemma 3.3.9. So together with the previous, $B$ is a subformula of $A$ or a subformula of a formula in $\Gamma$.

If $B = A_1$, then either $B \in \Gamma$ or $B$ is used as a case in an introduction rule, which is discharged by that introduction rule. If $B$ is a discharged assumption, then $B$ is a subformula of the corresponding introduced formula $c(B_1, \dots, B_m)$ with $B_i = B$ for some $i$. Formula $c(B_1, \dots, B_m)$ is contained in the I-part of $\pi$, or in a path $\pi'$ with $o(\pi') < o(\pi)$. So we can conclude that $B$ is a subformula of $A$ or subformula of a formula in $\Gamma$.

If $B$ is a formula in the E-part or the formula of the M-part, then $B$ is a subformula of $A_1$ by Lemma 3.3.9. This finishes the proof. □

Now we consider consistency and decidability. Compare it to the end of Section 1.3.

**Corollary 3.3.11** (Consistency)**.** *For any set of connectives $\mathcal{C}$, $\mathrm{IPC}_{\mathcal{C}}$ is consistent, that is, there are formulas $D$ such that $\nvdash_{\mathrm{IPC}_{\mathcal{C}}} D$.*

*Proof.* Let $D$ be an atom and suppose $\vdash D$. Then there is a normal deduction of $D$ without any open assumptions. There is a main path which contains the conclusion $D$. Formula $D$ is an atom, so it cannot be derived from an introduction rule. Because of the structure of a path, there are no introduction rules in the main path. This means that the first formula of the path is not discharged. This is a contradiction. ☐

**Corollary 3.3.12** (Decidability)**.** *For any set of connectives $\mathcal{C}$, $\mathrm{IPC}_{\mathcal{C}}$ is decidable, that is, there is an effective way to determine whether $\Gamma \vdash_{\mathrm{IPC}_{\mathcal{C}}} A$ or not.*

*Proof.* This can be done in the same strategy as the proof of Corollary 1.3.11. See [5] for a proof. ☐

## 3.4 Parallel simply typed $\lambda$-calculus

In Section 3.6, we prove strong normalization of $\mathrm{IPC}_{\mathcal{C}}$. First we have to examine the parallel simply typed $\lambda$-calculus in this section and the Curry Howard isomorphism for proof terms of the truth table system in the next section.

Type systems are used in proof theory and in studying the foundations of mathematics. There are many type systems. One of the easiest type systems is the simply typed $\lambda$-calculus, denoted by $\lambda^{\to}$. This system consists of types that only contain the connective $\to$ and terms can be reduced via $\beta$-reduction. This system adopts very nice properties such as the substitution Lemma, confluence, subject reduction property and strong normalization. See [8] for an extensive introduction on $\lambda^{\to}$.

In this section we extend $\lambda^{\to}$ with an extra typing rule. This makes it possible to combine different proofs of a formula. These proofs will stand parallel to each other, which justifies the name. We add a new term and one typing rule to $\lambda^{\to}$ in order to get the parallel simply typed $\lambda$-calculus, denoted by $\mathrm{p}\lambda^{\to}$, which results in the following definitions.

**Definition 3.4.1** (Parallel simply typed $\lambda$-calculus)**.** The types in the *parallel simply typed $\lambda$-calculus* are of the form

$$A ::= a \mid A \to A,$$

where $a$ is an atomic type. The abstract syntax for proof terms in the *parallel simply typed $\lambda$-calculus* is

$$M ::= x \mid (MM) \mid \lambda x.M \mid [M_1, \dots, M_n] \text{ for } (n > 1)$$

where $x$ ranges over variables. The terms are typed using the following derivation rules with *context* $\Gamma$.

$$\frac{}{\Gamma \vdash x_i : A_i} \text{ if } x_i : A_i \in \Gamma, \text{ axiom} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \text{ application}$$

$$\frac{\Gamma, x : A \vdash M : B}{\lambda x.M : A \to B} \text{ abstraction} \qquad \frac{\Gamma \vdash M_1 : A \qquad \dots \qquad \Gamma \vdash M_n : A}{\Gamma \vdash [M_1, \dots, M_n] : A} \; n > 1, \text{ parallel}$$

We use the following terminology and notations. We use capital letters $A, B, C$ to represent types. We reserve capital letters $M, N, O, P, Q$ for terms in $\mathrm{p}\lambda^{\to}$. When $M$ is of the form $[M_1, \dots, M_n]$, we call $M$ a *parallel term.*

Parallel terms can be applied when one want to store all information of a proof. For instance, it may be the case that a formula can be proved in several ways, and you want to have the different proofs.

**Example 3.4.2.** In the proof of $A \to B \to (A \to C) \to (B \to C) \to C$ you can apply the information from $A$ or the information from $B$. A parallel proof becomes

$$\dfrac{\dfrac{\dfrac{[A \to C] \qquad [A]}{C} \qquad \dfrac{[B \to C] \qquad [B]}{C}}{C}}{A \to B \to (A \to C) \to (B \to C) \to C}\ \text{4 abstractions}$$

The proof term that corresponds to this tree is $\lambda x.\lambda y.\lambda z.\lambda w.[zx, wy]$.

Such as in $\lambda^{\to}$, we can define substitutions and we can show that the substitution lemma holds in p$\lambda^{\to}$. The substitution lemma is proved by induction on derivations, such as many other lemmas in this section. When proving with induction we only show the statement for the parallel rule, since the lemmas are already proved for the axiom, abstraction and application rule [8].

**Definition 3.4.3** (Substitution)**.**

(1) (a) $x[x := N] \equiv N$,
    (b) $x[y := N] \equiv N$ if $x \not\equiv y$,
(2) $(PQ)[x := N] \equiv (P[x := N]Q[x := N])$,
(3) Rename variables in $N$, such that $y$ is not a free variable in $N$,
    then $(\lambda y.P)[x := N] \equiv \lambda y.(P[x := N])$,
(4) $[P_1, \ldots P_n][x := N] \equiv [P_1[x := N], \ldots, P_n[x := N]]$.

**Lemma 3.4.4** (Substitution lemma)**.** *Assume that* $\Gamma, x : B, \Delta \vdash M : A$ *and* $\Gamma \vdash N : B$, *then* $\Gamma, \Delta \vdash M[x := N] : A$.

*Proof.* We use induction on the derivation of $M$ in $\Gamma, x : B, \Delta \vdash M : A$. There are different possibilities: $M$ can be a variable, an application, an abstraction or a parallel term. See for the first three cases the proof of Lemma 2.11.1 in [8]. Now suppose $M$ is a parallel term, say $M \equiv [M_1, \ldots, M_n]$. Then $\Gamma, x : B, \Delta \vdash M_i : A$ for all $i$. The induction hypothesis gives us $\Gamma, \Delta \vdash M_i[x := N] : A$ for all $i$. Therefore $\Gamma, \Delta \vdash [M_1, \ldots, M_n][x := N] : A$. □

A central point in $\lambda^{\to}$ is the $\beta$-reduction. This makes it possible to avoid unnecessary combinations of an abstraction and an application. We get an altered definition of $\beta$-reduction, since we add a parallel typing rule for each $n > 1$.

**Definition 3.4.5** ($\beta$-reduction)**.** The *$\beta$-reduction* is defined as follows.

(1) $(\lambda x.M)N \to_\beta M[x := N]$,
(2) $[M_1, \ldots, M_n]N \to_\beta [M_1 N, \ldots, M_n N]$,
(3) If $M \to_\beta N$, then
    (a) $MP \to_\beta NP$,
    (b) $PM \to_\beta PN$,
    (c) $\lambda x.M \to_\beta \lambda x.N$,
    (d) $[P_1, \ldots, M, \ldots, P_n] \to_\beta [P_1, \ldots, N, \ldots, P_n]$.

The terms $(\lambda x.M)N \to_\beta M[x := N]$ and $[M_1, \dots, M_n]N \to_\beta [M_1 N, \dots, M_n N]$ are called *$\beta$-redexes*. If $M$ reduces in zero, one or more steps to $N$, we write $M \twoheadrightarrow_\beta N$. This means that $\twoheadrightarrow_\beta$ is the reflexive and transitive closure of $\to_\beta$. If $M$ reduces in one or more steps we write $M \xrightarrow{+}_\beta N$, which is the transitive closure of $\to_\beta$.

**Definition 3.4.6.** Let $R = \{\longrightarrow_1, \dots, \longrightarrow_n\}$ be a set of reduction relations. Well-formed term $M$ is *strongly normalizing with regard to* $R$ if there is no infinite reduction sequence of reductions from $R$ starting from $M$. We say that a logical system has the *strong normalization property with regard to* $R$ if every well-formed term is strongly normalizing with regard to $R$.

Here we have $R = \{\to_\beta\}$. Strong normalization of p$\lambda^\to$ is proved by constructing a model using the well-known saturated sets method of Tait [13]. We write $\mathsf{SN}$ for the set of strongly normalizing terms.

**Definition 3.4.7.** The interpretation of parallel simply typed terms is defined by

- $[\![a]\!] := \mathsf{SN}$ for atomic type $a$,
- $[\![A \to B]\!] := \{M \mid \forall N \in [\![A]\!]\ (MN \in [\![B]\!])\}$.

We have the following standard closure properties for $[\![A]\!]$. We write $\overline{P}$ to mean a sequence of terms $P_1 P_2 \dots P_n$.

**Lemma 3.4.8.** *For all types $A$, terms $M, M_1, \dots, M_n$ and list of terms $\overline{P}$ we have*

*(1)* $[\![A]\!] \subseteq \mathsf{SN}$,
*(2)* $x N_1 \dots N_k \in [\![A]\!]$ *for all $x$ and* $N_1, \dots, N_k \in \mathsf{SN}$,
*(3) if* $M[x := N]\overline{P} \in [\![A]\!]$ *and* $N \in \mathsf{SN}$, *then* $(\lambda x.M)N\overline{P} \in [\![A]\!]$,
*(4) if* $[M_1 N, \dots, M_n N]\overline{P} \in [\![A]\!]$, *then* $[M_1, \dots, M_n]N\overline{P} \in [\![A]\!]$.

*Proof.* All parts are proceeded by induction on the structure of type $A$. The first two are proved simultaneously. Here we prove point (4) by induction on type $A$. For atomic type $a$, we have $[\![a]\!] = \mathsf{SN}$, so suppose $[M_1 N, \dots, M_n N]\overline{P} \in \mathsf{SN}$. Because there are no infinite reductions in $N$, $\overline{P}$ and $M_i$ for all $i$, a reduction from $[M_1, \cdots M_n]N\overline{P}$ will reduce to $[M_1' N', \dots, M_n' N']\overline{P'}$ for $M_i'$, $N'$ and $\overline{P'}$ such that $M_i \twoheadrightarrow_\beta M_i'$ for all $i$, $N \twoheadrightarrow_\beta N'$ and $\overline{P} \twoheadrightarrow_\beta \overline{P'}$. By assumption we have $[M_1' N', \dots, M_n' N']\overline{P'} \in \mathsf{SN}$, so $[M_1, \dots, M_n]N\overline{P} \in \mathsf{SN}$. This completes the start of the induction. For the induction step, suppose $[M_1 N, \dots, M_n N]\overline{P} \in [\![A \to B]\!]$. Let $O \in [\![A]\!]$, then $[M_1 N, \dots, M_n N]\overline{P}O \in [\![B]\!]$ and with induction we have $[M_1, \dots, M_n]N\overline{P}O \in [\![B]\!]$. Therefore $[M_1, \dots, M_n]N\overline{P} \in [\![A \to B]\!]$. ☐

**Lemma 3.4.9.** *For all types $A$, terms $M_1, \dots, M_n$ and list of terms $\overline{P}$ we have*

$$[M_1, \dots, M_n]\overline{P} \in [\![A]\!] \text{ if and only if } M_i\overline{P} \in [\![A]\!] \text{ for all } i.$$

*Proof.* Proof by induction on the structure of $A$.

($\Rightarrow$): For atomic type $a$, we have $[\![a]\!] = \mathsf{SN}$, so suppose $[M_1, \dots, M_n]\overline{P} \in \mathsf{SN}$. This $\beta$-reduces in several steps to $[M_1\overline{P}, \dots, M_n\overline{P}]$. Hence the subterms $M_i\overline{P}$ are $\mathsf{SN}$. Now suppose $[M_1, \dots, M_n]\overline{P} \in [\![A \to B]\!]$ and $N \in [\![A]\!]$. By definition, $[M_1, \dots, M_n]\overline{P}N \in [\![B]\!]$. By induction hypothesis, $M_i\overline{P}N \in [\![B]\!]$ for all $i$. Hence $M_i\overline{P} \in [\![A \to B]\!]$ for all $i$.

($\Leftarrow$): For atomic type $a$, suppose $M_i\overline{P} \in \mathsf{SN}$ for all $i$. Because there are no infinite reductions in $M_i$ for all $i$ and $\overline{P}$, a reduction from $[M_1, \dots, M_n]\overline{P}$ reduces to $[Q_1, \dots Q_n]$ for $Q_i$ such that $M_i\overline{P} \twoheadrightarrow_\beta Q_i$. Since $Q_i$ is $\mathsf{SN}$ and $[Q_1, \dots, Q_n]$ does not add a new redex, we have that $[Q_1, \dots, Q_n] \in \mathsf{SN}$. So $[M_1, \dots, M_n]\overline{P}$ is $\mathsf{SN}$. The induction step is analogous to the induction step of ($\Rightarrow$). ☐

**Proposition 3.4.10.** *If* $y_1 : B_1, \ldots, y_m : B_m \vdash M : A$ *and* $N_1 \in [\![B_1]\!], \ldots, N_m \in [\![B_m]\!]$*, then* $M[y_1 := N_1, \ldots, y_m := N_m] \in [\![A]\!]$*.*

*Proof.* By induction on the derivation of $\Gamma \vdash M : A$. See Definition 3.4.1 for the rules. For the abstraction rule, that is $M = \lambda x.M'$, we have to use property (3) of Lemma 3.4.8. Here we only prove the statement for the parallel rule. Suppose for this case $y_1 : B_1, \ldots, y_m : B_m \vdash [M_1, \ldots, M_n] : A$ and $N_1 \in [\![B_1]\!], \ldots, N_m \in [\![B_m]\!]$. The induction hypothesis gives us $M_i[y_1 := N_1, \ldots, y_m := N_m] \in [\![A]\!]$ for all $i$. With Lemma 3.4.9 we conclude $[M_1, \ldots, M_n][y_1 := N_1, \ldots, y_m := N_m] \in [\![A]\!]$. □

**Theorem 3.4.11** (Strong normalization)**.** *Parallel simply typed λ-calculus is strongly normalizing.*

*Proof.* By taking $N_i := y_i$ in Proposition 3.4.10. (Note that $y_i \in [\![B_i]\!]$ by Lemma 3.4.8.) Then $M \in [\![A]\!] \subseteq \mathsf{SN}$. □

**Proposition 3.4.12** (Subject reduction property)**.** *Parallel simple typed λ-calculus satisfies the subject reduction property. That is, if* $\Gamma \vdash M : A$ *and* $M \to_\beta N$*, then* $\Gamma \vdash N : A$*.*

*Proof.* We do induction on the generation of $M \to_\beta N$. Definition 3.4.5 gives the different cases. See for case (1), (3a), (3b) and (3c) the proof of Lemma 2.11.5 in [8]. For case (1) we need the substitution lemma (Lemma 3.4.4).

For case (2), assume that $\Gamma \vdash [M_1, \ldots, M_n]N : A$. We want $\Gamma \vdash [M_1N, \ldots, M_nN] : A$. By $\Gamma \vdash [M_1, \ldots, M_n]N : A$, there must be a type $B$ such that $\Gamma \vdash [M_1, \ldots, M_n] : B \to A$ and $\Gamma \vdash N : B$. Then $\Gamma \vdash M_i : B \to A$ for all $i$, so $\Gamma \vdash M_iN : A$ for all $i$. Therefore $\Gamma \vdash [M_1N, \ldots, M_nN] : A$.

For (3d), suppose that $\Gamma \vdash [P_1, \ldots, M, \ldots, P_n] : A$ and $M \to_\beta N$. We have to prove that $\Gamma \vdash [P_1, \ldots, N, \ldots, P_n] : A$. We have that $\Gamma \vdash M : A$, so by induction we have also $\Gamma \vdash N : A$. Since $\Gamma \vdash P_i : A$ for all $i$ we conclude $\Gamma \vdash [P_1, \ldots, N, \ldots, P_n] : A$. □

**Proposition 3.4.13** (Church-Rosser property)**.** *Suppose that for a given* $\mathrm{p}\lambda^{\to}$*-term* $M$*, we have* $M \twoheadrightarrow_\beta N_1$ *and* $M \twoheadrightarrow_\beta N_2$*. Then there is a* $\mathrm{p}\lambda^{\to}$*-term* $N_3$ *such that* $N_1 \twoheadrightarrow_\beta N_3$ *and* $N_2 \twoheadrightarrow_\beta N_3$*.*

Takahashi developed a short proof method for the Church-Rosser property in $\lambda^{\to}$ [14]. We are going to apply her method to $\mathrm{p}\lambda^{\to}$. Before we can actually prove the statement, we have to introduce some terminology, analogously to Takahashi's method. She introduces a parallel β-reduction. Note that this has nothing to do with the parallel system $\mathrm{p}\lambda^{\to}$ we defined. Parallel β-reduction means that β-redexes are contracted simultaneously.

**Definition 3.4.14.** The *parallel β-reduction*, which is denoted by $\Rightarrow_\beta$, is defined inductively as follows.

(1) $x \Rightarrow_\beta x$,
(2) $\lambda x.M \Rightarrow_\beta \lambda x.M'$ if $M \Rightarrow_\beta M'$,
(3) $MN \Rightarrow_\beta M'N'$ if $M \Rightarrow_\beta M'$ and $N \Rightarrow_\beta N'$,
(4) $[M_1, \ldots, M_n] \Rightarrow_\beta [M_1', \ldots, M_n']$ if $M_i \Rightarrow_\beta M_i'$ for all $i$,
(5) $(\lambda x.M)N \Rightarrow_\beta M'[x := N']$ if $M \Rightarrow_\beta M'$ and $N \Rightarrow_\beta N'$,
(6) $[M_1, \ldots, M_n]N \Rightarrow_\beta [M_1'N', \ldots, M_n'N']$ if $M_i \Rightarrow_\beta M_i'$ for all $i$ and $N \Rightarrow_\beta N'$.

Based on the inductive definition of $\Rightarrow_\beta$, we have the following facts.

**Lemma 3.4.15.**

*(1)* $M \to_\beta M'$ *implies* $M \Rightarrow_\beta M'$*,*
*(2)* $M \Rightarrow_\beta M'$ *implies* $M \twoheadrightarrow_\beta M'$*,*
*(3)* $M \Rightarrow_\beta M'$*,* $N \Rightarrow_\beta N'$ *implies* $M[x := N] \Rightarrow_\beta M'[x := N']$*.*

*Proof.* Properties (2) and (3) can be verified by a straightforward induction on the structure of $M$. We have already seen several proofs by induction on the structure of a proof term, therefore we will skip the proof here for (2) and (3).

