SEMANTICS FOR A QUANTUM PROGRAMMING LANGUAGE BY OPERATOR ALGEBRAS
作用素環による量子プログラミング言語に対する意味論

by

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ABSTRACT

Operator algebras, specifically C^*-algebras and W^*-algebras (the latter are also known as von Neumann algebras), give a formulation for quantum theory that is alternative to the one by Hilbert spaces. They are successfully used in areas such as quantum field theory and quantum information theory, yielding many notable results. It seems, however, that the use of operator algebras is not common so far in the area of quantum computation.

As an application of operator algebras to quantum computation, the present thesis gives a denotational semantics for a quantum programming language by operator algebras. We show that the opposite category of the category of W^*-algebras and normal completely positive pre-unital maps is an elementary quantum flow chart category in the sense of Selinger. As a consequence, Selinger's quantum programming language QPL can be interpreted as a map between W^*-algebras.

論文要旨

作用素環，つまり C^*-環と W^*-環（後者は von Neumann 環としても知られる）は Hilbert 空間による定式化に代わる，量子論の新たな定式化を与える．作用素環は量子場理論や量子情報理論の分野において効果的に用いられ，様々な成果をもたらしている．しかしながら，量子計算の分野においては，作用素環はこれまであまり利用されていなかったようである．

作用素環の量子計算への応用として，本論文では量子プログラミング言語への表示的意味論を作用素環によって与える．W^*-環と正規完全正前単位的写像の圏の双対圏が Selinger の意味での elementary quantum flow chart 圏であることが示され，その結果として，Selinger の量子プログラミング言語 QPL が W^*-環の間の写像として解釈できることがわかる．

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Chapter 1

Introduction

1.1 Background

1.1.1 Quantum computation and quantum programming language

Quantum computation is a new paradigm of computation that is done in the framework of quantum theory. In other words, it is a computation making use of quantum phenomena such as *superposition* and *entanglement*. Such quantum phenomena enable us to design *quantum algorithms* and *quantum protocols* which realize what is impossible in a classical way. For instance, Shor's algorithm [58] performs integer factorization in polynomial time, which is believed to be classically impossible. Another example is quantum cryptography protocols such as BB84 [6]. It guarantees unconditional security without relying the computational hardness, while the security of the classical cryptographic system RSA, for instance, relies on the hardness of integer factorization.

Quantum algorithms and protocols are actively studied, but they are often designed via low level machinery such as quantum gates and circuits. Coupled with the counter-intuitive nature of quantum phenomena, it is difficult to design quantum algorithms and protocols correctly. To alleviate such difficulties, *quantum programming languages* have been recently studied. As one of pioneering works, Selinger proposed a first-order functional language for quantum computation and its denotational semantics [53]. He (jointly with Valiron) successively started to study a *higher-order* functional language for quantum computation, also known as a *quantum lambda calculus* [55–57]. The first denotational semantics for a quantum lambda calculus with full features (especially, the ! modality) was given by Hasuo and Hoshino via Geometry of Interaction [25]. Recently, Pagani, Selinger and Valiron gave a denotational semantics by a different approach [45].

1.1.2 Operator algebras for quantum theory

It is now standard that quantum theory is formulated in terms of Hilbert spaces. This rigorous formulation of quantum theory is, after enormous efforts by physicists, finally given by John von Neumann [64]. It seems lesser-known, however, that quantum theory can be formulated in more abstract and general way by *operator algebras*. This formulation is sometimes called the *algebraic* formulation [35]. In fact, von Neumann also played a major role in developing the theory of operator algebras, and he preferred the formulation of quantum theory by operator algebras to the one by Hilbert spaces [48].

An operator algebra, in general, refers to an algebra of bounded operators on a Hilbert space, but there are two important classes. Von Neumann (jointly with Murray) studied *rings of operators* [41–43, 62, 63], a certain class of operator algebras which are now called *von Neumann algebras*. Gelfand and Neumark characterized *C^*-algebras* abstractly (i.e. space-freely) [20], which is another class of operator algebras that includes von Neumann algebras. Abstract characterization of von Neumann algebras, called *W^*-algebras*, was later obtained by Sakai [51]. Currently, the theory of operator algebras is usually concerned with C^*-algebras and W^*-algebras, or von Neumann algebras [52, 60].

C^*-algebras can be seen as noncommutative geometry [12], while W^*-algebras can be thought of as noncommutative measure (or probability), which in fact coincides with measure (probability) in quantum theory [23, 36, 39, 49]. Operator algebras are, indeed, successfully used in the wide area of quantum theory such as quantum statistical mechanics [8, 9], quantum information theory [5, 31] and quantum field theory [2,21,22]. It is commonly said that the algebraic formulation has an advantage in handling a system with infinitely many degrees of freedom over the Hilbert space formulation.

1.2 Our work

Despite its success in many areas of quantum theory, the use of operator algebras is not common so far in the area of quantum computation. One of the reasons is that to study quantum computation, usually a finite level system is sufficient, i.e. we can work in just a finite dimensional Hilbert space $\mathbb{C}^n$. Therefore, it might seem useless and meaningless to use operator algebras to study quantum computation. One of motivations of our work is to demonstrate that operator algebras *are* indeed useful and meaningful to study quantum computation.

In the present paper, as an application of operator algebras to quantum computation, we give a *denotational semantics* for a quantum programming language by operator algebras. More specifically, we show that the category $\mathbf{Wstar}_{\text{CP-PU}}$ of W^*-algebras and normal completely positive pre-unital maps is a symmetric monoidal $\mathbf{Dcppo}_\perp$-enriched category with products, and hence the opposite category of $\mathbf{Wstar}_{\text{CP-PU}}$ is an *elementary quantum flow chart category* [53]. As a consequence, it gives rise to a denotational semantics for the quantum programming language QPL [53].

