Test exercises Semantics, (Course Fall 2007)

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1. Which of the following are ccpos?

   1. $\{X \subset \mathbb{R} \mid X$ is countable or finite$\}$, ordered by inclusion.
   2. $\mathbb{N} \to \{0,1\}$, where $f \sqsubset g$ iff $\forall x \in \mathbb{N}(f(x) \leq g(x))$.
   3. $\mathbb{N} \to \mathbb{N}$, where $f \sqsubset g$ iff $\forall x \in \mathbb{N}(f(x) \leq g(x))$.

2. Let $(D_1, <_1)$, $(D_2, <_2)$ and $(E, <)$ be ccpos. For $F : D_1 \times D_2 \to E$, define

$$\begin{array}{rcl} F_a(y) & := & F(a,y), \text{ for every } a \in D_1 \\ F^b(x) & := & F(x,b), \text{ for every } b \in D_2. \end{array}$$

   Prove that F is continuous iff both F_a and F^b are continuous for every $a \in D_1$ and $b \in D_2$.

3. Which of the following maps $F : \mathcal{P}(A) \times \mathcal{P}(A) \to \mathcal{P}(A)$ is continuous? (Hint: Use the previous exercise.)

   1. $F(X,Y) := X \cap Y$.
   2. $F(X,Y) := X \cup Y$.
   3. $F(X,Y) := X \setminus Y$.

4. Consider the extension of the flat ccpo of booleans with an "undefined" value u. So u is an undefined boolean value that we can compute, e.g. $\mathcal{B}[\frac{1}{0} = \frac{1}{0}] = u$. The ordering is now $\bot \sqsubset u \sqsubset \mathbf{tt}, \mathbf{ff}$.
   Give a "sensible" (at least monotone) definition of the $\vee$ function on these booleans:

   1. as a strict function.
   2. as a function that first evaluates its left argument.

5. Let $f \in \mathbb{N} \to \mathbb{N}_\bot$ satisfy

$$\begin{array}{rcl} f(0) & = & 1 \\ f(1) & = & f(3) \\ f(k) & = & f(k-2) \text{ otherwise} \end{array}$$

   Prove that each of the following is a solution of this equation.

1. $k \mapsto 1$
2. $k \mapsto k (\text{mod } 2) + 1$
3. $k \mapsto 1$ if k is even and $k \mapsto \bot$ otherwise

Determine the *least* solution to the above equation.

6. We define the program

$$S := \textbf{while}\, x > y\, \textbf{do}\, (x := x - 1; y := y + 1)$$

1. Determine the functional F that we need in order to compute the denotational semantics of S.
2. Determine a general form for $F^n(\bot)$ and prove it.
   It should be something like

$$F^n(\bot)(s) \quad = \quad \begin{cases} \text{id} & \text{if } s(x) \leq s(y) \\ s[x \mapsto \lfloor \frac{s(x)+s(y)}{2} \rfloor, y \mapsto \lceil \frac{s(x)+s(y)}{2} \rceil] & \text{if } s(y) < s(x) \leq s(y) + n \\ \bot & \text{if } s(x) > s(y) + n \end{cases}$$

3. Determine the least fixed oint of F.
4. Determine the denotational semantics of S.

7. Determine the denotational semantics of the program construct

$$\textbf{while}\, b_1\, \textbf{do}\, S\, \textbf{until}\, b_2$$

which operates as follows: Check b_1; if false, then terminate, if true then execute S, then check b_2; if true, terminate, if false return to the begin of the program.

8. Determine the denotational semantics of the following functional program.

$$f(3)\, \textbf{where}\, f(x) = g(f(x-1))\, \textbf{where}\, g(y) = g(y-1) + g(y-1).$$

(Hint: first determine the semantics of $g(f(x-1))$.)

9. Define the following extension of Combinatory Logic, CL^+. The terms are built from application and the constants $\mathbf{I}, \mathbf{K}, \mathbf{S}, \mathbf{B}$ and $\mathbf{C}$. Prove that CL^+ is combinatory complete by using the followiing (adapted) abstraction function:

$$\begin{array}{rcl} \lambda^* x.x & := & \mathbf{I} \\ \lambda^* x.P & := & \mathbf{K}P \text{ if } x \notin \text{FV}(P) \\ \lambda^* x.PQ & := & \mathbf{B}(\lambda^* x.P)Q \text{ if } x \notin \text{FV}(Q) \\ \lambda^* x.PQ & := & \mathbf{C}P(\lambda^* x.Q) \text{ if } x \notin \text{FV}(P) \\ \lambda^* x.PQ & := & \mathbf{S}(\lambda^* x.P)(\lambda^* x.Q) \text{ else.} \end{array}$$

10. For an arbitrary set A, compute the interpreations of $\omega := \lambda x.xx$, $\omega\omega$ and of $\omega\omega\mathbf{I}$ in the model D_A. Show that $\omega\omega\mathbf{I}$ and $\omega\omega$ have the same interpretation.