Statement (1) is proved by induction on the generation of $M \to_\beta M'$, see Definition 3.4.5. Case (1), (3a), (3b) and (3c) of Definition 3.4.5 are already verified by Takahashi [14]. For the second case, we have to prove that $[M_1, \ldots, M_n]N \Rightarrow_\beta [M_1 N, \ldots, M_n N]$. Point (1) and (6) of Definition 3.4.14 gives immediately the desired result. For case (3d) of Definition 3.4.5, assume that $M \to_\beta N$ and we want to prove $[P_1, \ldots, M, \ldots, P_n] \Rightarrow_\beta [P_1, \ldots, N, \ldots, P_n]$. With induction hypothesis we have $M \Rightarrow_\beta N$. Since $P_i \Rightarrow_\beta P_i$ for all $i$, we conclude $[P_1, \ldots, M, \ldots, P_n] \Rightarrow_\beta [P_1, \ldots, N, \ldots, P_n]$. □

Property (1) and (2) of Lemma 3.4.15 induce that $\twoheadrightarrow_\beta$ is the reflexive and transitive closure of $\Rightarrow_\beta$. This means that when proving the Church-Rosser property, it is sufficient to show the 'diamond property' of $\Rightarrow_\beta$, that is,

$$\text{if } N_1 \;{}_\beta\!\Leftarrow M \Rightarrow_\beta N_2, \text{ then } N_1 \Rightarrow_\beta N_3 \;{}_\beta\!\Leftarrow N_2 \text{ for some } N_3.$$

But it is possible to prove even a stronger statement,

$$\text{if } M \Rightarrow_\beta N, \text{ then } N \Rightarrow_\beta M^* \text{ for some } M^*. \qquad (*)$$

At this point we have enough information to prove the Church-Rosser property.

*Proof of Proposition 3.4.13.* We prove statement $(*)$. We are going to define $M^*$ by induction on term $M$. Term $M^*$ does not depend on the form of $N$.

(1*) $x^* = x$,
(2*) $(\lambda x.M)^* = \lambda x.M^*$,
(3*) $(M_1 M_2)^* = M_1^* M_1^*$ if $M_1 M_2$ is not a $\beta$-redex.
(4*) $[M_1, \ldots, M_n]^* = [M_1^*, \ldots, M_n^*]$,
(5*) $((\lambda x.M_1)M_2)^* = M_1^*[x := M_2^*]$,
(6*) $([M_1, \ldots, M_n]P)^* = [M_1^* P^*, \ldots, M_n^* P^*]$.

Property $(*)$ can be verified by induction on $M$. Proofs of (1*), (2*), (3*) and (5*) are written down in [14]. For case (4*), suppose that $[M_1, \ldots, M_n] \Rightarrow_\beta N$. We have $N = [N_1, \ldots, N_n]$ with $M_i \Rightarrow_\beta N_i$ for all $i$. By the induction hypothesis we know $N_i \Rightarrow_\beta M_i^*$ for all $i$, therefore $N = [N_1, \ldots, N_n] \Rightarrow_\beta [M_1^*, \ldots, M_n^*] = M^*$. Now we prove case (5*). If $M = [M_1, \ldots, M_n]P \Rightarrow_\beta N$, then either $N = [N_1, \ldots, N_n]Q$ or $N = [N_1 Q, \ldots, N_n Q]$ with in both cases $M_i \Rightarrow_\beta N_i$ for all $i$ and $P \Rightarrow_\beta Q$. By induction hypothesis, $N_i \Rightarrow_\beta M_i^*$ for all $i$ and $Q \Rightarrow_\beta P^*$.

- If $N = [N_1, \ldots, N_n]Q$: We have $[N_1, \ldots, N_n] \Rightarrow_\beta [M_1^*, \ldots, M_n^*] = [M_1, \ldots, M_n]^*$. Then $[N_1, \ldots, N_n]Q \Rightarrow_\beta [M_1^* P^*, \ldots, M_n^* P^*] = M^*$.
- If $N = [N_1 Q, \ldots, N_n Q]$: We have $N_i Q \Rightarrow_\beta M_i^* P^*$, so $[N_1 Q, \ldots N_n Q] \Rightarrow_\beta [M_1^* P^*, \ldots, M_n^* P^*]$.

This completes the proof of $(*)$, and hence we have proved the Church-Rosser property for p$\lambda^\to$. □

We have shown that p$\lambda^\to$ satisfies the same important properties as $\lambda^\to$, such as the substitution lemma, strong normalization and the Church-Rosser property. The advantage of p$\lambda^\to$ is that the parallel terms are proofs which stores several possible (normal) derivations. In the proof of strong normalization of the natural deduction derived from truth tables in Chapter 3.6, we will see that it is useful to choose a possible derivation in a parallel term.

**Definition 3.4.16.** We give an inductive definition of the notion of parallel subterm. We write $M' \sqsubseteq M$ for $M'$ is *parallel subterm of* $M$. Relation $\sqsubseteq$ is defined by the following rules.

- $M \sqsubseteq M$,
- If $N \sqsubseteq M_i$ for some $i$, then $N \sqsubseteq [M_1, \ldots, M_n]$,
- If $N_i \sqsubseteq M_i$ for all $i$, then $[N_1, \ldots, N_n] \sqsubseteq [M_1, \ldots, M_n]$,
- If $P \sqsubseteq Q$, then $\lambda x.P \sqsubseteq \lambda x.Q$,
- If $P \sqsubseteq Q$ and $M \sqsubseteq N$, then $PM \sqsubseteq QN$.

**Example 3.4.17.** Recall the proof term $\lambda x.\lambda y.\lambda z.\lambda w.[zx, wy]$ in Example 3.4.2. It has the following subterms: the term itself, $\lambda x.\lambda y.\lambda z.\lambda w.zx$ and $\lambda x.\lambda y.\lambda z.\lambda w.wy$.

**Lemma 3.4.18.** *The relation $\sqsubseteq$ is a partial order, that is, $\sqsubseteq$ is reflexive, antisymmetric and transitive.*

*Proof.* Reflexivity follows directly from the definition. For antisymmetry suppose $M \sqsubseteq N$ and $N \sqsubseteq M$ and suppose $M \neq N$. By inspection of the definition of $\sqsubseteq$, we know that $M$ is derived from less or the same amount of typing rules as $N$, because $M \sqsubseteq N$. If $M$ and $N$ consists of the same number of typing rules, then $M = N$. This is guaranteed by the fact that components in a parallel term cannot be swapped by the subterm relation. But we assumed that $M \neq N$. So $M$ is derived from less rules then $N$, but then $N \not\sqsubseteq M$ which leads to a contradiction.

For transitivity suppose $M \sqsubseteq N$ and $N \sqsubseteq P$. We prove this by induction on the generation of $N \sqsubseteq P$. If $N = P$ then indeed $M \sqsubseteq P$. Now let $N \neq P$.

- If $N \sqsubseteq [P_1, \ldots, P_n]$, then $N \sqsubseteq P_i$ for some $i$, or $N = [N_1, \ldots, N_n]$ with $N_i \sqsubseteq P_i$ for all $i$. For the first case we have immediately $M \sqsubseteq N \sqsubseteq P_i$, so $M \sqsubseteq [P_1, \ldots, P_n]$. For the second, the induction hypothesis gives $M \sqsubseteq [N_1, \ldots, N_n]$. Again there are two possibilities. If $M \sqsubseteq N_j$ for some $j$, then $M \sqsubseteq N_j \sqsubseteq P_j$ and therefore $M \sqsubseteq [P_1, \ldots, P_n]$. If $M = [M_1, \ldots, M_n]$ with $M_i \sqsubseteq N_i$ for all $i$ (it may me the case that $M_i = N_i$ for all $i$), then $M_i \sqsubseteq P_i$ for all $i$ and hence $M = [M_1, \ldots, M_n] \sqsubseteq [P_1, \ldots, P_n]$.
- If $N \sqsubseteq \lambda x.P'$, then $N$ of the form $\lambda x.N'$ with $N' \sqsubseteq P'$. But then $M = \lambda x.M'$ for some $M'$ with $M' \sqsubseteq N'$ (it may be the case that $M' = N'$). Applying the induction hypothesis to $P'$ we have $M' \sqsubseteq P'$, so $M \sqsubseteq P$.
- If $N \sqsubseteq P_1P_2$, then $N = N_1N_2$ for some $N_1$ and $N_2$ such that $N_1 \sqsubseteq P_1$ and $N_2 \sqsubseteq P_2$. This implies that $M$ is also of the form $M_1M_2$ with $M_1 \sqsubseteq N_1$ and $M_2 \sqsubseteq N_2$ (it may be the case that $M = N$). With induction hypothesis we conclude $M_1M_2 \sqsubseteq P_1P_2$.

We have treated all cases for $N \sqsubseteq P$, hence $\sqsubseteq$ is transitive. □

**Lemma 3.4.19.** *Let $M$, $N$ and $P$ be* p$\lambda^{\rightarrow}$*-terms such that $M \rightarrow_\beta N$ and $M \sqsubseteq P$, then there is a* p$\lambda^{\rightarrow}$*-term $Q$ such that $P \stackrel{+}{\rightarrow}_\beta Q$ and $N \sqsubseteq Q$.*

*Proof.* We proceed by induction on the generation of $M \sqsubseteq P$. We look at two interesting cases with redex $M = M_1M_2 = (\lambda x.M')M_2$. Let $P = P_1P_2$ with $M_1 \sqsubseteq P_1$, $M_2 \sqsubseteq P_2$ and $M = (\lambda x.M')M_2 \rightarrow_\beta M'[x := M_2] = N$. There are two possibilities for $M_1 \sqsubseteq P_1$.

- If $P_1 = \lambda x.P'$ for some $P'$, define $Q = P'[x := P_2]$. Then we have $P = (\lambda x.P')P_2 \rightarrow_\beta Q$ and $N = M'[x := M_2] \sqsubseteq Q$, because $M' \sqsubseteq P'$ and $M_2 \sqsubseteq P_2$.
- If $P_1 = [P'_1, \ldots, P'_n]$ such that $M_1 \sqsubseteq P'_i$ for some $i$. Again there are two cases. If $P'_i$ is a parallel term, then repeat the process. If $P'_i = \lambda x.P''$ with $M' \sqsubseteq P''$, then define

  $$Q = [P'_1P_2, \ldots, P''[x := P_2], \ldots, P'_nP_2].$$

  We have $P = [P'_1, \ldots, P'_n]P_2 \rightarrow_\beta [P'_1P_2, \ldots, (\lambda x.P'')P_2, \ldots, P'_nP_2] \rightarrow_\beta Q$. And we also have $N = M'[x := M_2] \sqsubseteq Q$, because $M' \sqsubseteq P''$ and $M_2 \sqsubseteq P_2$. □

## 3.5 The Curry-Howard isomorphism

The Curry-Howard isomorphism is a general notion to define a direct relationship between proof theory and type theory. Proof theory is in the field of mathematics, whereas type theory is a subject of computer science. In short, due to the Curry-Howard isomorphism, formulas correspond to types and derivations correspond to proof terms. So far, we considered the truth table system in a proof theoretical context. The short notation of proof terms forms a great advantage of studying proof terms over derivation trees. In addition, detour and permutation conversions correspond to certain reductions in the type sytem. This makes it easier to prove strong normalization in Section 3.6.

The type system that we define is based on the $\lambda$-calculus. For each $\mathcal{C}$ we define a system $\lambda^{\mathcal{C}}$, where for each derivation rule in $\mathrm{IPC}_C$, we give a typing rule in $\lambda^{\mathcal{C}}$. The types in $\lambda^{\mathcal{C}}$ are exactly the formulas. In addition, we define reductions in $\lambda^{\mathcal{C}}$ that correspond to detour and permutation conversions. [5]

**Definition 3.5.1.** Let $\mathcal{C}$ be a set of connectives. The *types* in $\lambda^{\mathcal{C}}$ are the formulas involving connectives from $\mathcal{C}$. The abstract syntax for *proof terms* in $\lambda_{\mathcal{C}}$ is

$$M ::= x \mid M \cdot_r [\overline{M}; \overline{\lambda x.M}] \mid \{\overline{M}; \overline{\lambda x.M}\}_r$$

where $x$ ranges over variables and $r$ ranges over the rules of all connectives in $\mathcal{C}$. We write $\overline{M}$ to mean a finite sequence of terms. Let $\Gamma$ be a set of type declarations of the form $x : A$. $\Gamma$ is called the *context*. The terms are typed using the derivation rules of $\mathrm{IPC}_{\mathcal{C}}$.

$$\frac{}{\Gamma \vdash x : A}\ \text{axiom, if } x : A \in \Gamma$$

$$\frac{\Gamma \vdash M : \Phi \qquad \ldots \Gamma \vdash N_k : A_k \ldots \qquad \ldots \Gamma, x_l : A_l \vdash O_l : D \ldots}{\vdash M \cdot_r [\overline{N}; \overline{\lambda x.O}] : D}\ r, \text{el}$$

$$\frac{\ldots \Gamma \vdash N_j : A_j \ldots \qquad \ldots \Gamma, y_i : A_i \vdash M_i : \Phi \ldots}{\vdash \{\overline{N}; \overline{\lambda y.M}\}_r : \Phi}\ r, \text{in}^{\text{i}}$$

Both in the elimination and introduction rules, we prefer to use capital letter $M$ for terms of type $\Phi$, where $\Phi = c(A_1, \ldots, A_n)$ for the concerned connective $c$. In the elimination rules, $\overline{N}$ is the sequence of terms $N_k$ for the 1-entries in the truth table $t_c$ and $\overline{\lambda x.O}$ is the sequence of the $\lambda x_l.O_l$'s for the 0-entries. Identically, for the introduction rules, $\overline{N}$ is a sequence containing $N_j$ for 1-entries and $\overline{\lambda y.M}$ contains terms $\lambda y_i.M_i$ for 0-entries.

If it is clear from the context which rule is applied, we omit $r$ in the elimination and introduction term. The method of Definition 3.5.1 can also applied to the optimized rules in a straightforward way.

**Example 3.5.2.** The typing rules of disjunction are as follows, with their corresponding terms.

$$\frac{\vdash M : A \vee B \quad x : A \vdash O_1 : D \quad y : B \vdash O_2 : D}{\vdash M \cdot [\ ; \lambda x.O_1, \lambda y.O_2] : D}\ \vee\text{-el} \qquad \frac{\vdash N_A : A \quad y : B \vdash M : A \vee B}{\vdash \{N_A\ ;\ \lambda y.M\} : A \vee B}\ \vee\text{-in}_{10}$$

$$\frac{x : A \vdash M : A \vee B \qquad \vdash N_B : B}{\vdash \{N_B\ ;\ \lambda x.M\} : A \vee B}\ \vee\text{-in}_{01} \qquad \frac{\vdash N_A : A \qquad \vdash N_B : B}{\vdash \{N_A, N_B;\ \} : A \vee B}\ \vee\text{-in}_{11}$$

With Definition 3.5.1, we see that derivations in $\mathrm{IPC}_{\mathcal{C}}$ directly correspond to proof terms in $\lambda^{\mathcal{C}}$. Now we define term reduction rules that correspond to detour and permutation conversion. Just as in [5], we write $\overline{N, N_{j'}}$ to mean the sequence $N_1, \ldots, N_{j'}, \ldots, N_m$. We use this notation if $N_{j'}$ plays a role in a matching case of a detour convertibility. In the definition of the detour reduction we use the notion of substituting term $N$ into $\lambda x.M$, writing $M[x := N]$. Substitution in $\lambda^{\mathcal{C}}$ is defined in a similar way as for simply typed lambda calculus (see Definition 3.4.3), so we leave out the formal definition and turn directly to the definition of detour reduction.

**Definition 3.5.3** (Detour reduction)**.** Consider a term of a detour convertibility as defined in Definition 3.1.1. Define detour reduction in $\lambda_{\mathcal{C}}$ as follows.

(1) $l' = j'$ for some $l', j'$, that is, $x_{l'} : A_{l'}$ and $N_{j'} : A_{j'}$ with $A_{l'} = A_{j'}$:

$$\{\overline{N, N_{j'}} \;;\; \overline{\lambda y.M}\} \cdot [\overline{P} \;;\; \overline{\lambda x.Q, \lambda x_{l'}.Q_{l'}}] \;\longrightarrow_{\mathrm{D}}\; Q_{l'}[x_{l'} := N_{j'}]$$

(2) $k' = i'$ for some $k', i'$, that is, $y_{i'} : A_{i'}$ and $P_{k'} : A_{k'}$ with $A_{k'} = A_{i'}$:

$$\{\overline{N} \;;\; \overline{\lambda y.M, \lambda y_{i'}.M_{i'}}\} \cdot [\overline{P, P_{k'}} \;;\; \overline{\lambda x.Q}] \;\longrightarrow_{\mathrm{D}}\; M_{i'}[y_{i'} := P_{k'}] \cdot [\overline{P, P_{k'}} \;;\; \overline{\lambda x.Q}]$$

(3) and if $P \longrightarrow_{\mathrm{D}} Q$, then

  (a) $P \cdot [\overline{N} \;;\; \overline{\lambda x.O}] \;\longrightarrow_{\mathrm{D}}\; Q \cdot [\overline{N} \;;\; \overline{\lambda x.O}]$

  (b) $M \cdot [\overline{N}, P, \overline{N'} \;;\; \overline{\lambda x.O}] \;\longrightarrow_{\mathrm{D}}\; M \cdot [\overline{N}, Q, \overline{N'} \;;\; \overline{\lambda x.O}]$

  (c) $M \cdot [\overline{N} \;;\; \overline{\lambda x.O}, \lambda x.P, \overline{\lambda x'.O'}] \;\longrightarrow_{\mathrm{D}}\; M \cdot [\overline{N} \;;\; \overline{\lambda x.O}, \lambda x.Q, \overline{\lambda x'.O'}]$

  (d) $\{\overline{N}, P, \overline{N'} \;;\; \overline{\lambda y.M}\} \;\longrightarrow_{\mathrm{D}}\; \{\overline{N}, Q, \overline{N'} \;;\; \overline{\lambda y.M}\}$

  (e) $\{\overline{N} \;;\; \overline{\lambda y.M}, \lambda y.P, \overline{\lambda y'.M'}\} \;\longrightarrow_{\mathrm{D}}\; \{\overline{N} \;;\; \overline{\lambda y.M}, \lambda y.Q, \overline{\lambda y.M'}\}$

Numbers (1) and (2) are base cases of the detour reduction. In these cases, the left term is called a *redex.* Number (3) represents the extension of the reduction to subterms. Points (a)-(c) are concerned with the elimination term, and points (d) and (e) are concerned with the introduction term.

**Example 3.5.4.** There are four detour reductions of terms with disjunction from Example 3.5.2, namely

$$\begin{aligned}
\{N_B \;;\; \lambda x.M\}_{01} \cdot_{\vee} [\;;\lambda x.O_1, \lambda y.O_2] &\;\longrightarrow_{\mathrm{D}}\; O_2[y := N_B] \\
\{N_A \;;\; \lambda y.M\}_{10} \cdot_{\vee} [\;;\lambda x.O_1, \lambda y.O_2] &\;\longrightarrow_{\mathrm{D}}\; O_1[y := N_A] \\
\{N_A, N_B;\;\}_{11} \cdot_{\vee} [\;;\lambda x.O_1, \lambda y.O_2] &\;\longrightarrow_{\mathrm{D}}\; O_1[y := N_A] \\
\{N_A, N_B;\;\}_{11} \cdot_{\vee} [\;;\lambda x.O_1, \lambda y.O_2] &\;\longrightarrow_{\mathrm{D}}\; O_2[y := N_B]
\end{aligned}$$

We see that there are two possible reductions starting from $\{N_A, N_B;\;\}_3$ in two different terms, which means that the detour reduction is non-deterministic. We already saw this in Section 3.1.

Now we define the permutation reductions.

**Definition 3.5.5.** Consider a term of a permutation convertibility as defined in Definition 3.1.5. Define permutation reduction in $\lambda_{\mathcal{C}}$ as follows.

(1) $(M \cdot [\overline{N}; \overline{\lambda x.O}]) \cdot [\overline{P}; \overline{\lambda y.Q}] \;\longrightarrow_{\mathrm{P}}\; M \cdot [\overline{N}; \overline{\lambda x.(O \cdot [\overline{P}; \overline{\lambda y.Q}])}]$

(2) Extend the definition on subterms in the same way as in Definition 3.5.3 (3) with $\longrightarrow_{\mathrm{D}}$ replaced by $\longrightarrow_{\mathrm{P}}$ .

Number (1) is the base case, where we call the left term a *redex.*

**Definition 3.5.6.** A term is in *normal form* if it contains no redex.

Due to the Curry-Howard isomorphism, normal derivations in $\mathrm{IPC}_{\mathcal{C}}$ correspond to normal forms in $\lambda^{\mathcal{C}}$. Since an introduction followed by an elimination is always a redex, we have the following lemma about normal forms [5].

**Lemma 3.5.7.** *Term $P$ is in normal form if one of the following holds.*

- $P = x$, *where $x$ is a variable,*
- $P = x \cdot [\overline{N}; \overline{\lambda x.O}]$ *with all $N_k$ and $O_l$ in normal form and $x$ a variable,*
- $P = \{\overline{N}; \overline{\lambda y.M}\}$ *with all $N_j$ and $M_i$ in normal form.*

## 3.6 Strong normalization

There are several proofs for weak normalization of $\text{IPC}_{\mathcal{C}}$ with permutation and detour conversions, but proving strong normalization is rather difficult. In [5], strong normalization of detour conversion is proven. Also strong normalization for permutation conversion. Here we present a detailed proof of strong normalization of the union of detour and permutation conversion. We recall the general definition of strong normalization.

**Definition 3.6.1.** Let $R = \{\longrightarrow_1, \ldots, \longrightarrow_n\}$ be a set of reduction relations. Well-formed term $M$ is *strongly normalizing with regard to $R$* if there is no infinite reduction sequence of reductions from $R$ starting from $M$. We say that a logical system has the *strong normalization property with regard to $R$* if every well-formed term is strongly normalizing with regard to $R$.

So in this section we have $R = \{ \longrightarrow_{\mathrm{D}} , \longrightarrow_{\mathrm{P}} \}$.

Our proof is based on the work of Philippe de Groote [6] who gives an extensive proof for strong normalization of the Prawitz natural deduction (Chapter 1). His proof is based on a translation from natural deduction to simply typed $\lambda$-calculus, which is strongly normalizing. The clue of the proof is that permutation conversions in the Prawitz system (Definition 1.2.8) correspond to syntactic $\beta$-equivalence in simply typed $\lambda$-calculus and that detour conversions (Definition 1.2.7) correspond to $\beta$-reduction in simply typed $\lambda$-calculus. In [6], it is not always made explicit how the definitions arise. This makes it difficult to generalize for arbitrary connectives. Another difficulty is that De Groote excludes the conversion of a cut formula of $\bot^{\mathrm{i}}$-rule. However, we have created a method to apply his approach to $\text{IPC}_{\mathcal{C}}$.

Analoguous to De Groote, we define a translation from the truth table system to parallel simply typed lambda calculus, $\mathrm{p}\lambda^{\rightarrow}$. Since De Groote does not encounter conversions of the $\bot^{\mathrm{i}}$-rule, this approach only leads to strong normalization for certain permutation conversions in the truth table system. But this is enough to show strong normalization for all conversions. Recall the definitions of detour and permutation conversions from Definition 3.5.3 and Definition 3.5.5. We now distinguish between two base cases for permutation conversions.

*Permutation reductions:*

(1) 'Positive permutation': assume there is a case $\lambda x_l.O_l$.

$$(M \cdot [\overline{N}; \overline{\lambda x.O}]) \cdot_{r_s} [\overline{P}; \overline{\lambda y.Q}] \longrightarrow_{\mathrm{Ppos}} M \cdot [\overline{N}; \overline{\lambda x.(O \cdot_{r_s} [\overline{P}; \overline{\lambda y.Q}])}]$$

(2) 'Negative permutation':

$$(M \cdot [\overline{N};\ ]) \cdot_{r_s} [\overline{P}; \overline{\lambda y.Q}] \longrightarrow_{\mathrm{Pneg}} M \cdot [\overline{N};\ ]$$

*Detour reductions:*

(1) $l' = j'$ : $\quad \{\overline{N, N_{j'}}; \overline{\lambda y.M}\} \cdot_{r_s} [\overline{P}; \overline{\lambda x.Q, \lambda x_{l'}.Q_{l'}}] \longrightarrow_{\mathrm{D}} Q_l[x_{l'} := N_{j'}]$

(2) $k' = i'$ : $\quad \{\overline{N}; \overline{\lambda y.M, \lambda y_{i'}.M_{i'}}\} \cdot_{r_s} [\overline{P, P_{k'}}; \overline{\lambda x.Q}] \longrightarrow_{\mathrm{D}} M_{i'}[y_{i'} := P_{k'}] \cdot [\overline{P, P_{k'}}; \overline{\lambda x.Q}]$

We will see that the approach of De Groote proves strong normalization for detour and positive permutation conversions of the truth table system. Theorem 3.6.23 shows that this leads to strong normalization for detour and all permutation conversions.

We make a translation for each type (formula) of the truth table system and we define a corresponding translation of terms. First we translate formulas. We use a negative translation of formulas. De Groote bases his negative translation on the translation induced by Plotkin's call-by-name CPS-translation [10]. In [6], it is not made explicit how the translation arises. However, after a close reading one can observe that the translation can be derived from the elimination rules of the concerned connective. We generalizes this strategy and it turns out to be effective in order to prove strong normalization.

**Definition 3.6.2** (Type translation)**.** Let $o$ be a distinguished atomic proposition in $\mathrm{p}\lambda^{\rightarrow}$ and for every type $A$ in $\mathrm{p}\lambda^{\rightarrow}$, denote $\sim A := A \to o$. The negative translation $\overline{\Phi}$ of any formula $\Phi$ of $\mathrm{IPC}_{\mathcal{C}}$ is defined inductively as follows.

- For $\Phi = A$ a proposition letter, $\overline{A} := \sim\sim A$.
- For $\Phi = c(A_1, \dots, A_n)$ for some connective $c \in \mathcal{C}$, with elimination rules $r_1, \dots, r_t$,

  $$\overline{\Phi} := \sim(E_1 \to \cdots \to E_t \to o),$$

  with elimination pattern for rule $r_s$

  $$E_s = \overline{A}_{k_1} \to \cdots \to \overline{A}_{k_m} \to \ \sim\overline{A}_{l_1} \to \cdots \to \sim\overline{A}_{l_{n-m}} \to o,$$

  where the $A_k$'s are the formulas where $a_k = 1$ and the $A_l$'s are the formulas where $a_l = 0$ in the row of the truth table $t_c$ that corresponds with elimination rule $r_s$.

  There are two special cases. If there is no elimination rule for $c$, then $\overline{\Phi} := \sim o$. If $c$ is a 0-ary connective with an elimination rule, then $\overline{\Phi} := \sim\sim o$. Note that $\bot$ is the only connective with this property.

If we look at the definition, $\overline{\Phi}$ is of the form $\sim\Phi^{\circ}$, where $\Phi^{\circ} = \sim\Phi$ if $\Phi$ is a propostion letter and $\Phi^{\circ} = E_1 \to \cdots \to E_t \to o$ if $\Phi = c(A_1, \dots, A_n)$ for some connective $c$.

It is possible to apply this translation to the optimized rules defined in Section 2.2. To do this, we only consider rules optimized by Lemma 2.2.3. In Appendix A, the optimized rules are presented for both Lemma 2.2.3 and Lemma 2.2.4. When an elimination rule is optimized by Lemma 2.2.4, the conclusion of that rule is not an arbitrary formula $D$, but a subformula of the major premise. We must rewrite those rules using Lemma 2.2.4 by replacing the conclusion $A$ by an arbitrary formula $D$ and adding the case $A \vdash D$. We give an example of $\wedge$ and see how the translation works.

**Example 3.6.3.** Conjunction $\wedge$ has the following two optimized elimination rules, as denoted in Appendix A.

$$\frac{A \wedge B}{\vdash A}\ \wedge\text{-el}_1 \qquad \text{and} \qquad \frac{A \wedge B}{\vdash B}\ \wedge\text{-el}_2$$

First we rewrite those rules with Lemma 2.2.4 to the following.

$$\frac{A \wedge B \qquad A \vdash D}{\vdash D}\ \wedge\text{-el}_1 \qquad \text{and} \qquad \frac{A \wedge B \qquad B \vdash D}{\vdash D}\ \wedge\text{-el}_2$$

We have that $E_1 = \sim\overline{A} \to o = \sim\sim\overline{A}$ and $E_2 = \sim\overline{B} \to o = \sim\sim\overline{B}$. So

$$\overline{A \wedge B} = \sim(\sim\sim\overline{A} \to \sim\sim\overline{B} \to o).$$

This is almost similar to the definition of De Groote. He defines a translation $\overline{A} = \sim\sim A^{\circ}$ with $(A \wedge B)^{\circ} = \sim(\overline{A} \to \overline{B} \to o)$.

**Example 3.6.4.** Here we state the negative translation of the optimized formulas with connectives $\vee$, $\to$, $\neg$, $\bot$ and $\top$. See Appendix A for the optimized elimination rules. Note that some rules have to be rewritten with Lemma 2.2.4 before translating to p$\lambda^{\to}$.

- Disjunction: $\overline{A \vee B} = \sim((\sim\overline{A} \to \sim\overline{B} \to o) \to o)$
- Implication: $\overline{A \to B} = \sim((\overline{A} \to \sim\overline{B} \to o) \to o)$
- Negation: $\overline{\neg A} = \sim(\sim\overline{A} \to o) = \sim\sim\sim\overline{A}$
- Bottom: $\overline{\bot} = (o \to o) \to o = \sim\sim o$
- Top: $\overline{\top} = o \to o$

In [6], the term translation is established in two steps. First a translation $\overline{M}$ and then refining this definition to $\overline{\overline{M}}$. We proceed in the same way of translating a term $M$ in the truth table system to terms $\overline{M}$ and $\overline{\overline{M}}$ in p$\lambda^{\to}$. We prefer to write indices $i, j, k, l$ in the sequences of terms, for example, we write $\vec{N_j}$ instead of $\vec{N}$, to make clear that it is a sequences of $N_j$'s.

**Definition 3.6.5** (Term translation $\overline{M}$)**.** For $c$ an $n$-ary connective with $t$ elimination rules $r_1, \dots, r_t$ we define $\overline{M}$ inductively.

(1) (Axiom)

$$\overline{x} := \lambda h.xh$$

(2) (Elimination) We distinguish between elimination terms with case or without any case.

$$\overline{M \cdot_{r_s} [\vec{N_k}; \vec{\lambda x_l.O_l}]} := \lambda h.\overline{M}(\lambda g_1 \dots \lambda g_t.g_s\, \vec{\overline{N_k}}(\vec{\lambda x_l.\overline{O_l}h})) \text{ if there is a case } \lambda x_l.O_l$$

$$\overline{M \cdot_{r_s} [\vec{N_k};\ ]} := \lambda h.\overline{M}\big(\lambda p.(\lambda g_1, \dots, \lambda g_t.g_s\, \vec{\overline{N_k}})h\big)$$

(3) (Introduction)

$$\overline{\{\vec{N_j}; \vec{\lambda y_i.M_i}\}} := \lambda h.\big((\vec{\lambda q_j}\, .\, \vec{\lambda q_i}\, .h)\, \vec{\overline{N_j}}\, (\vec{\lambda y_i.\overline{M_i}h})\,\big)e_1^h \dots e_t^h,$$

where $\vec{\lambda q_j}$ should be understood as a sequences of lambda abstractions corresponding to $N_j$'s and $\vec{\lambda q_i}$ correspond to $(\lambda y_i.\overline{M_i}h)$'s. Term $e_s^h$ is the possibly parallel term $[\dots]$ defined as follows:

If $E_s = \overline{A}_{k_1} \to \dots \to \overline{A}_{k_m} \to\ \sim\overline{A}_{l_1} \to \dots \to \sim\overline{A}_{l_{n-m}} \to o$ is the elimination pattern for $r_s$, then $e_s^h$ contains

- $\vec{\lambda h_k}\, .\, \vec{\lambda h_l}\, .h_{l'}\overline{N}_{j'}$ for all $j'$ and $l'$ in $r_s$ such that $j' = l'$,
- and $\vec{\lambda h_k}\, .\, \vec{\lambda h_l}\, .(\lambda y_{i'}.\overline{M}_{i'}h)h_{k'}$ for all $i'$ and $k'$ in $r_s$ such that $i' = k'$.

Here, $\vec{\lambda h_k}$ quantifies over all $\overline{A_k}$ in $E_s$ and $\vec{\lambda h_l}$ over all $\sim\overline{A_l}$. In the proof of Proposition 3.6.8, we see that the terms $e_s^h$ are well-defined of type $E_s$, by checking the types of its components.

In the definition we see that there are a lot of redexes, which we call *dummy redexes*. In the elimination term $\overline{M \cdot_{r_s} [\vec{N_k};\ ]}$ redex $(\lambda p.\dots)h$ is a dummy redex. In the introduction term we have for each $N_j$ and $M_i$ a dummy redex $(\lambda p_j \dots)\overline{N_j}$ and $(\lambda p_i \dots)(\lambda y_i.\overline{M_i}h)$. These are necessary in order to not lose any information. In the elimination term we preserve $h$ in this manner. For the introduction rule we make sure that each subterm appears in the translation.

In this definition, we use parallel terms in the introduction term. This is a new idea which is not present in the approach of De Groote. This modification of the term translation is essential, because detour conversion in the truth table system is non-deterministic. With this solution, we see that term $e_s^h$ in the translation of an introduction term stores each possible combination of a detour conversion. This makes it possible to identify each possible detour conversion with a $\beta$-reduction in the translated p$\lambda^{\to}$-term.

Just like the type translation, the term translation can also be applied to the optimized rules. This can be done for terms which are optimized by both Lemma 2.2.3 and Lemma 2.2.4. When an elimination term is optimized by Lemma 2.2.4, the term translation has to be slightly modified which proceeds in the following way. A rule can be optimized by Lemma 2.2.4, only when it has exactly one case, say $x : A \vdash O : D$. Then the term $(\lambda x.\overline{O}h)$ in the elimination translation is replaced by $(\lambda f.fh)$ with $f : \overline{A}$.

**Example 3.6.6.** In this example, we give the term translations of the optimized rules for conjunction, $\wedge$. We already gave the type translation in Example 3.6.3. We have the following optimized rules from Appendix A with the corresponding terms.

$$\frac{M : A \wedge B}{\vdash M \cdot_{\mathrm{el1}} [\,] : A}\wedge\text{-el}_1 \;,\qquad \frac{M : A \wedge B}{\vdash M \cdot_{\mathrm{el2}} [\,] : B}\wedge\text{-el}_2 \qquad \text{and} \qquad \frac{\vdash N_A : A \qquad \vdash N_B : B}{\vdash \{N_A, N_B;\ \}_{\mathrm{in}} : A \wedge B}\wedge\text{-in}$$

The p$\lambda^{\to}$-terms after term translation are as follows. We omit the dummy redexes in the introduction term, because each subterm is already translated by a matching case $j' = l'$.

- $\overline{M \cdot_{\mathrm{el1}} [\,]} = \lambda h.\overline{M}(\lambda g_1.\lambda g_2.g_1(\lambda f.fh))$,
- $\overline{M \cdot_{\mathrm{el2}} [\,]} = \lambda h.\overline{M}(\lambda g_1.\lambda g_2.g_2(\lambda f.fh))$,
- $\overline{\{N_A, N_B;\ \}_{\mathrm{in}}} = \lambda h.h(\lambda h_1.h_1\overline{N_A})(\lambda h_2.h_2\overline{N_B})$.

The detour conversions for the optimized rules for $\wedge$ are deterministic, so there is no parallel term in the introduction rule. With the following derivations, we show that the type and term translations for $\wedge$ commute with the typing relation. That is, if $M : A$ in the truth table system, then $\overline{M} : \overline{A}$ in p$\lambda^{\to}$. We represent the derivations for $\wedge$-el$_1$ and $\wedge$-in,

$$\dfrac{\dfrac{\overline{M} : \overline{A \wedge B} \qquad \dfrac{\dfrac{\dfrac{[g_1 : \sim\sim\overline{A}] \qquad \dfrac{\dfrac{[f : \overline{A}] \qquad [h : A^\circ]}{fh : o}}{\lambda f.fh : \sim\overline{A}}}{g_1(\lambda f.fh) : o}}{\lambda g_2.g_1(\lambda f.fh) : \sim\sim\overline{B} \to o}}{\lambda g_1.\lambda g_2.g_1(\lambda f.fh) : \sim\sim\overline{A} \to \sim\sim\overline{B} \to o}}{\overline{M}(\lambda g_1.\lambda g_2.g_1(\lambda f.fh)) : o}}{\lambda h.\overline{M}(\lambda g_1.\lambda g_2.g_1(\lambda f.fh)) : \overline{A}}$$

and

$$\dfrac{\dfrac{\dfrac{[h : \sim\sim\overline{A} \to \sim\sim\overline{B} \to o] \qquad \dfrac{\dfrac{[h_1 : \sim\overline{A}] \qquad \overline{N_A} : \overline{A}}{h_1\overline{N_A} : o}}{\lambda h_1.h_1\overline{N_A} : \sim\sim\overline{A}}}{h(\lambda h_1.h_1\overline{N_A}) : \sim\sim\overline{B} \to o} \qquad \dfrac{\dfrac{h_2 : \sim\overline{B} \qquad \overline{N_B} : \overline{B}}{h_2\overline{N_B} : o}}{\lambda h_2.h_2\overline{N_B} : \sim\sim\overline{B}}}{h(\lambda h_1.h_1\overline{N_A})(\lambda h_2.h_2\overline{N_B}) : o}}{\lambda h.h(\lambda h_1.h_1\overline{N_A})(\lambda h_2.h_2\overline{N_B}) : \overline{A \wedge B}}$$

**Example 3.6.7.** We continue from Example 3.6.4. We give the term translations of the connectives $\vee, \to, \neg, \bot$ and $\top$ with the optimized rules. See Appendix A for the rules. In each introduction rule we can omit the dummy redexes.

- *Disjunction:* Let $M : A \vee B$, $M_i : A \vee B$, $N_A : A$ and $N_B : B$. Then

$$\overline{M \cdot_{\mathrm{el}} [\,; \lambda x_A.O_1, \lambda x_B.O_2]} = \lambda h.\overline{M}(\lambda g_1.g_1(\lambda x_A.\overline{O_1}h)(\lambda x_B.\overline{O_2}h)),$$
$$\overline{\{N_A;\ \}_{\mathrm{in1}}} = \lambda h.h(\lambda h_1 \lambda h_2.h_1\overline{N_A}),$$
$$\overline{\{N_B;\ \}_{\mathrm{in2}}} = \lambda h.h(\lambda h_1 \lambda h_2.h_2\overline{N_B}).$$

- *Implication:* Let $M : A \to B$, $N_A : A$ and $N_B : B$. Then

$$\overline{M \cdot_{\mathrm{el}} [N_A;\ ]} = \lambda h.\overline{M}(\lambda g_1.g_1\overline{N_A}(\lambda f.fh)),$$
$$\overline{\{N_B;\ \}_{\mathrm{in1}}} = \lambda h.h(\lambda h_1 \lambda h_2.h_2\overline{N_B}),$$
$$\overline{\{\ ; \lambda y_A.M\}_{\mathrm{in2}}} = \lambda h.h(\lambda h_1 \lambda h_2.(\lambda y_A.\overline{M}h)h_1).$$

- *Negation:* Let $M : \neg A$ and $N_A : A$. Then

$$\overline{M \cdot_{\mathrm{el}} [N_A;\ ]} = \lambda h.\overline{M}\big(\lambda p.(\lambda g_1.g_1\overline{N_A})h\big),$$
$$\overline{\{\ ; \lambda y_A.M\}_{\mathrm{in}}} = \lambda h.h(\lambda h_1.(\lambda y_A.\overline{M}h)h_1).$$

- *Bottom:* Let $M : \bot$. Then

$$\overline{M \cdot_{\mathrm{el}} [\ ;\ ]} = \lambda h.\overline{M}\big(\lambda p.(\lambda g_1.g_1)h\big)$$

- *Top:*

$$\overline{\{\ ;\ \}_{\mathrm{in}}} = \lambda h.h$$

For the optimized rules, the detour conversion is deterministic. This means that there is no parallel term in each of the introduction terms.

The following proposition shows that the translations from Definition 3.6.2 and Definition 3.6.5 commute with the typing relation.

**Proposition 3.6.8.** *If $\Gamma \vdash M : A$ in $\mathrm{IPC}_{\mathcal{C}}$, then $\overline{\Gamma} \vdash \overline{M} : \overline{A}$ in parallel simply typed $\lambda$-calculus, where $\overline{B} \in \overline{\Gamma}$ iff $B \in \Gamma$.*

*Proof.* We proceed by induction on the derivation of $\Gamma \vdash M : A$.

1. (Axiom)

$$\dfrac{\dfrac{x : \overline{A} \qquad [h : \sim A^\circ]}{xh : o}}{\lambda h.xh : \overline{A}}$$

2. (Elimination) Let $\Phi = c(A_1, \dots A_n)$ with $t$ elimination rules. The added dummy redex in a term $\overline{M \cdot [N_k;\ ]}$ does not influence the type, so we can prove without such a redex. We look at the $s$-th elimination rule. Induction hypotheses are $\vdash \overline{M} : \overline{\Phi}$, $\vdash \overline{N_k} : \overline{A_k}$ and $x_l : \overline{A_l} \vdash \overline{O_l} : \overline{D}$ if the elimination contains a case. For every $u \leq t$, let

$$E_u = \overline{A}_{k_1} \to \dots \to \overline{A}_{k_m} \to\ \sim\overline{A}_{l_1} \to \dots \to \sim\overline{A}_{l_{n-m}} \to o,$$

where the $A_k$'s are the formulas where $a_k = 1$ and the $A_l$'s are the formulas where $a_l = 0$ in the truth table $t_c$ of the corresponding elimination rule $r_u$. Then

$$\dfrac{\overline{M} : \overline{\Phi} \qquad \dfrac{\vdots}{\dfrac{\dfrac{[g_s : E_s] \quad \dots \overline{N}_k : \overline{A}_k \dots \qquad \dots \dfrac{\dfrac{x_l : A_l \vdash \overline{O}_l : \overline{D} \qquad [h : D^\circ]}{o}}{\lambda x_l.\overline{O}_l h \ : \ \sim\overline{A}_l} \dots}{\dfrac{o}{E_t \to o}}}{E_1 \to \dots \to E_t \to o}}{\dfrac{o}{\overline{D}}}$$

We conclude that indeed $\lambda h.\overline{M}(\lambda g_1 \dots \lambda g_t.g_s\,\overline{N}_k(\lambda x_l.\overline{O}_l h))$ has type $\overline{D}$.

3. (Introduction) Dummy redexes do not influence the type, so we prove without these redexes. Let $\Phi = c(A_1, \dots A_n)$ with $t$ elimination rules. Induction gives $\vdash \overline{N_j} : \overline{A_j}$ and $x_i : \overline{A_i} \vdash \overline{M_i} : \overline{\Phi}$.

We have to prove $\lambda h.he^h_1 \dots e^h_t : \overline{\Phi}$. But first we prove for every $s \leq t$ that

$$E_s = \overline{A}_{k_1} \to \dots \to \overline{A}_{k_m} \to\ \sim\overline{A}_{l_1} \to \dots \to \sim\overline{A}_{l_{n-m}} \to o,$$

is inhabited by the term $e^h_s$ with $h : \Phi^\circ$. Term $e^h_s$ could be a parallel term $[\dots]$ with elements that belong to one of the following cases. Note that $e^h_s$ always exists, since there is at least one matching case.

- $j' = l'$ for some $j'$ and some $l'$ in $r_s$:

$$\dfrac{\dfrac{[h_{l'} :\ \sim\overline{A}_{l'}] \qquad \overline{N}_{j'} : \overline{A}_{j'}}{o}}{\overline{\lambda h_k}\,.\,\overline{\lambda h_l}\,.h_{l'}\overline{N}_{j'} : \overline{A}_{k_1} \to \dots \to \overline{A}_{k_m} \to\ \sim\overline{A}_{l_1} \to \dots \to \sim\overline{A}_{l_{n-m}} \to o}$$

  So we conclude that such an element in $e^h_s$ has type $E_s$.

- $i' = k'$ for some $i'$ and some $k'$ in $r_s$:

$$\dfrac{\dfrac{\dfrac{\dfrac{y_{i'} : A_{i'} \vdash \overline{M}_{i'} : \overline{\Phi} \qquad h : \Phi^\circ}{\overline{M}_{i'}h : o}}{\lambda y_{i'}.\overline{M}_{i'}h :\ \sim\overline{A}_{i'}} \qquad [h_{k'} : \overline{A}_{k'}]}{o}}{\overline{\lambda h_k}\,.\,\overline{\lambda h_l}\,.(\lambda y_{i'}.\overline{M}_{i'}h)h_{k'} : \overline{A}_{k_1} \to \dots \to \overline{A}_{k_m} \to\ \sim\overline{A}_{l_1} \to \dots \to \sim\overline{A}_{l_{n-m}} \to o}$$

  We conclude that in this matching case also the element in $e^h_s$ has type $E_s$.

Each element in parallel term $e^h_s$ has type $E_s$, so $e^h_s$ is well-defined and has type $E_s$. Now we conclude that $\lambda h.he^h_1, \dots, e^h_t$ has type $\overline{\Phi}$, since $h$ has type $\Phi^\circ = E_1 \to \dots \to E_t \to o$.

□

Just as in [6], this translation $\overline{M}$ does not suffices, but we have to modify it to a translation $\overline{\overline{M}}$. Definition 3.6.5 does not suffice, since permutation conversions do not correspond to the right $\beta$-reductions. It would be sufficient if we would have a scheme of the following shape:

$$\begin{array}{ccc} M & \longrightarrow & \overline{M} \\ \downarrow{\scriptstyle P} & & {\scriptstyle \beta}\downarrow{\scriptstyle +} \\ N & \longrightarrow & \overline{N} \end{array}$$

But this is not always the case. Take for instance

$$M = (M' \cdot_{\vee\text{-el}} [\lambda x_A.P_1, \lambda x_B.P_2]) \cdot_{\vee\text{-el}} [\lambda y_A.Q_1, \lambda y_B.Q_2],$$

which is a permutation convertibility with two times the $\vee$-el rule. This term permutation reduces to

$$N = M' \cdot_{\vee\text{-el}} \Big[\lambda x_A.(P_1 \cdot_{\vee\text{-el}} [\lambda y_A.Q_1, \lambda y_B.Q_2]), \lambda x_B.(P_2 \cdot [\lambda y_A.Q_1, \lambda y_B.Q_2])\Big].$$

The term translations of both, with

$$L = \lambda g_1.g_1(\lambda y_A.\overline{Q_1}h))(\lambda y_B.\overline{Q_2}h)) \text{ and } L' = \lambda g_1.g_1(\lambda y_A.\overline{Q_1}h'))(\lambda y_B.\overline{Q_2}h'))$$

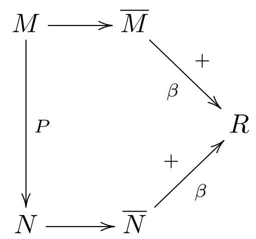
are

$$\begin{aligned} \overline{M} &= \lambda h.\Big(\lambda h'.\overline{M}(\lambda g_1.g_1(\lambda x_A.\overline{P_1}h')(\lambda x_B.\overline{P_2}h'))\Big)L \\ \overline{N} &= \lambda h.\overline{M}\Big(\lambda g_1.g_1\big(\lambda x_A.((\lambda h'.\overline{P_1}L')h)\big(\lambda x_B.((\lambda h'.\overline{P_2}L')h)\Big). \end{aligned}$$

We do not have the situation that $\overline{M} \xrightarrow{+}_\beta \overline{N}$, but both do reduce to a third term $R$, where

$$R = \lambda h.\overline{M}\Big(\lambda g_1.g_1(\lambda x_A.\overline{P_1}L)(\lambda x_B.\overline{P_2}L)\Big).$$

This means that, instead, we have the following diagram:




This is not a commuting diagram, therefore, we make a modified translation such that $M$ translates to $R$ and $N$ translates to $R$. This translation does not avoid all such diagrams, but later we will show that for positive permutation conversions we have indeed that if $M \longrightarrow_{\mathrm{P}} N$, then $\overline{\overline{M}} \equiv R \equiv \overline{\overline{N}}$ in the modified translation. In short, following De Groote [6], the modified translation performs certain $\beta$-reductions in order to circumvent the wrong diagram for positive permutation conversions.

**Definition 3.6.9** (Modified term translation $\overline{\overline{M}}$)**.** For every term $M$ in $\mathrm{IPC}_{\mathcal{C}}$, we define term $\overline{\overline{M}}$ in parallel simply typed $\lambda$-calculus by

$$\overline{\overline{M}} = \lambda h.(M : h),$$

where $h$ is a fresh variable and where the operator : is defined as follows (do not confuse it with the typing relation).

1. (Axiom)
$$x : H := xH$$
2. (Elimination) For connective $c$, let $r_1, \ldots, r_t$ be its elimination rules. We distinguish between elimination terms with case or without any case.
$$M \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}] : H := M : \big(\lambda g_1 \ldots \lambda g_t.g_s\, \overline{\overline{N_k}}(\overline{\lambda x_l.(O_l : H)})\big) \text{ if there is a case } \lambda x_l.O_l$$
$$M \cdot_{r_s} [\overline{N_k};\ ] : H := M : \big(\lambda p.(\lambda g_1, \ldots, \lambda g_t.g_s\, \overline{\overline{N_k}})H\big)$$
3. (Introduction) For connective $c$, let $r_1, \ldots, r_t$ be its elimination rules.
$$\{\overline{N_j}; \overline{\lambda y_i.M_i}\} : H := \big((\overline{\lambda q_j}\,.\,\overline{\lambda q_i}\,.H)\,\overline{\overline{N_j}}\,(\overline{\lambda y_i.(M_i : H)})\,\big)\overline{\overline{e_1^H}} \ldots \overline{\overline{e_t^H}},$$
where $\overline{\lambda q_j}$ should be understood as a sequences of dummy lambda abstractions corresponding to $N_j$'s and $\overline{\lambda q_i}$ correspond to $\lambda y_i.(M_i : H)$'s. Term $\overline{\overline{e_s^H}}$ is the possibly parallel term $[\ldots]$ defined as follows:

   If $E_s = \overline{A}_{k_1} \to \cdots \to \overline{A}_{k_m} \to \ {\sim}\overline{A}_{l_1} \to \cdots \to {\sim}\overline{A}_{l_{n-m}} \to o$ is the elimination pattern for $r_s$, then $\overline{\overline{e_s^H}}$ contains

   - $\overline{\lambda h_k}\,.\,\overline{\lambda h_l}\,.h_{l'}\overline{\overline{N}}_{j'}$ for all $j'$ and $l'$ in $r_s$ such that $j' = l'$,
   - and $\overline{\lambda h_k}\,.\,\overline{\lambda h_l}\,.(\lambda y_{i'}.(M_{i'} : H))h_{k'}$ for all $i'$ and $k'$ in $r_s$ such that $i' = k'$.

   Here, $\overline{\lambda h_k}$ quantifies over all $\overline{A_k}$ in $E_s$ and $\overline{\lambda h_l}$ over all ${\sim}\overline{A_l}$.

Up to this point, we have defined the type translation in Definition 3.6.2 and the right modified term translation in Definition 3.6.9. We also showed that the type translation and the first term translation commute with the typing relation. Now we have to show that this is also the case for the modified term translation. This can be shown using the following lemma.

**Lemma 3.6.10.** *Let $M$ be a term in* $\mathrm{IPC}_{\mathcal{C}}$ *and let $H$ be a* $\mathrm{p}\lambda^{\to}$*-term, then:*

1. $\overline{M} \twoheadrightarrow_\beta \overline{\overline{M}}$,
2. $\overline{M}H \twoheadrightarrow_\beta M : H$.

*Proof.* These statements are proved simultaneously by induction on the structure of $M$. Here, we only look at the introduction case $\{\vec{N}_j; \lambda\vec{y}_i.\vec{M}_i\}$. The other cases are proved in a similar way. Let $c$ be a connective with introduction rule $r_{\text{in}}$ and elimination rules $r_1, \ldots, r_t$. In this proof we use the following induction hypotheses for all $j$ and $i$:

$$\overline{N_j} \twoheadrightarrow_\beta \overline{\overline{N_j}} \text{ and } \overline{M_i}H \twoheadrightarrow_\beta M_i : H.$$

First we look at the $\mathrm{p}\lambda^{\to}$-terms $e^h_s = [\ldots]$ and $\overline{\overline{e^h_s}} = [\ldots]$ in the definitions of the normal term translation (Definition 3.6.5) and the modified translation (Definition 3.6.9) for an introduction term. If $\lambda\vec{h}_k \,.\, \lambda\vec{h}_l\,.h_{l'}\overline{N}_{j'}$ is in the parallel term $e^h_s$, then

$$\lambda\vec{h}_k \,.\, \lambda\vec{h}_l\,.h_{l'}\overline{N}_{j'} \twoheadrightarrow_\beta \lambda\vec{h}_k \,.\, \lambda\vec{h}_l\,.h_{l'}\overline{\overline{N}}_{j'}$$

is in the parallel term $\overline{\overline{e^h_s}}$. And if $\lambda\vec{h}_k \,.\, \lambda\vec{h}_l\,.(\lambda y_{i'}.\overline{M_{i'}}h)h_{k'}$ is included in $e^h_s$, then

$$\lambda\vec{h}_k \,.\, \lambda\vec{h}_l\,.(\lambda y_{i'}.\overline{M_{i'}}h)h_{k'} \twoheadrightarrow_\beta \lambda\vec{h}_k \,.\, \lambda\vec{h}_l\,.(\lambda y_{i'}.(M_{i'} : h))h_{k'}$$

is in the parallel term $\overline{\overline{e^h_s}}$. This means that $e^h_s \twoheadrightarrow_\beta \overline{\overline{e^h_s}}$ for all $s$. Now we can conclude that

$$\begin{aligned}\overline{\{\vec{N}_j; \lambda\vec{y}_i.\vec{M}_i\}} &= \lambda h.\big((\lambda\vec{q}_j \,.\, \lambda\vec{q}_i\,.h)\,\vec{\overline{N}}_j\,\overrightarrow{(\lambda y_i.\overline{M}_i h)}\big)e^h_1 \ldots e^h_t \\ &\twoheadrightarrow_\beta \lambda h.\big((\lambda\vec{q}_j \,.\, \lambda\vec{q}_i\,.h)\,\vec{\overline{\overline{N}}}_j\,\overrightarrow{(\lambda y_i.(M_i : h))}\big)\overline{\overline{e^h_1}} \ldots \overline{\overline{e^h_t}} = \overline{\overline{\{\vec{N}_j; \lambda\vec{y}_i.\vec{M}_i\}}}\end{aligned}$$

and

$$\overline{\{\vec{N}_j; \lambda\vec{y}_i.\vec{M}_i\}}H \twoheadrightarrow_\beta \{\vec{N}_j; \lambda\vec{y}_i.\vec{M}_i\} : H.$$

□

**Proposition 3.6.11.** *If $\Gamma \vdash M : A$ in* $IPC_{\mathcal{C}}$*, then $\overline{\Gamma} \vdash \overline{\overline{M}} : \overline{A}$ in parallel simply typed $\lambda$-calculus, where $\overline{B} \in \overline{\Gamma}$ iff $B \in \Gamma$.*

*Proof.* This follows from Proposition 3.6.8, Lemma 3.6.10, and the subject reduction property of the parallel simply typed $\lambda$-calculus (Proposition 3.4.12). □

Now we see that positive permutation reductions correspond to syntactic equality.

**Proposition 3.6.12.** *Let $M$ and $N$ be terms in* $\mathrm{IPC}_{\mathcal{C}}$ *such that $M \longrightarrow_{\mathrm{Ppos}} N$. Then*

1. *$M : H = N : H$, for any parallel simple term $H$,*
2. *$\overline{\overline{M}} = \overline{\overline{N}}$.*

*Proof.* Statement (2.) is a direct consequence of (1.). For (1.), we proceed by induction on the generation of $M \longrightarrow_{\mathrm{Ppos}} N$. We only treat the base case, since the induction steps are easily verified.

We consider the permutation convertibility

$$(M \cdot [\vec{N}; \lambda\vec{x}.\vec{O}]) \cdot_{r_s} [\vec{P}; \lambda\vec{y}.\vec{Q}] \longrightarrow_{\mathrm{Ppos}} M \cdot [\vec{N}; \overrightarrow{\lambda x.(O \cdot_{r_s} [\vec{P}; \lambda\vec{y}.\vec{Q}])}],$$

where there is at least one case of the form $\lambda x.O$. Write $L = \big(\lambda g_1 \dots \lambda g_t.g_s\, \overline{\overline{\overline{P}}}_k(\overline{\lambda x.(Q_l : H)})\big)$, then

$$
\begin{aligned}
&(M\cdot[\overline{N};\overline{\lambda x.O}])\cdot_{r_s}[\overline{P};\overline{\lambda y.Q}] : H\\
&\quad = (M\cdot[\overline{N};\overline{\lambda x.O}]) : L\\
&\quad = M : \big(\lambda g_1 \dots \lambda g_t.g_s\, \overline{\overline{\overline{N}}}(\overline{\lambda x.(O : L)})\big)\\
&\quad = M : \big(\lambda g_1 \dots \lambda g_t.g_s\, \overline{\overline{\overline{N}}}(\overline{\lambda x.((O\cdot_{r_s}[\overline{P};\overline{\lambda y.Q}]) : H)})\big)\\
&\quad = (M\cdot[\overline{N};\overline{\lambda x.(O\cdot_{r_s}[\overline{P};\overline{\lambda y.Q}])}]) : H
\end{aligned}
$$

□

The following lemmas are useful to prove that detour conversion steps in the truth table system correspond to $\beta$-reduction in $\mathrm{p}\lambda^{\to}$ which we establish in Proposition 3.6.17.

**Lemma 3.6.13.** *Let $M$ and $P$ be terms in $\mathrm{IPC}_{\mathcal{C}}$ and $H$ be a $\mathrm{p}\lambda^{\to}$-term in which there is no free occurrence of $z$, then:*

1. $(M : H)[z := \overline{\overline{P}}] \twoheadrightarrow_\beta (M[z := P]) : H$,
2. $\overline{\overline{M}}[z := \overline{\overline{P}}] \twoheadrightarrow_\beta \overline{\overline{M[z := P]}}$.

*Proof.* Property (2.) is a direct consequence of (1.). For (1.) we proceed by induction on the structure of $M$. We only look at the elimination case $R\cdot_{r_s}[\overline{N_k};\overline{\lambda x_l.O_l}]$ for an elimination term which contains a case $\lambda x_l.O_l$. The other cases can be proved in a similar way. For simplicity, we assume that $c$ has one elimination rule. In general, for arbitrary $\mathrm{p}\lambda^{\to}$-term $H$ and every $M$ we have

$$(M : H)[z := \overline{\overline{P}}] = (M : H[z := \overline{\overline{P}}])[z := \overline{\overline{P}}]$$

assuming that $z$ does not occur in $\overline{\overline{P}}$ (which can be reached by renaming variables). When applying this to the elimination term we get

$$(R\cdot_{r_s}[\overline{N_k};\overline{\lambda x.O_l}] : H)[z := \overline{\overline{P}}] = \Big(R : \big(\lambda g_1.g_1\, \overline{\overline{\overline{N_k}}[z := \overline{\overline{P}}]}\ \ \overline{(\lambda x.(O_l : H)[z := \overline{\overline{P}}])}\big)\Big)[z := \overline{\overline{P}}].$$

Now it is possible to apply the induction hypothesis to $R$, because $z$ does not occur as a free variable in the term after $R$ anymore. Before we do the induction for $R$, we look at $N_k$ and $O_l$. When applying the induction hypothesis to $\overline{\overline{N_k}}[z := \overline{\overline{P}}]$ we get

$$\overline{\overline{N_k}}[z := \overline{\overline{P}}] \ = \ (\lambda h.(N_k : h)[z := \overline{\overline{P}}]) \twoheadrightarrow_\beta (\lambda h.(N_k[z := P]) : h) = \overline{\overline{N_k[z := P]}}.$$

When applying the induction hypothesis to $\lambda x.(O_l : H)[z := \overline{\overline{P}}]$ we get

$$\lambda x.(O_l : H)[z := \overline{\overline{P}}] \twoheadrightarrow_\beta \lambda x.((O_l[z := P]) : H).$$

Now we derive

$$
\begin{aligned}
(R\cdot_{r_s}[\overline{N_k};\overline{\lambda x.O_l}] : H)[z := \overline{\overline{P}}] &= \Big(M : \big(\lambda g_1.g_1\, \overline{\overline{\overline{N_k}}[z := \overline{\overline{P}}]}\ \ \overline{(\lambda x.(O_l : H)[z := \overline{\overline{P}}])}\big)\Big)[z := \overline{\overline{P}}]\\
&\twoheadrightarrow_\beta \Big(R : \big(\lambda g_1.g_1\, \overline{\overline{\overline{N_k[z := P]}}}\ \ \overline{(\lambda x.((O_l[z := P]) : H))}\big)\Big)[z := \overline{\overline{P}}]\\
&\twoheadrightarrow_\beta R[z := P] : \big(\lambda g_1.g_1\, \overline{\overline{\overline{N_k[z := P]}}}\ \ \overline{(\lambda x.((O_l[z := P]) : H))}\big)\\
&= \big(R[z := P]\cdot_{r_s}[\overline{N_k[z := P]};\overline{\lambda x.O_l[z := P]}]\big) : H\\
&= \big(R\cdot_{r_s}[\overline{N_k};\overline{\lambda x.O_l}]\big)[z := P] : H
\end{aligned}
$$

□

Next lemma is based on Lemma 14 of De Groote [6].

**Lemma 3.6.14.** *Let $H$ and $L$ be* p$\lambda^{\rightarrow}$*-terms such that $H \xrightarrow{+}_\beta L$. Then, for any term $M$ in* $\mathrm{IPC}_\mathcal{C}$*, we have $M : H \xrightarrow{+}_\beta M : L$.*

*Proof.* By induction on $M$. Here we show it for an elimination term $R \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}]$ containing a case $\lambda x_l.O_l$. Suppose $H \xrightarrow{+}_\beta L$. By induction hypothesis we have that $O_l : H \xrightarrow{+}_\beta O_l : L$, so $\big(\lambda g_1 \ldots \lambda g_t.g_s\overline{\overline{N_k}}(\lambda x.(O_l : H))\big) \xrightarrow{+}_\beta \big(\lambda g_1 \ldots \lambda g_t.g_s\overline{\overline{N_k}}(\lambda x.(O_l : L))\big)$. Now, by induction hypothesis,

$$\begin{aligned} R \cdot_{r_s} [\overline{N_k}; \overline{\lambda x.O_l}] : H &:= R : \big(\lambda g_1 \ldots \lambda g_t.g_s\, \overline{\overline{\overline{N}}}_k\, \overline{(\lambda x.(O_l : H))}\,\big) \\ &\xrightarrow{+}_\beta R : \big(\lambda g_1 \ldots \lambda g_t.g_s\, \overline{\overline{\overline{N}}}_k\, \overline{(\lambda x.(O_l : L))}\,\big) \\ &= R \cdot_{r_s} [\overline{N_k}; \overline{\lambda x.O_l}] : L. \end{aligned}$$

Other cases of $M$ are proved in a similar way. It is important to mention that $H$ is always present in any modified translation of Definition 3.6.9, which makes sure that we always have a $\xrightarrow{+}_\beta$ step and we never get $\beta$-equivalence. □

Recall the definition of parallel subterm of Definition 3.4.16, $K \sqsubseteq L$. Since we are working in the parallel simply typed $\lambda$-calculus, we need the following lemma.

**Lemma 3.6.15.** *Let $H$ and $L$ be* p$\lambda^{\rightarrow}$*-terms, such that $H \sqsubseteq L$. Then, for any term $M$ in* $\mathrm{IPC}_\mathcal{C}$*, we have $(M : H) \sqsubseteq (M : L)$.*

*Proof.* This is proved by induction on the structure of $M$. The axiom rule is easily verified.

For an elimination term with at least one case $\lambda x_l.O_l$ we have

$$(M \cdot_{r_s} [\overline{N_k}; \overline{\lambda x.O_l}]) : H = M : \big(\lambda g_1 \ldots \lambda g_t.g_s\, \overline{\overline{\overline{N}}}_k\, \overline{(\lambda x.(O_l : H))}\,\big).$$

By induction hypothesis we have $O_l : H \sqsubseteq O_l : L$ for each $l$. By definition and transitivity of $\sqsubseteq$, we see that

$$\lambda g_1 \ldots \lambda g_t.g_s\, \overline{\overline{\overline{N}}}_k\, \overline{(\lambda x.(O_l : H))} \sqsubseteq \lambda g_1 \ldots \lambda g_t.g_s\, \overline{\overline{\overline{N}}}_k\, \overline{(\lambda x.(O_l : L))}\,.$$

Now we can apply the induction hypothesis to $M$ to conclude

$$M : \big(\lambda g_1 \ldots \lambda g_t.g_s\, \overline{\overline{\overline{N}}}_k\, \overline{(\lambda x.(O_l : H))}\,\big) \sqsubseteq M : \big(\lambda g_1 \ldots \lambda g_t.g_s\, \overline{\overline{\overline{N}}}_k\, \overline{(\lambda x.(O_l : L))}\,\big).$$

Note that the lemma also holds for elimination rules without a case.

Now we consider a introduction term. From definition we have

$$\{\overline{N_j}; \overline{\lambda y_i.M_i}\} : H = \big((\overline{\lambda q_j}\,.\,\overline{\lambda q_i}\,.H)\, \overline{\overline{\overline{N_j}}}\, \overline{(\lambda y_i.(M_i : H))}\,\big)\overline{\overline{e_1^H}} \ldots \overline{\overline{e_t^H}},$$

where parallel term $\overline{\overline{e_s^H}}$ contains components of the form

$$\overline{\lambda h_k}\,.\,\overline{\lambda h_l}\,.h_{l'}\overline{\overline{N}}_{j'} \quad \text{or} \quad \overline{\lambda h_k}\,.\,\overline{\lambda h_l}\,.(\lambda y_{i'}.(M_{i'} : H))h_{k'}.$$

With the induction hypothesis we obtain for $\overline{\overline{e_s^H}}$ that $(M_{i'} : H) \sqsubseteq (M_{i'} : L)$ for each $i'$. This means that

$$\overline{\lambda h_k}\,.\,\overline{\lambda h_l}\,.(\lambda y_{i'}.(M_{i'} : H))h_{k'} \sqsubseteq \overline{\lambda h_k}\,.\,\overline{\lambda h_l}\,.(\lambda y_{i'}.(M_{i'} : L))h_{k'}$$

for each $i'$. From the definition of $\sqsubseteq$ we have $e_s^H \sqsubseteq e_s^L$ for each $s$. Also $H \sqsubseteq L$ so

$$\big((\overline{\lambda q_j}\,.\,\overline{\lambda q_i}\,.H)\, \overline{\overline{\overline{N_j}}}\, \overline{(\lambda y_i.(M_i : H))}\,\big)\overline{\overline{e_1^H}} \ldots \overline{\overline{e_t^H}} \sqsubseteq \big((\overline{\lambda q_j}\,.\,\overline{\lambda q_i}\,.L)\, \overline{\overline{\overline{N_j}}}\, \overline{(\lambda y_i.(M_i : L))}\,\big)\overline{\overline{e_1^L}} \ldots \overline{\overline{e_t^L}}.$$

□

**Proposition 3.6.16.** *Let $M$ and $N$ be terms in $\mathrm{IPC}_{\mathcal{C}}$ such that $M \longrightarrow_{\mathrm{D}} N$. Then*

1. *for every $\mathrm{p}\lambda^{\to}$-term $H$ there exists a $\mathrm{p}\lambda^{\to}$-term $K$ such that $(M : H) \xrightarrow{+}_{\beta} K$ and $(N : H) \sqsubseteq K$,*
2. *there exists a $\mathrm{p}\lambda^{\to}$-term $K$ such that $\overline{\overline{M}} \xrightarrow{+}_{\beta} K$ and $\overline{\overline{N}} \sqsubseteq K$.*

*Proof.* We prove (1.) by induction on the generation of $M \longrightarrow_{\mathrm{D}} N$. See Definition 3.5.3 for all possible detour reductions. The induction steps are proven in Appendix B. Here we focus on the base cases of the detour reductions. We adopt the numbering of Definition 3.5.3.

(1) $j' = l'$ :
We consider $\{\overline{N, N_{j'}}; \overline{\lambda y.M}\} \cdot_{r_s} [\overline{P}; \overline{\lambda x.Q, \lambda x_{l'}.Q_{l'}}] \longrightarrow_{\mathrm{D}} Q_{l'}[x_{l'} := N_{j'}]$.
In this case we should have a case in the elimination term of the form $\lambda x_l.Q_l$.
Write $L = \big(\lambda g_1 \dots g_t.g_s \overline{\overline{\overline{P}}}_k(\overline{\lambda x.(Q_l : H)})\big)$, then

$$
\begin{aligned}
\{\overline{N, N_{j'}}; \overline{\lambda y.M}\} \cdot_{r_s} [\overline{P}; \overline{\lambda x.Q, \lambda x_{l'}.Q_{l'}}] : H \\
&= \{\overline{N, N_{j'}}; \overline{\lambda y.M}\} : L \\
&= \big((\overline{\lambda q_j} \,.\, \overline{\lambda q_i}\, L)\, \overline{N_j}\, \overline{\lambda y_i.(M_i : L)}\,\big)\overline{\overline{e_1^L}} \dots \overline{\overline{e_t^L}} \\
&\xrightarrow{+}_{\beta} L\overline{\overline{e_1^L}}, \dots, \overline{\overline{e_t^L}} && \text{(Delete dummy redexes)} \\
&\xrightarrow{+}_{\beta} \overline{\overline{e_s^L}}\,\overline{\overline{\overline{P}}}\,(\overline{\lambda x.(Q : H)}) \\
&= [\dots, (\overline{\lambda h_k} \,.\, \overline{\lambda h_l}\, .h_{l'}\overline{\overline{N}}_{j'}), \dots]\, \overline{\overline{\overline{P}}}\,(\overline{\lambda x.(Q : H)}) && \text{(Definition of } \overline{\overline{e_s^L}}) \\
&\rightarrow_{\beta} [\dots, (\overline{\lambda h_k} \,.\, \overline{\lambda h_l}\, .h_{l'}\overline{\overline{N}}_{j'})\, \overline{\overline{\overline{P}}}\,(\overline{\lambda x.(Q : H)}), \dots] \\
&\xrightarrow{+}_{\beta} [\dots, (\lambda x_{l'}.(Q_{l'} : H))\overline{\overline{N}}_{j'}, \dots] \\
&\xrightarrow{+}_{\beta} [\dots, (Q_{l'} : H)[x_{l'} := \overline{\overline{N}}_{j'}], \dots] \\
&\twoheadrightarrow_{\beta} [\dots, (Q_{l'}[x_{l'} := N_{j'}]) : H, \dots] && \text{(Lemma 3.6.13)}
\end{aligned}
$$

Define $K = [\dots, (Q_{l'}[x_{l'} := N_{j'}]) : H, \dots]$, then we can conclude that $(Q_l[x_{l'} := N_{j'}] : H) \sqsubseteq K$.

(2) $i' = k'$ :
We consider $\{\overline{N}; \overline{\lambda y.M, \lambda y_i.M_i}\} \cdot_{r_s} [\overline{P, P_{k'}}; \overline{\lambda x.Q}] \longrightarrow_{\mathrm{D}} M_{i'}[y_{i'} := P_{k'}] \cdot [\overline{P, P_{k'}}; \overline{\lambda x.Q}]$.
Now there are two possibilities for the elimination term. There is a case $\lambda x_l.Q_l$ or there is not. First suppose we have such a case.
Write $L = \big(\lambda g_1 \dots g_t.g_s \overline{\overline{\overline{P}}}_k(\overline{\lambda x.(Q_l : H)})\big)$, then

$$
\begin{aligned}
\{\overline{N}; \overline{\lambda y.M, \lambda y_{i'}.M_{i'}}\} \cdot_{r_s} [\overline{P, P_{k'}}; \overline{\lambda x.Q}] : H \\
&= \{\overline{N}; \overline{\lambda y.M, \lambda y_{i'}.M_{i'}}\} : L \\
&= \big((\overline{\lambda q_j} \,.\, \overline{\lambda q_i}\, L)\, \overline{N_j}\, \overline{\lambda y_i.(M_i : L)}\,\big)\overline{\overline{e_1^L}} \dots \overline{\overline{e_t^L}} \\
&\xrightarrow{+}_{\beta} L\overline{\overline{e_1^L}}, \dots, \overline{\overline{e_t^L}} && \text{(Delete dummy redexes)} \\
&\xrightarrow{+}_{\beta} \overline{\overline{e_s^L}}\,\overline{\overline{\overline{P}}}\,(\overline{\lambda x.(Q : H)}) \\
&= [\dots, (\overline{\lambda h_k} \,.\, \overline{\lambda h_l}\, .(\lambda y_{i'}.(M_{i'} : L))h_{k'}), \dots]\, \overline{\overline{\overline{P}}}\,(\overline{\lambda x.(Q : H)}) \\
&\rightarrow_{\beta} [\dots, (\overline{\lambda h_k} \,.\, \overline{\lambda h_l}\, .(\lambda y_{i'}.(M_{i'} : L))h_{k'})\, \overline{\overline{\overline{P}}}\,(\overline{\lambda x.(Q : H)}), \dots] \\
&\xrightarrow{+}_{\beta} [\dots, (\lambda y_{i'}.(M_{i'} : L))\overline{\overline{P}}_{k'}, \dots] \\
&\xrightarrow{+}_{\beta} [\dots, (M_{i'} : L)[y_{i'} := \overline{\overline{P}}_{k'}], \dots] \\
&\twoheadrightarrow_{\beta} [\dots, (M_{i'}[y_{i'} := P_{k'}]) : L, \dots] && \text{(Lemma 3.6.13)} \\
&= [\dots, M_{i'}[y_{i'} := P_{k'}] \cdot [\overline{P, P_{k'}}; \overline{\lambda x.Q}] : H, \dots]
\end{aligned}
$$

Define $K = [\ldots, M_{i'}[y_{i'} := P_{k'}] \cdot [\overline{P, P_{k'}}; \overline{\lambda x.Q}] : H, \ldots]$, then we can conclude that indeed $(M_{i'}[y_{i'} := P_{k'}] \cdot [\overline{P, P_{k'}}; \overline{\lambda x.Q}] : H) \sqsubseteq K$.

Now suppose that we do not have a case $\lambda x_l.O_l$. Write $L = \big(\lambda p.(\lambda g_1 \ldots g_t.g_s \overline{\overline{P}})H\big)$, then

$$
\begin{aligned}
\{\overline{N}; \overline{\lambda y.M, \lambda y_{i'}.M_{i'}}\} \cdot_{r_s} [\overline{P, P_{k'}};\ ] : H & \\
&= \{\overline{N}; \overline{\lambda y.M, \lambda y_{i'}.M_{i'}}\} : L \\
&= \big((\overline{\lambda q_j} \,.\, \overline{\lambda q_i}\, L)\, \overline{N_j}\, \overline{\lambda y_i.(M_i : L)}\big)\overline{\overline{e_1^L}} \ldots \overline{\overline{e_t^L}} \\
&\xrightarrow{+}_\beta L\overline{\overline{e_1^L}}, \ldots, \overline{\overline{e_t^L}} \qquad \text{(Delete dummy redexes)} \\
&= \big(\lambda p.(\lambda g_1 \ldots g_t.g_s \overline{\overline{P}})H\big)\overline{\overline{e_1^L}}, \ldots, \overline{\overline{e_t^L}} \\
&\twoheadrightarrow_\beta (\lambda g_1 \ldots g_t.g_s \overline{\overline{P}})\overline{\overline{e_1^L}}, \ldots, \overline{\overline{e_t^L}} \qquad \text{(Delete dummy redex)} \\
&\xrightarrow{+}_\beta \overline{\overline{e_s^L}}\, \overline{\overline{P}}\, \overline{(\lambda x.(Q : H))} \\
&= [\ldots, (\overline{\lambda h_k} \,.\, \overline{\lambda h_l} \,.(\lambda y_{i'}.(M_{i'} : L))h_{k'}), \ldots]\, \overline{\overline{P}}\, \overline{(\lambda x.(Q : H))} \\
&\rightarrow_\beta [\ldots, (\overline{\lambda h_k} \,.\, \overline{\lambda h_l} \,.(\lambda y_{i'}.(M_{i'} : L))h_{k'})\, \overline{\overline{P}}\, \overline{(\lambda x.(Q : H))}, \ldots] \\
&\xrightarrow{+}_\beta [\ldots, (\lambda y_{i'}.(M_{i'} : L))\overline{\overline{P}}_{k'}, \ldots] \\
&\xrightarrow{+}_\beta [\ldots, (M_{i'} : L)[y_{i'} := \overline{\overline{P}}_{k'}], \ldots] \\
&\twoheadrightarrow_\beta [\ldots, (M_{i'}[y_{i'} := P_{k'}]) : L, \ldots] \qquad \text{(Lemma 3.6.13)} \\
&= [\ldots, M_{i'}[y_{i'} := P_{k'}] \cdot [\overline{P, P_{k'}};\ ] : H, \ldots]
\end{aligned}
$$

Define $K = [\ldots, M_{i'}[y_{i'} := P_{k'}] \cdot [\overline{P, P_{k'}};\ ] : H, \ldots]$, then we can conclude that indeed $(M_{i'}[y_{i'} := P_{k'}] \cdot [\overline{P, P_{k'}};\ ] : H) \sqsubseteq K$.

This completes the proof of (1.).

For the proof of (2.), we need (1.). $\overline{\overline{M}} = \lambda h.(M : h)$. By (1.) we know that there exists a $K'$, such that $(M : h) \xrightarrow{+}_\beta K'$ and $(N : h) \sqsubseteq K'$. Define $K = \lambda h.K'$. Then $\overline{\overline{M}} \xrightarrow{+}_\beta K$ and $\overline{\overline{N}} = \lambda h.(N : h) \sqsubseteq K$. ☐

**Proposition 3.6.17.** *Let $M$ and $N$ be terms in $\mathrm{IPC}_\mathcal{C}$ such that $M \longrightarrow_\mathrm{D} N$ and $\overline{\overline{M}} \sqsubseteq K$ for a $\mathrm{p}\lambda^\rightarrow$-term $K$. Then there exists a $K'$ such that $K \xrightarrow{+}_\beta K'$ and $\overline{\overline{N}} \sqsubseteq K'$. This statement is shown in the following diagram.*

$$
\begin{array}{ccccc}
M & \longrightarrow & \overline{\overline{M}} \sqsubseteq & K \\
\big\downarrow \mathrm{D} & & & \beta \big\downarrow + \\
N & \longrightarrow & \overline{\overline{N}} \sqsubseteq & \exists K'
\end{array}
$$

*Proof.* From Proposition 3.6.16, we know that there exists a term $L$ such that $\overline{\overline{M}} \xrightarrow{+}_\beta L$ and $\overline{\overline{N}} \sqsubseteq L$. By Lemma 3.4.19 we know that we can reduce $K$ to some term $K'$ such that $L \sqsubseteq K'$, because $\overline{\overline{M}} \sqsubseteq K$. Since relation $\sqsubseteq$ is transitive we conclude $\overline{\overline{N}} \sqsubseteq K'$. ☐

Now we look at an example on how the parallel terms are used in a detour reduction.

**Example 3.6.18.** Consider the following detour convertibility with non-optimized rules of conjunction with corresponding proof terms.

$$\dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ \Gamma \vdash N_A : A\end{matrix} \qquad \begin{matrix}\Sigma_2 \\ \Gamma \vdash N_B : B\end{matrix}}{\Gamma \vdash \{N_A, N_B\} : A \wedge B}\wedge\text{-in} \qquad \begin{matrix}\Pi_1 \\ \Gamma, x_A : A \vdash O_1 : D\end{matrix} \qquad \begin{matrix}\Pi_2 \\ \Gamma, x_B : B \vdash O_2 : D\end{matrix}}{\Gamma \vdash \{N_A, N_B\} \cdot [\lambda x_A.O_1, \lambda x_B.O_1] : D}\wedge\text{-el}$$

There are two possibilities to reduce this derivation.

(1) $\{N_A, N_B\} \cdot [\lambda x_A.O_1, \lambda x_B.O_1] \longrightarrow_{\mathrm{D}} O_1[x_A := N_A]$
(2) $\{N_A, N_B\} \cdot [\lambda x_A.O_1, \lambda x_B.O_1] \longrightarrow_{\mathrm{D}} O_2[x_B := N_B]$

The term translation is as follows. Note that $\wedge$ has three elimination rules

$$\overline{\overline{\{N_A, N_B\} \cdot [\lambda x_A.O_1, \lambda x_B.O_1]}} = \lambda h.\Big( (\lambda g_1, g_2, g_3.g_1(\lambda x_A.(O_1 : h))(\lambda x_B.(O_2 : h)) \; \overline{\overline{e_1}} \; \overline{\overline{e_2}} \; \overline{\overline{e_3}} \Big)$$

with

$$\begin{aligned}\overline{\overline{e_1}} &= \big[\lambda h_1, h_2.h_1\overline{\overline{N}}_A, \lambda h_1, h_2.h_2\overline{\overline{N}}_B\big] \\ \overline{\overline{e_2}} &= \lambda h_2, h_1.h_1\overline{\overline{N}}_A \\ \overline{\overline{e_3}} &= \lambda h_1, h_2.h_2\overline{\overline{N}}_B\end{aligned}$$

The translated term $\beta$-reduces to

$$K = \lambda h.\big[(O_1[x_A := N_A]) : h \;,\; (O_2[x_B := N_B]) : h\big].$$

If we chose to do the first detour reduction, then indeed

$$\overline{\overline{O_1[x_A := N_A]}} = \lambda h.(O_1[x_A := N_A]) : h) \sqsubseteq K.$$

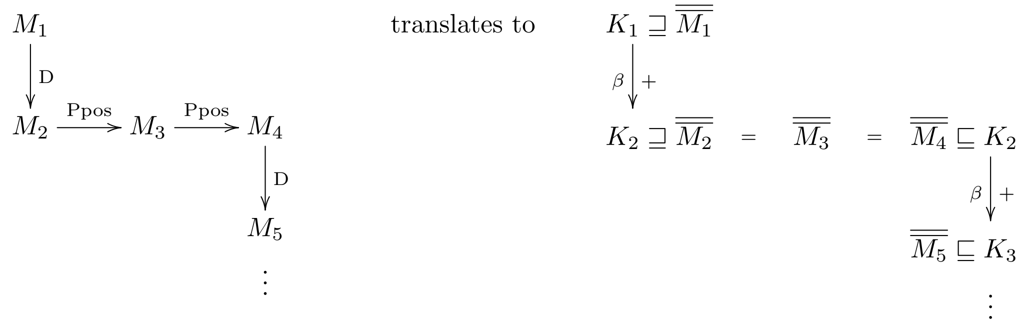
When we would have picked the second possibility, then

$$\overline{\overline{O_2[x_B := N_B]}} = \lambda h.(O_2[x_B := N_B]) : h) \sqsubseteq K.$$

Up to this point, we have shown that detour conversion corresponds to one or more $\beta$-reduction steps in a parallel term. This means that an infinite detour reduction in the truth tables system leads to an infinite $\beta$-reduction sequence in $\mathrm{p}\lambda^{\rightarrow}$, which is not possible. We also have seen that positive permutation conversion corresponds to syntactic equality. Now we can prove the following important theorem, which proves strong normalization with regard to $\{ \longrightarrow_{\mathrm{D}} , \longrightarrow_{\mathrm{Ppos}} \}$. This is the generalization of Theorem 18 of De Groote [6].

**Theorem 3.6.19.** *For any set of connectives $\mathcal{C}$, $\mathrm{IPC}_{\mathcal{C}}$ is strongly normalizing with regard to detour and positive permutation conversions $\{ \longrightarrow_{\mathrm{D}} , \longrightarrow_{\mathrm{Ppos}} \}$.*

*Proof.* Suppose there is an infinite sequence of detour and positive permutation conversions starting from term $M$ in $\mathrm{IPC}_{\mathcal{C}}$. We draw the following picture.



If the sequence contains infinitely many detour steps, then there must be an infinite sequence $K_1, K_2, \ldots$ of $\beta$-reduction steps, by Proposition 3.6.17 and Proposition 3.6.12. But this is in contradiction with the strong normalization property of $\mathrm{p}\lambda^{\rightarrow}$. This means that the sequence may only contain a finite number of detour reductions. But in that case it would contain an infinite sequence of consecutive permutation conversions, which contradicts the fact that our system is strongly normalizing with regard to permutation reductions (which is shown in [5]). □

This is the end of the proof strategy of De Groote in [6]. We have now strong normalization of $\mathrm{IPC}_{\mathcal{C}}$ for $\{\ \longrightarrow_{\mathrm{D}}\ ,\ \longrightarrow_{\mathrm{Ppos}}\ \}$. Mention that if each connective $c \in \mathcal{C}$ has the following row in its truth table, $t_c(1, \ldots, 1) = 1$, then each elimination rule has at least one case. This means that there are no negative permutations. So for these connectives we already have strong normalization for all reductions.

For arbitrary connectives we also have strong normalization for all conversions $\{\ \longrightarrow_{\mathrm{D}}\ ,\ \longrightarrow_{\mathrm{P}}\ \}$ which is the same as $\{\ \longrightarrow_{\mathrm{D}}\ ,\ \longrightarrow_{\mathrm{Ppos}}\ ,\ \longrightarrow_{\mathrm{Pneg}}\ \}$. This follows from Theorem 3.6.19. We need two more lemmas.

**Lemma 3.6.20.** *Let $M_1, M_2, M_3$ be terms in $\mathrm{IPC}_{\mathcal{C}}$ such that $M_1 \ \longrightarrow_{\mathrm{Pneg}}\ M_2 \ \longrightarrow_{\mathrm{D}}\ M_3$. Then there is a term $F$ in $\mathrm{IPC}_{\mathcal{C}}$ such that $M_1 \ \longrightarrow_{\mathrm{D}}\ F \ \twoheadrightarrow_{\mathrm{Pneg}}\ M_3$. This statement can be illustrated by the following diagram.*

$$\begin{array}{ccc} M_1 & \xrightarrow[\mathrm{D}]{} & \exists F \\ \downarrow{\scriptstyle \mathrm{Pneg}} & & \downarrow{\scriptstyle \mathrm{Pneg}} \\ M_2 & \xrightarrow[\mathrm{D}]{} & M_3 \end{array}$$

*Proof.* Proceed by induction on the generation of $M_1 \ \longrightarrow_{\mathrm{Pneg}}\ M_2$. See Appendix B for a detailed proof. □

For positive permutations we want to prove a similar statement. We define the following special positive permutation steps.

**Definition 3.6.21.** We define a relation $\Longrightarrow^n_{\mathrm{Ppos}}$ in the following way. Consider permutation convertibility of the form

$$M = T \cdot [\overline{R}; \overline{\lambda y.S}] \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \cdots \cdot [\overline{U_n}; \overline{\lambda w.V_n}],$$

with at least one case of the form $\lambda y.S$. This convertibility reduces with $n$ positive permutations to

$$N = T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \cdots \cdot [\overline{U_n}; \overline{\lambda w.V_n}])}].$$

Then we say that $M \Longrightarrow^n_{\mathrm{Ppos}} N$.

Note that $\Longrightarrow^1_{\mathrm{Ppos}}$ is the same as $\ \longrightarrow_{\mathrm{Ppos}}\ $.

**Lemma 3.6.22.** *Let $M_1, M_2, M_3$ be terms in $\mathrm{IPC}_{\mathcal{C}}$ such that $M_1 \ \longrightarrow_{\mathrm{Pneg}}\ M_2 \Longrightarrow^n_{\mathrm{Ppos}} M_3$. Then there is a term $F$ in $\mathrm{IPC}_{\mathcal{C}}$ such that $M_1 \Longrightarrow^m_{\mathrm{Ppos}} F \ \twoheadrightarrow_{\mathrm{Pneg}}\ M_3$ with $m = n$ or $m = n + 1$. This statement can be illustrated by the following diagram.*

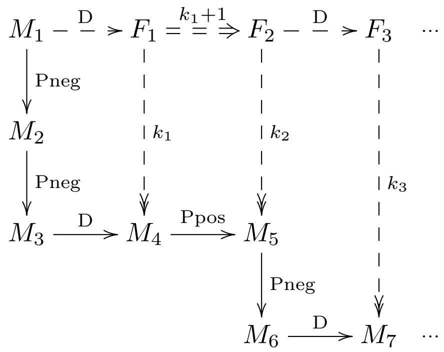
$$\begin{array}{ccc} M_1 & \overset{m}{\underset{\mathrm{Ppos}}{\Longrightarrow}} & \exists F \\ \downarrow{\scriptstyle \mathrm{Pneg}} & & \downarrow{\scriptstyle \mathrm{Pneg}} \\ M_2 & \overset{n}{\underset{\mathrm{Ppos}}{\Longrightarrow}} & M_3 \end{array}$$

*Proof.* Proceed by induction on the generation of $M_1 \longrightarrow_{\text{Pneg}} M_2$. See Appendix B for a detailed proof. $\square$

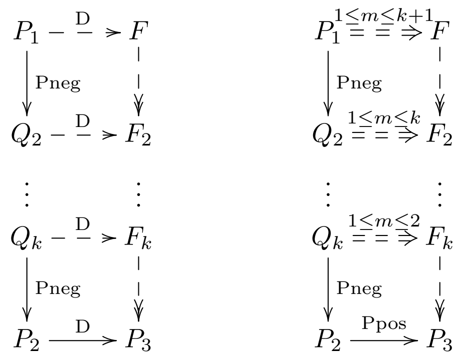
Finally, we conclude with our main theorem.

**Theorem 3.6.23** (Strong normalization)**.** *For any set of connectives $\mathcal{C}$, $\text{IPC}_{\mathcal{C}}$ is strongly normalizing with regard to detour and permutation conversions $\{ \longrightarrow_{\text{D}} , \longrightarrow_{\text{P}} \}$.*

*Proof.* Suppose there is an infinite sequence of detour and permutation conversions starting from term $M$ in $\text{IPC}_{\mathcal{C}}$. Distinguish between negative and positive permutation conversions. There cannot be an infinite sequence of consecutive negative permutation conversions, because permutation conversion is strongly normalizing. So we have a following diagram where the vertical direction indicates negative permutation conversions and the horizontal direction the other conversions, where each vertical part consists of finitely many reductions. We show that the dashed arrows exist such that we get an infinite sequence on top of the figure of only detour and positive permutation conversions. This, then, contradicts Theorem 3.6.19.




From Lemma 3.6.20 we can immediately conclude that if $P_1 \longrightarrow^k_{\text{Pneg}} P_2$ in $k$ steps and $P_2 \longrightarrow_{\text{D}} P_3$, then there exists a $F$ such that $P_1 \longrightarrow_{\text{D}} F \twoheadrightarrow_{\text{Pneg}} P_3$. For positive permutations we have a similar claim. From Lemma 3.6.22 we can conclude that if $P_1 \longrightarrow^k_{\text{Pneg}} P_2$ in $k$ steps and $P_2 \longrightarrow_{\text{Ppos}} P_3$, then there exists a $F$ such that $P_1 \Longrightarrow^m_{\text{Ppos}} F \twoheadrightarrow_{\text{Pneg}} P_3$, with $1 \leq m \leq k+1$. Both are shown in following diagrams.




So we can construct an infinite sequence of detour and positive permutation conversions, which is impossible. Therefore $\text{IPC}_{\mathcal{C}}$ is strongly normalizing with regard to detour and permutation conversions $\{ \longrightarrow_{\text{D}} , \longrightarrow_{\text{P}} \}$. $\square$

# Chapter 4 | Discussion

We have studied the truth table system, a natural deduction system for which the derivation rules follow from the truth table as defined in [4]. The truth table system is a manner to define natural deduction rules for arbitrary connectives using a standard format. There are other ways to generalize standard natural deduction. Studies of these other approaches could be a source of inspiration for further research on the truth table system.

## 4.1 Related work

Closely related to the truth table system is the work of Milne [7], but his strategy is slightly different. He starts from the introduction rules which define a certain truth table. From these truth tables, the elimination rules are derived. Milne defines his method for classical logic.

The idea that introduction rules are the basis for the definition of the elimination rules is rooted in the inversion principle of Gentzen and Prawitz: 'the introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.' [3] This idea can be generalized to so-called 'general elimination rules'. This is studied by various researchers, such as Von Plato, Read, Francez and Dyckhoff [9, 12, 2]. The idea is that elimination rules are naturally determined by the introduction rules. The method with general elimination rules makes it possible to define deduction rules for arbitrary connectives, where the meaning of the connective lies in the introduction rules. Compare this to the truth table system where connectives arise from truth tables.

The elimination rules that arise from the 'general elimination' method have a similar shape as the elimination rules we derive from truth tables. They look the same in the sense that the conclusion of an elimination rule is an arbitrary formula $D$ instead of a subformula of the major premise. In this way, the standard $\vee$-E rule in the Prawitz system is a general elimination rule. However, the general elimination rules differ from our elimination rules for some connectives, such as for $\wedge$ [5].

Von Plato focuses on general elimination rules for the well-known connectives [9]. He studies normalization of intuitionistic logic where he also uses a form of segments, which he calls threads. He defines classical logic by adding a rule of excluded middle. This is a big difference to the truth table system where the classical property lies in the rules of the connectives.

Frances, Dyckhoff and Read [2, 12] study the rules on a more philosophical level. They study notions of harmony of general natural deduction systems. The inversion principle can be seen as a form of harmony, but it can be reversed. Frances and Dyckhoff suggest a harmony based on the elimination rules, where the introduction rules arise from the elimination rules. In addition, they try to define harmony on a local and a global level.

Read shows that without any constraints on the introduction rules, weird connectives can be defined using 'general elimination rules'. He illustrates it with a zero placed connective $\bullet$ with one introduction rule. He shows that the derived elimination rule together with the introduction rule yield an inconsistent system. But the system is harmonious! Read claims that the notion of harmony is not the problem, but the definition of the introduction rule causes the problem. Fortunately, this cannot happen in the truth table system, since we proved consistency.

## 4.2 Future research

We examined the truth table system. A lot of results were already established in [4, 5]. In this thesis we proved two new results on the relation between IPC and CPC. We showed that IPC and CPC are equivalent for monotone connectives and we showed Glivenko's theorem. In our study on normalization we gave a new proof of weak normalization based on ideas of Prawitz, which enabled us to study the form of normal derivations. At the end, we established our main result, the strong normalization for intuitionistic propositional logic with regard to detour and permutation conversions.

The work done in this thesis does not complete the whole study of the truth table system. There are some open questions. For example, is the reverse statement about monotone connectives true? In other words, if IPC and CPC are equivalent for some connective, must this connective be monotone? We would believe so, but it is hard to find a strategy to prove it. Another challenge is to find a 'general' Glivenko's translation from IPC to CPC for arbitrary connectives.

Another technical open problem is how to define detour conversion for the classical rules. There is no intuitive notion of reducing an introduction followed by an elimination due to the format of the classical introduction rules. The conclusion of a classical introduction of connective $c$ can be an arbitrary formula $D$ which has no relation with the elimination rules of $c$.

Finally, we can examine the concept of harmony in the truth table system. We may wonder how we should interpret harmony, since there is no preference for the elimination or the introduction rules. This would be interesting to study in order to get more insights in the relationship between introduction and elimination rules.

# Appendix A | Rules from Truth Tables

We represent the truth table rules for the well-known connectives $\wedge, \neg, \to, \vee, \bot$ and $\top$. Both in plain form from the definition (left column) and in optimized form by Lemmas 2.2.3 and 2.2.4 (right column). We present both the intuitionistic rules and the classical rules. The rules derived from the definition are labeled by the corresponding entries in the truth table. The intuitionistic optimized rules for the connectives $\wedge$ and $\vee$ are the same as the classical rules, because those are monotone connectives. We also present the intuitionistic rules for if-then-else and most.

| $A$ | $B$ | $A \vee B$ | $A \wedge B$ | $A \to B$ | $\neg A$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | |
| 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | |

| $\bot$ | $\top$ |
|---|---|
| 0 | 1 |

**Disjunction $\vee$**

| $\vee$ | From definition | Optimized rules |
|---|---|---|
| Elim | $\dfrac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \vee\text{-el}$ | $\dfrac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \vee\text{-el}$ |
| Intro | $\dfrac{\vdash A \quad B \vdash A \vee B}{\vdash A \vee B}\ \vee\text{-in}^{\text{i}}{}_{10}$ <br> $\dfrac{\vdash A \quad \vdash B}{\vdash A \vee B}\ \vee\text{-in}^{\text{i}}{}_{11}$ <br> $\dfrac{A \vdash A \vee B \quad \vdash B}{\vdash A \vee B}\ \vee\text{-in}^{\text{i}}{}_{01}$ | $\dfrac{\vdash A}{\vdash A \vee B}\ \vee\text{-in}_1$ <br> $\dfrac{\vdash B}{\vdash A \vee B}\ \vee\text{-in}_2$ |
| | $\dfrac{A \vee B \vdash D \quad \vdash A \quad B \vdash D}{\vdash D}\ \vee\text{-in}^{\text{c}}{}_{10}$ <br> $\dfrac{A \vee B \vdash D \quad \vdash A \quad \vdash B}{\vdash D}\ \vee\text{-in}^{\text{c}}{}_{11}$ <br> $\dfrac{A \vee B \vdash D \quad A \vdash D \quad \vdash B}{\vdash D}\ \vee\text{-in}^{\text{c}}{}_{01}$ | |

**Conjunction** $\wedge$

| $\wedge$ | From definition | Optimized rules |
|---|---|---|
| Elim | $\dfrac{\vdash A \wedge B \qquad A \vdash D \qquad B \vdash D}{\vdash D}\ \wedge\text{-el}_{00}$<br>$\dfrac{\vdash A \wedge B \qquad A \vdash D \qquad \vdash B}{\vdash D}\ \wedge\text{-el}_{01}$<br>$\dfrac{\vdash A \wedge B \qquad \vdash A \qquad B \vdash D}{\vdash D}\ \wedge\text{-el}_{10}$ | $\dfrac{\vdash A \wedge B}{\vdash A}\ \wedge\text{-el}_1$<br>$\dfrac{\vdash A \wedge B}{\vdash B}\ \wedge\text{-el}_2$ |
| Intro | $\dfrac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\ \wedge\text{-in}$ | $\dfrac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\ \wedge\text{-in}$ |
| | $\dfrac{A \wedge B \vdash D \qquad \vdash A \qquad \vdash B}{\vdash D}\ \wedge\text{-in}^{\mathrm{c}}$ | |

**Implication** $\rightarrow$

| $\rightarrow$ | From definition | Optimized rules |
|---|---|---|
| Elim | $\dfrac{\vdash A \rightarrow B \qquad \vdash A \qquad B \vdash D}{\vdash D}\ \rightarrow\text{-el}$ | $\dfrac{\vdash A \rightarrow B \qquad \vdash A}{\vdash B}\ \rightarrow\text{-el}$ |
| Intro | $\dfrac{A \vdash A \rightarrow B \qquad B \vdash A \rightarrow B}{\vdash A \rightarrow B}\ \rightarrow\text{-in}^{\mathrm{i}}{}_{00}$<br>$\dfrac{\vdash A \qquad \vdash B}{\vdash A \rightarrow B}\ \rightarrow\text{-in}^{\mathrm{i}}{}_{11}$<br>$\dfrac{A \vdash A \rightarrow B \qquad \vdash B}{\vdash A \rightarrow B}\ \rightarrow\text{-in}^{\mathrm{i}}{}_{01}$ | $\dfrac{\vdash B}{\vdash A \rightarrow B}\ \rightarrow\text{-in}^{\mathrm{i}}{}_{1}$<br>$\dfrac{A \vdash A \rightarrow B}{\vdash A \rightarrow B}\ \rightarrow\text{-in}^{\mathrm{i}}{}_{2}$ |
| | $\dfrac{A \rightarrow B \vdash D \qquad A \vdash D \qquad B \vdash D}{\vdash D}\ \rightarrow\text{-in}^{\mathrm{c}}{}_{00}$<br>$\dfrac{A \rightarrow B \vdash D \qquad \vdash A \qquad \vdash B}{\vdash D}\ \rightarrow\text{-in}^{\mathrm{c}}{}_{11}$<br>$\dfrac{A \rightarrow B \vdash D \qquad A \vdash D \qquad \vdash B}{\vdash D}\ \rightarrow\text{-in}^{\mathrm{c}}{}_{01}$ | $\dfrac{\vdash B}{\vdash A \rightarrow B}\ \rightarrow\text{-in}^{\mathrm{c}}{}_{1}$<br>$\dfrac{A \rightarrow B \vdash D \qquad A \vdash D}{\vdash D}\ \rightarrow\text{-in}^{\mathrm{c}}{}_{2}$ |

**Negation ¬**

| $\neg$ | From definition = optimized |
|---|---|
| Elim | $\dfrac{\vdash \neg A \qquad \vdash A}{\vdash D}\ \neg\text{-el}$ |
| Intro | $\dfrac{A \vdash \neg A}{\vdash \neg A}\ \neg\text{-in}^{\mathrm{i}}$ |
| | $\dfrac{\neg A \vdash D \qquad A \vdash D}{\vdash D}\ \neg\text{-in}^{\mathrm{c}}$ |

**Bottom ⊥**

| $\bot$ | From definition |
|---|---|
| Elim | $\dfrac{\vdash \bot}{\vdash D}\ \bot\text{-el}$ |

**Top ⊤**

| $\top$ | From definition | Optimized rules |
|---|---|---|
| Intro | $\dfrac{}{\vdash \top}\ \top\text{-in}^{\mathrm{i}}$ | $\dfrac{}{\vdash \top}\ \top\text{-in}$ |
| | $\dfrac{\top \vdash D}{\vdash D}\ \top\text{-in}^{\mathrm{c}}$ | |

Now we present intuitionistic optimized rules for if-then-else and most. For the rules directly derived from the truth tables see [4]. Note that there are more possible optimized rules reduced with Lemma 2.2.3 and Lemma 2.2.4. First we state the truth tables of if-then-else and most. We use the notation $A \to B/C$ to mean 'if $A$ then $B$ else $C$'.

| $A$ | $B$ | $C$ | $A \to B/C$ | most$(A, B, C)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

**If-then-else** $A \to B/C$

| if-then-else | From definition | Optimized rules |
|---|---|---|
| Elim | $\dfrac{\vdash A \to B/C \quad A \vdash D \quad B \vdash D \quad C \vdash D}{\vdash D}\,\mathrm{el}_{000}$ <br> $\dfrac{\vdash A \to B/C \quad A \vdash D \quad \vdash B \quad C \vdash D}{\vdash D}\,\mathrm{el}_{010}$ <br> $\dfrac{\vdash A \to B/C \quad \vdash A \quad B \vdash D \quad C \vdash D}{\vdash D}\,\mathrm{el}_{100}$ <br> $\dfrac{\vdash A \to B/C \quad \vdash A \quad B \vdash D \quad \vdash C}{\vdash D}\,\mathrm{el}_{101}$ | $\dfrac{\vdash A \to B/C \quad A \vdash D \quad C \vdash D}{\vdash D}\,\text{else-el}$ <br> $\dfrac{\vdash A \to B/C \qquad \vdash A}{\vdash B}\,\text{then-el}$ |
| Intro | $\dfrac{A \vdash A \to B/C \quad B \vdash A \to B/C \quad \vdash C}{\vdash A \to B/C}\,\mathrm{in}_{001}$ <br> $\dfrac{A \vdash A \to B/C \quad \vdash B \quad \vdash C}{\vdash A \to B/C}\,\mathrm{in}_{011}$ <br> $\dfrac{\vdash A \quad \vdash B \quad C \vdash A \to B/C}{\vdash A \to B/C}\,\mathrm{in}_{110}$ <br> $\dfrac{\vdash A \quad \vdash B \quad \vdash C}{\vdash A \to B/C}\,\mathrm{in}_{111}$ | $\dfrac{A \vdash A \to B/C \qquad \vdash C}{\vdash A \to B/C}\,\text{else-in}$ <br> $\dfrac{\vdash A \qquad \vdash B}{\vdash A \to B/C}\,\text{then-in}$ |

**Most** $\mathsf{most}(A, B, C)$

We only state the optimized rules. These are the same for the classical and intuitionistic rules.

| Elim | Intro |
|---|---|
| $\dfrac{\vdash \mathsf{most}(A,B,C) \qquad B \vdash D \qquad C \vdash D}{\vdash D}\,\mathsf{most\text{-}el}_1$ | $\dfrac{\vdash A \qquad \vdash B}{\vdash \mathsf{most}(A,B,C)}\,\mathsf{most\text{-}in}_1$ |
| $\dfrac{\vdash \mathsf{most}(A,B,C) \qquad A \vdash D \qquad C \vdash D}{\vdash D}\,\mathsf{most\text{-}el}_2$ | $\dfrac{\vdash A \qquad \vdash C}{\vdash \mathsf{most}(A,B,C)}\,\mathsf{most\text{-}in}_2$ |
| $\dfrac{\vdash \mathsf{most}(A,B,C) \qquad A \vdash D \qquad B \vdash D}{\vdash D}\,\mathsf{most\text{-}el}_3$ | $\dfrac{\vdash B \qquad \vdash C}{\vdash \mathsf{most}(A,B,C)}\,\mathsf{most\text{-}in}_3$ |

# Appendix B | Induction Proofs

**Proof of Proposition 3.6.16:** Let $M$ and $N$ be terms in $\mathrm{IPC}_{\mathcal{C}}$ such that $M \longrightarrow_{\mathrm{D}} N$. Then

1. for every $\mathrm{p}\lambda^{\rightarrow}$-term $H$ there exists a $\mathrm{p}\lambda^{\rightarrow}$-term $K$ such that $(M : H) \xrightarrow{+}_{\beta} K$ and $(N : H) \sqsubseteq K$,
2. there exists a $\mathrm{p}\lambda^{\rightarrow}$-term $K$ such that $\overline{\overline{M}} \xrightarrow{+}_{\beta} K$ and $\overline{\overline{N}} \sqsubseteq K$.

*Proof.* Statement (1.) is proved by induction on the generation of $M \longrightarrow_{\mathrm{D}} N$. Here we show induction steps using Lemma 3.6.14 and 3.6.15 at some places. Let $H$ be a $\mathrm{p}\lambda^{\rightarrow}$-term and suppose $P \longrightarrow_{\mathrm{D}} Q$. Numbering corresponds to Definition 3.5.3.

(3a) $P \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} Q \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}]$:
By induction hypothesis on $P \longrightarrow_{\mathrm{D}} Q$, we immediately can find a $K$ such that

$$P : \big(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N}}}_k \, \overline{(\lambda x_l.(O_l : H))}\big) \xrightarrow{+}_{\beta} K$$

with $Q : \big(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N}}}_k \, \overline{(\lambda x_l.(O_l : H))}\big) \sqsubseteq K$. This is also true for an elimination term without any case $\lambda x_l.O_l$.

(3b) $M \cdot_{r_s} [\overline{N_k}, P, \overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} M \cdot_{r_s} [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}]$: First we assume that the elimination term contains a case $\lambda x_l.O_l$.

$$\begin{aligned}(M \cdot_{r_s} [\overline{N_k}, P, \overline{N_k}; \overline{\lambda x_l.O_l}]) : H &= M : \big(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N_k}}} \, \overline{\overline{P}} \, \overline{\overline{\overline{N_k}}} \, \overline{(\lambda x_l.(O_l : H))}\big) \\ &= M : \big(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N_k}}}(\lambda h.(P : h)) \, \overline{\overline{\overline{N_k}}} \, \overline{(\lambda x_l.(O_l : H))}\big).\end{aligned}$$

By induction hypothesis, there is a $K'$ such that $(P : h) \xrightarrow{+}_{\beta} K'$ and $(Q : h) \sqsubseteq K'$. Now

$$\begin{aligned}&\big(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N_k}}}(\lambda h.(P : h)) \, \overline{\overline{\overline{N_k}}} \, \overline{(\lambda x_l.(O_l : H))}\big) \\ &\quad \xrightarrow{+}_{\beta} \big(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N_k}}}(\lambda h.K') \, \overline{\overline{\overline{N_k}}} \, \overline{(\lambda x_l.(O_l : H))}\big)\end{aligned}$$

and

$$\begin{aligned}&\big(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N_k}}}(\lambda h.(Q : h)) \, \overline{\overline{\overline{N_k}}} \, \overline{(\lambda x_l.(O_l : H))}\big) \\ &\quad \sqsubseteq \big(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N_k}}}(\lambda h.K') \, \overline{\overline{\overline{N_k}}} \, \overline{(\lambda x_l.(O_l : H))}\big).\end{aligned}$$

Define $K = M : \big(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N_k}}}(\lambda h.K') \, \overline{\overline{\overline{N_k}}} \, \overline{(\lambda x_l.(O_l : H))}\big)$. Applying Lemma 3.6.14 and Lemma 3.6.15 results in

$$(M \cdot_{r_s} [\overline{N_k}, P, \overline{N_k}; \overline{\lambda x_l.O_l}]) : H \twoheadrightarrow_{\beta} K$$

with

$$(M \cdot_{r_s} [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}]) : H \sqsubseteq K.$$

If there is no case $\lambda x_l.O_l$, then we have

$$\begin{aligned}(M \cdot_{r_s} [\overline{N_k}, P, \overline{N_k};\ ]) : H &= M : \big(\lambda p.(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N_k}}} \, \overline{\overline{P}} \, \overline{\overline{\overline{N_k}}})H\big) \\ &= M : \big(\lambda p.(\lambda g_1 \ldots \lambda g_t.g_s \overline{\overline{\overline{N_k}}}(\lambda h.(P : h)) \, \overline{\overline{\overline{N_k}}})H\big).\end{aligned}$$

Apply the same strategy to obtain the desired result.

(3c) $M \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}, \lambda x.P, \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} M \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}, \lambda x.Q, \overline{\lambda x_l.O_l}]$:
Use the same strategy as the previous case. Note that the elimination term has at least one case, namely $\lambda x.P$.

(3d) $\{\overline{N_j}, P, \overline{N_j}; \overline{\lambda y_i.M_i}\} \longrightarrow_{\mathrm{D}} \{\overline{N_j}, Q, \overline{N_j}; \overline{\lambda y_i.M_i}\}$:

$$\{\overline{N_j}, P, \overline{N_j}; \overline{\lambda y_i.M_i}\} : H = \big((\overline{\lambda q_j} \,.\, \overline{\lambda q_i} \,.H)\, \overline{\overline{\overline{N_j}}}\, \overline{\overline{P}}\, \overline{\overline{\overline{N_j}}}\, \overline{(\lambda y_i.(M_i : H))}\,\big)\overline{\overline{e_1^H}} \dots \overline{\overline{e_t^H}},$$

where parallel term $\overline{\overline{e_s^H}}$ may contain subterms of the form

$$\overline{\lambda h_k} \,.\, \overline{\lambda h_l} \,.h_{l'} \overline{\overline{N}}_{j'} \quad \text{or} \quad \overline{\lambda h_k} \,.\, \overline{\lambda h_l} \,.h_{l'} \overline{\overline{P}} \quad \text{or} \quad \overline{\lambda h_k} \,.\, \overline{\lambda h_l} \,.(\lambda y_{i'}.(M_{i'} : H))h_{k'}.$$

Note that the middle one does not have to be present, since it is not necessary that $P$ belongs to a matching case with some $l'$. But $\overline{\overline{P}}$ is always present in the whole translation of the introduction term due to the dummy redexes. By induction hypothesis, there is a $K'$ such that $(P : h) \xrightarrow{+}_{\beta} K'$ and $(Q : h) \sqsubseteq K'$. If the middle one exists then

$$\overline{\lambda h_k} \,.\, \overline{\lambda h_l} \,.h_{l'}(\lambda h.(P : h)) \xrightarrow{+}_{\beta} \overline{\lambda h_k} \,.\, \overline{\lambda h_l} \,.h_{l'}(\lambda h.K')$$

and

$$\overline{\lambda h_k} \,.\, \overline{\lambda h_l} \,.h_{l'}(\lambda h.(Q : h)) \sqsubseteq \overline{\lambda h_k} \,.\, \overline{\lambda h_l} \,.h_{l'}(\lambda h.K').$$

Therefore $\overline{\overline{e_s^H}} \twoheadrightarrow_{\beta} \overline{\overline{e_s'^H}}$, where $\overline{\overline{e_s'^H}}$ is parallel term $\overline{\overline{e_s^H}}$ where $(P : h)$ is replaced by $K'$. This holds for all $s$. Define $K = \big((\overline{\lambda q_j} \,.\, \overline{\lambda q_i} \,.H)\, \overline{\overline{\overline{N_j}}}(\lambda h.K')\, \overline{\overline{\overline{N_j}}}\, \overline{(\lambda y_i.(M_i : H))}\,\big)\overline{\overline{e_1'^H}} \dots \overline{\overline{e_t'^H}}$, then

$$\begin{aligned}\{\overline{N_j}, P, \overline{N_j}; \overline{\lambda y_i.M_i}\} : H &= \big((\overline{\lambda q_j} \,.\, \overline{\lambda q_i} \,.H)\, \overline{\overline{\overline{N_j}}}\, \overline{\overline{P}}\, \overline{\overline{\overline{N_j}}}\, \overline{(\lambda y_i.(M_i : H))}\,\big)\overline{\overline{e_1^H}} \dots \overline{\overline{e_t^H}} \\ &\xrightarrow{+}_{\beta} K\end{aligned}$$

and

$$\{\overline{N_j}, Q, \overline{N_j}; \overline{\lambda y_i.M_i}\} : H \sqsubseteq K.$$

Note that it indeed amounts to a $\xrightarrow{+}_{\beta}$ step to $K$ and not a $\beta$-equivalence, because $\overline{\overline{P}}$ is present in the translation due to the dummy redex.

(3e) $\{\overline{N_j}; \overline{\lambda y_i.M_i}, \lambda y.P, \overline{\lambda y_i.M_i}\} \longrightarrow_{\mathrm{D}} \{\overline{N_j}; \overline{\lambda y_i.M_i}, \lambda y.Q, \overline{\lambda y_i.M_i}\}$:
Use the same strategy as the previous case.

□

**Proof of Proposition 3.6.20:** Let $M_1, M_2, M_3$ be terms in IPC$_{\mathcal{C}}$ with $M_1 \longrightarrow_{\mathrm{Pneg}} M_2 \longrightarrow_{\mathrm{D}} M_3$. Then there is a term $F$ in IPC$_{\mathcal{C}}$ such that $M_1 \longrightarrow_{\mathrm{D}} F \twoheadrightarrow_{\mathrm{Pneg}} M_3$.

*Proof.* We use induction on the generation of $M_1 \longrightarrow_{\mathrm{Pneg}} M_2$. We show some cases in detail. First we look at the base case.

(1) $(M \cdot [\overline{N_k}]) \cdot [\overline{P}; \overline{\lambda x.Q}] \longrightarrow_{\mathrm{Pneg}} M \cdot [\overline{N_k}]$:
There are several possibilities of a detour reduction starting from $M \cdot [\overline{N_k}]$.
- If the detour reduction is in the subterm $M \longrightarrow_{\mathrm{D}} M'$, define $F = (M' \cdot [\overline{N_k}]) \cdot [\overline{P}; \overline{\lambda x.Q}]$ to get the desired result. Idem for detour reductions in $N_k$'s.
- If direct detour on $M \cdot [\overline{N_k}]$, then $M$ of the form $\{\overline{R}, \overline{\lambda y_i.S_i}\}$. The only matching case is $k' = i'$, so then $M \cdot [\overline{N}] \longrightarrow_{\mathrm{D}} S_{i'}[y_{i'} := N_{k'}] \cdot [\overline{N_k}] = M_3$. Define

$$F = (S_{i'}[y_{i'} := N_{k'}] \cdot [\overline{N_k}]) \cdot [P, \overline{\lambda x.Q}],$$

then $M_1 = (M \cdot [\overline{N_k}]) \cdot [\overline{P}; \overline{\lambda x.Q}] \longrightarrow_{\mathrm{D}} F \longrightarrow_{\mathrm{Pneg}} S_{i'}[y_{i'} := N_{k'}] \cdot [\overline{N_k}] = M_3$.

For the induction steps we assume $P \longrightarrow_{\mathrm{Pneg}} Q$.

(2a) $P \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{Pneg}} Q \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}]$:

- For detour in terms $N_k$ or $O_l$, it is easily verified.
- If detour on $Q$, say $Q \longrightarrow_{\mathrm{D}} R$. With the induction hypothesis we can find an $F'$ such that $P \longrightarrow_{\mathrm{D}} F' \twoheadrightarrow_{\mathrm{Pneg}} R$. Define $F = F' \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}]$. Then we have

$$M_1 = P \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} F \twoheadrightarrow_{\mathrm{Pneg}} R \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}] = M_3.$$

- If direct detour in $Q \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}]$, then $Q$ of the form $\{\overline{R_j}, \overline{\lambda y_i.S_i}\}$. We treat both matching cases.
  If $l' = j'$, then $\{\overline{R_j}, \overline{\lambda y_i.S_i}\} \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} O_{l'}[x_{l'} := R_{j'}] = M_3$. We had a negative permutation from $P$ to $Q = \{\overline{R_j}, \overline{\lambda y_i.S_i}\}$, which means that $P$ is also of the form $\{\overline{R'_j}, \overline{\lambda y'_i.S'_i}\}$ with $R'_j \longrightarrow_{\mathrm{Pneg}} R_j$ or $S'_i \longrightarrow_{\mathrm{Pneg}} S_i$ for some $j$ or $i$ and $R'_j = R_j$, $S'_i = S_i$ for all other $j$ and $i$. Define $F = O_{l'}[x_{l'} := R'_{j'}]$. Then we have

$$M_1 = \{\overline{R'_j}, \overline{\lambda y'_i.S'_i}\} \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} F \twoheadrightarrow_{\mathrm{Pneg}} O_{l'}[x_{l'} := R_j] = M_3.$$

  Note that there may be zero negative permutation steps.
  If $k' = i'$, then $\{\overline{R_j}, \overline{\lambda y_i.S_i}\} \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} S_{i'}[y_{i'} := N_{k'}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}]$. Again we have $P$ of the form $\{\overline{R'_j}, \overline{\lambda y'_i.S'_i}\}$ with the same conditions as in case $l' = j'$. Define $F = S'_{i'}[y_{i'} := N_{k'}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}]$, then

$$M_1 = \{\overline{R'_j}, \overline{\lambda y'_i.S'_i}\} \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} F \twoheadrightarrow_{\mathrm{Pneg}} S_{i'}[y_{i'} := N_{k'}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}] = M_3.$$

(2b) $M \cdot_{r_s} [\overline{N_k}, P, \overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{Pneg}} M \cdot_{r_s} [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}]$:

- For detour in terms $M$, $N_k$ or $O_l$, it is easily verified.
- If detour in $Q$, use the same induction strategy as in (2a).
- If direct detour on $M \cdot_{r_s} [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}]$, then $M$ of the form $\{\overline{R_j}, \overline{\lambda y_i.S_i}\}$.
  If $l' = j'$, then $M \cdot_{r_s} [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} O_{l'}[x_{l'} := R_{j'}] = M_3$. Defining $F = M_3$ gives the desired result with zero negative permutation steps.
  If $k' = i'$, then

$$M \cdot_{r_s} [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} S_{i'}[y_{i'} := N_{k'}] \cdot [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}] = M_3$$

  or

$$M \cdot_{r_s} [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} S_{i'}[y_{i'} := Q] \cdot [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}] = M_3.$$

  Define $F = S_{i'}[y_{i'} := N_{k'}] \cdot [\overline{N_k}, P, \overline{N_k}; \overline{\lambda x_l.O_l}]$ and $F = S_{i'}[y_{i'} := P] \cdot [\overline{N_k}, P, \overline{N_k}; \overline{\lambda x_l.O_l}]$ respectively, then for both cases

$$M_1 = M \cdot_{r_s} [\overline{N_k}, P, \overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{D}} F \longrightarrow_{\mathrm{Pneg}} M_3.$$

(2c) $M \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}, \lambda x.P, \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{Pneg}} M \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}, \lambda x.Q, \overline{\lambda x_l.O_l}]$:
Use the same strategy as the previous case.

(2d) $\{\overline{N_j}, P, \overline{N_j}; \overline{\lambda y_i.M_i}\} \longrightarrow_{\mathrm{Pneg}} \{\overline{N_j}, Q, \overline{N_j}; \overline{\lambda y_i.M_i}\}$:

- For detour in terms $N_j$ and $M_i$ it is easily verified.
- If detour in $Q$, use same induction strategy as in (2a).
- There are no more possibilities for a detour reduction from $\{\overline{N_j}, Q, \overline{N_j}; \overline{\lambda y_i.M_i}\}$.

(2e) $\{\overline{N_j}; \overline{\lambda y_i.M_i}, \lambda y.P, \overline{\lambda y_i.M_i}\} \longrightarrow_{\mathrm{Pneg}} \{\overline{N_j}; \overline{\lambda y_i.M_i}, \lambda y.Q, \overline{\lambda y_i.M_i}\}$:
Use same strategy as previous case.

□

**Proof of Proposition 3.6.22:** Let $M_1, M_2, M_3$ be terms in $\mathrm{IPC}_{\mathcal{C}}$ with $M_1 \longrightarrow_{\mathrm{Pneg}} M_2 \Longrightarrow^n_{\mathrm{Ppos}} M_3$. Then there is a term $F$ in $\mathrm{IPC}_{\mathcal{C}}$ such that $M_1 \Longrightarrow^m_{\mathrm{Ppos}} F \twoheadrightarrow_{\mathrm{Pneg}} M_3$ with $m = n$ or $m = n+1$.

*Proof.* We use induction on the generation of $M_1 \longrightarrow_{\mathrm{Pneg}} M_2$. We present some cases in detail. We start with the base case.

(1) $(M \cdot [\overline{N_k}]) \cdot [\overline{P}; \overline{\lambda x.Q}] \longrightarrow_{\mathrm{Pneg}} M \cdot [\overline{N_k}]$:

- If $\Longrightarrow^n_{\mathrm{Ppos}}$ in the subterm $M \Longrightarrow^n_{\mathrm{Ppos}} M'$, define $F = (M' \cdot [\overline{N_k}]) \cdot [\overline{P}; \overline{\lambda x.Q}]$ to get the desired result. Same strategy for reductions in $N_k$'s.
- If $\Longrightarrow^n_{\mathrm{Ppos}}$ on $M \cdot [\overline{N_k}]$, then $M = T \cdot [\overline{R}; \overline{\lambda y.S}] \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}}; \overline{\lambda w.V_{n-1}}]$ with positive reductions

$$M \cdot [\overline{N_k}] \Longrightarrow^n_{\mathrm{Ppos}} T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}}; \overline{\lambda w.V_{n-1}}] \cdot [\overline{N_k}])}] = M_3.$$

Now define $F = T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}}; \overline{\lambda w.V_{n-1}}] \cdot [\overline{N_k}] \cdot [\overline{P}; \overline{\lambda x.Q}])}]$. Then

$$\begin{aligned} M_1 &= T \cdot [\overline{R}; \overline{\lambda y.(S)}] \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}}; \overline{\lambda w.V_{n-1}}] \cdot [\overline{N_k}] \cdot [\overline{P}; \overline{\lambda x.Q}] \\ &\Longrightarrow^{n+1}_{\mathrm{Ppos}} T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}}; \overline{\lambda w.V_{n-1}}] \cdot [\overline{N_k}] \cdot [\overline{P}; \overline{\lambda x.Q}])}] = F \\ &\twoheadrightarrow_{\mathrm{Pneg}} T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}}; \overline{\lambda w.V_{n-1}}] \cdot [\overline{N_k}])}] = M_3. \end{aligned}$$

In short notation we have $M_1 \Longrightarrow^{n+1}_{\mathrm{Ppos}} F \twoheadrightarrow_{\mathrm{Pneg}} M_3$. This completes the base case.

Now we turn to the induction step. Assume $P \longrightarrow_{\mathrm{Pneg}} Q$.

(2a) $P \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\mathrm{Pneg}} Q \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}]$:

- For $\Longrightarrow^n_{\mathrm{Ppos}}$ in terms $N_k$ or $O_l$, it is easily verified.
- If $\Longrightarrow^n_{\mathrm{Ppos}}$ in $Q$, say $Q \Longrightarrow^n_{\mathrm{Ppos}} R$. With the induction hypothesis we can find an $F'$ such that $P \Longrightarrow^m_{\mathrm{Ppos}} F' \twoheadrightarrow_{\mathrm{Pneg}} R$ with $m = n$ or $m = n+1$. Define $F = F' \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}]$. Then we have

$$M_1 = P \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}] \Longrightarrow^m_{\mathrm{Ppos}} F \twoheadrightarrow_{\mathrm{Pneg}} R \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}] = M_3.$$

- If $\Longrightarrow^n_{\mathrm{Ppos}}$ direct on $Q \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}]$, then $Q$ of the form

$$T \cdot [\overline{R}; \overline{\lambda y.S}] \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}}; \overline{\lambda w.V_{n-1}}],$$

with

$$Q \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}] \Longrightarrow^n_{\mathrm{Ppos}} T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}}; \overline{\lambda w.V_{n-1}}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}])}] = M_3.$$

There are two possibilities: there is a case of the form $\lambda w.V_{n-1}$ or not.
In the first case we have $P = T' \cdot [\overline{R'}; \overline{\lambda y'.S'}] \cdot [\overline{U_1'}; \overline{\lambda w.V_1'}] \cdot \dots \cdot [\overline{U_{n-1}'}; \overline{\lambda w.V_{n-1}'}]$ with for some subterm $W'$ of $P$ we have $W' \longrightarrow_{\mathrm{Pneg}} W$ for $W$ in $Q$, for all others $W' = W$. Define

$$F = T' \cdot [\overline{R'}; \overline{\lambda y'.(S' \cdot [\overline{U_1'}; \overline{\lambda w.V_1'}] \cdot \dots \cdot [\overline{U_{n-1}'}; \overline{\lambda w.V_{n-1}'}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}])}].$$

Then

$$\begin{aligned} M_1 &= T' \cdot [\overline{R'}; \overline{\lambda y'.S'}] \cdot [\overline{U_1'}; \overline{\lambda w.V_1'}] \cdot \dots \cdot [\overline{U_{n-1}'}; \overline{\lambda w.V_{n-1}'}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}] \\ &\Longrightarrow^n_{\mathrm{Ppos}} T' \cdot [\overline{R'}; \overline{\lambda y'.(S' \cdot [\overline{U_1'}; \overline{\lambda w.V_1'}] \cdot \dots \cdot [\overline{U_{n-1}'}; \overline{\lambda w.V_{n-1}'}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}])}] = F \\ &\twoheadrightarrow_{\mathrm{Pneg}} T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}}; \overline{\lambda w.V_{n-1}}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}])}] = M_3. \end{aligned}$$

In the second case, $P$ can have the same form or

$$P = T \cdot [\overline{R}; \overline{\lambda y.S}] \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}};\ ] \cdot [\overline{X}; \overline{\lambda z.Z}],$$

with $P \longrightarrow_{\text{Pneg}} Q$. Define

$$F = T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}};\ ] \cdot [\overline{X}; \overline{\lambda z.Z}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}])}].$$

Then

$$\begin{aligned} M_1 &= T \cdot [\overline{R}; \overline{\lambda y.S}] \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}};\ ] \cdot [\overline{X}; \overline{\lambda z.Z}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}] \\ &\Longrightarrow^{n+1}_{\text{Ppos}} T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}};\ ] \cdot [\overline{X}; \overline{\lambda z.Z}] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}])}] = F \\ &\longrightarrow\!\!\!\!\rightarrow_{\text{Pneg}} T \cdot [\overline{R}; \overline{\lambda y.(S \cdot [\overline{U_1}; \overline{\lambda w.V_1}] \cdot \dots \cdot [\overline{U_{n-1}};\ ] \cdot [\overline{N_k}; \overline{\lambda x_l.O_l}])}] = M_3. \end{aligned}$$

(2b) $M \cdot_{r_s} [\overline{N_k}, P, \overline{N_k}; \overline{\lambda x_l.O_l}] \longrightarrow_{\text{Pneg}} M \cdot_{r_s} [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}]$:

- For $\Longrightarrow^{n}_{\text{Ppos}}$ in terms $M$, $N_k$ or $O_l$, it is easily verified.
- If $\Longrightarrow^{n}_{\text{Ppos}}$ in $Q$, use the same induction strategy as in (2a).
- If $\Longrightarrow^{n}_{\text{Ppos}}$ direct on $M \cdot_{r_s} [\overline{N_k}, Q, \overline{N_k}; \overline{\lambda x_l.O_l}]$, proceed in the same way as in case (2a).

(2c) $M \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}, \lambda x.P, \overline{\lambda x_l.O_l}] \longrightarrow_{\text{Pneg}} M \cdot_{r_s} [\overline{N_k}; \overline{\lambda x_l.O_l}, \lambda x.Q, \overline{\lambda x_l.O_l}]$:
Use the same strategy as the previous case.

(2d) $\{\overline{N_j}, P, \overline{N_j}; \overline{\lambda y_i.M_i}\} \longrightarrow_{\text{Pneg}} \{\overline{N_j}, Q, \overline{N_j}; \overline{\lambda y_i.M_i}\}$:

- For $\Longrightarrow^{n}_{\text{Ppos}}$ in terms $N_j$ and $M_i$ it is easily verified.
- If $\Longrightarrow^{n}_{\text{Ppos}}$ in $Q$, use same induction strategy as in (2a).
- There are no more possibilities for a $\Longrightarrow^{n}_{\text{Ppos}}$ reduction from $\{\overline{N_j}, Q, \overline{N_j}; \overline{\lambda y_i.M_i}\}$.

(2e) $\{\overline{N_j}; \overline{\lambda y_i.M_i}, \lambda y.P, \overline{\lambda y_i.M_i}\} \longrightarrow_{\text{Pneg}} \{\overline{N_j}; \overline{\lambda y_i.M_i}, \lambda y.Q, \overline{\lambda y_i.M_i}\}$:
Use same strategy as previous case.

□

# Bibliography


[1] D. van Dalen. *Logic and Structures.* Springer-Verlag, Berlin Heidelberg, 1994.

[2] N. Francez and R. Dyckhoff. A note on harmony. *Journal of Philosophical Logic*, 41(3):613–628, Jun 2012.

[3] G. Gentzen. Untersuchungen über das logische Schliessen. In *The Collected Papers of Gerhard Gentzen.* tr. M. Szabo, North-Holland, Amsterdam, 1969.

[4] H. Geuvers and T. Hurkens. Deriving natural deduction rules from truth tables. In *Logic and Its Applications - 7th Indian Conference, ICLA 2017, Kanpur, India, January 5-7, 2017, Proceedings*, pages 123–138, 2017.

[5] H. Geuvers and T. Hurkens. Proof terms for generalized natural deduction. 2018, to appear in Proceedings of Types 2017, http://www.cs.ru.nl/H.Geuvers/PUBS/prooftermsnaturaldeduction.pdf.

[6] P. de Groote. On the strong normalisation of intuitionistic natural deduction with permutation-conversions. *Inf. Comput.*, 178(2):441–464, 2002.

[7] P. Milne. Inversion principles and introduction rules. *Dag Prawitz on Proofs and Meaning*, Outstanding Contributions to Logic, vol 7:190–201, 2015.

[8] R. Nederpelt and H. Geuvers. *Type Theory and Formal Proof: An Introduction.* Cambridge University Press, 2014.

[9] J. von Plato. Natural deduction: some recent developments, summary of four lectures at the summer school on proof theory, computation, and complexity. *Technische Universität Dresden*, June 23 - July 4, 2003, http://www.helsinki.fi/∼negri/dresum.pdf.

[10] G.D. Plotkin. Call-by-name, call-by-value and the λ-calculus. *Theoretical Computer Science*, 1(2):125 – 159, 1975.

[11] D. Prawitz. *Natural Deduction: a proof-theoretical study.* Almqvist and Wiksell, Stockholm, 1965.

[12] S. Read. Harmony and autonomy in classical logic. *Journal of Philosophical Logic*, 29(2):123–154, 2000.

[13] W. W. Tait. Intensional interpretations of functionals of finite type I. *J. Symb. Log.*, 32(2):198–212, 1967.

[14] M. Takahashi. Parallel reductions in λ-calculus. *Information and Computation*, 118(1):120 – 127, 1995.

[15] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, New York, NY, USA, 1996.