Our contributions can be summarized as follows. On the mathematical side, we examine categories of operator algebras (C^*-algebras or W^*-algebras) with various morphisms, and especially we show the category $\mathbf{Wstar}_{\text{CP-PU}}$ is a $\mathbf{Dcppo}_\perp$-enriched category with suitably enriched (categorical) products and symmetric monoidal structure (i.e. tensor products). To the author's knowledge, this fact is not previously observed. We also prove some missing results in the literature such as the distribution of tensor products over direct sums. On the quantum computational side, we propose a novel denotational semantics of the quantum programming language QPL by operator algebras. Due to Selinger's work [53], it suffices to show the opposite category of $\mathbf{Wstar}_{\text{CP-PU}}$ is an elementary quantum flow chart category, which is immediate from the fact that $\mathbf{Wstar}_{\text{CP-PU}}$ is a symmetric monoidal $\mathbf{Dcppo}_\perp$-enriched category with finite products.

In comparison to Selinger's original semantics, our semantics by operator algebras has an advantage in handling infinite data and classical data well. As a consequence, we demonstrate that operator algebras (especially, W^*-algebras) give a good model for quantum computation.

1.3 Related work

First of all, our work depends to a great extent on Selinger's work [53], where he designed a first-order functional quantum programming language QPL[1] and gave its denotational semantics. The denotational semantics is given by (matrices of) *quantum operations*, but he also established what (abstract) categorical structures suffice to interpret the language QPL. Namely, he clarified that any *elementary quantum flow chart category* gives rise to a denotational semantics of the language QPL (without recursion). Hence we can give a denotational semantics of QPL, just by showing that a category satisfies the conditions of a elementary quantum flow chart category.

D'Hondt and Panangaden also gave a denotational semantics of QPL by the *weakest precondition* semantics [15]. We will discuss in Chap. 5 the difference between Selinger's original semantics, D'Hondt and Panangaden's weakest precondition semantics, and our semantics by operator algebras.

Recently, there are much more works on *higher-order* functional quantum programming languages, or quantum lambda calculi. Selinger and Valiron first proposed a quantum lambda calculus with classical controls and its operational semantics in [55]. They then gave a denotational semantics of a fragment of the quantum lambda calculus in [56]; it is given in essentially the same approach as in [53]. Most recently, Pagani, Selinger and Valiron have succeeded in giving a denotational semantics of the "full" quantum lambda calculus by the extending the approach of [53, 55]. In advance of them, however, the first denotational semantics of the "full" quantum lambda calculus is given by Hasuo and Hoshino [25] via Geometry of Interaction. Their semantics has a flavor of game semantics, and hence is significantly different from Selinger et al. [45, 53, 56] and our work.

Jacobs' work [30] is one of a few works using operator algebras for quantum computation. The work axiomatized, in categorical terms, *block structures* that often appears in programming languages. Applying it to the quantum setting, it turns out that such a block structure does not exist on Hilbert spaces, but does exist on C^*-algebras. It led him to use operator algebras. His work does not overlap directly with ours, but they should be related in some way. The exploration of the relationship between his and our works will be future work. Actually, Jacobs' work [30] and Furber and Jacobs' works [18, 19], which also study C^*-algebras from a categorical perspective, were what inspired the author to use operator algebras for quantum computation.

1.4 Organization of the thesis

In Chap. 2, we will fix notations and terminologies, and collect basic results we need later. Chapter 3 presents the basics on C^*-algebras, almost all of which are well-known. Chapter 4 first presents the basics on W^*-algebras, and then shows the order-enrichment of categories of W^*-algebras. In Chap. 5, we will present a denotational semantics of the language QPL by operator algebras. We conclude the thesis with future work in Chap. 6.

[1] Strictly speaking, he defined two languages: a flow chart language and a textual language. The name QPL is in fact reserved for the textual one. Because the denotation of the textual language is given by reducing to the flow chart language, we do not distinguish the two languages.

Chapter 2

Preliminaries

This chapter is intended to fix notations and terminologies, and collect some elementary results we will use later.

2.1 Functional analysis

In this section, we will fix notations and terminologies and present some basic results in the theory of functional analysis. Our primary references are [13, 47].

Definition 2.1.1. Let X and Y be normed spaces. A linear map $f\colon X \to Y$ is said to be

1. *bounded* if there exists $M \in \mathbb{R}^+$ such that $\|f(x)\| \leq M\|x\|$ for all $x \in X$.
2. *(weakly) contractive* (or *non-expansive*, *short*) if it does not increase the norm, i.e. $\|f(x)\| \leq \|x\|$ for all $x \in X$.
3. *isometric* if it preserves the norm, i.e. $\|f(x)\| = \|x\|$ for all $x \in X$. ◁

Definition 2.1.2. Let $f\colon X \to Y$ be a bounded linear map between normed space. An *operator norm* $\|f\|$ of f is defined by:

$$\begin{aligned}\|f\| &\coloneqq \sup\{\|f(x)\| \mid x \in X, \|x\| \leq 1\} \\ &= \inf\{M \in \mathbb{R}^+ \mid \|f(x)\| \leq M\|x\| \text{ for all } x \in X\} \ .\end{aligned}$$ ◁

Here are some elementary properties of normed spaces.

Proposition 2.1.3. *Let X, Y and Z be normed spaces.*

1. *A linear map $f\colon X \to Y$ is bounded if and only if it is continuous (wrt. the norms).*
2. *A bounded linear map $f\colon X \to Y$ is contractive if and only if $\|f\| \leq 1$.*
3. *The set $\mathcal{B}(X, Y)$ of bounded linear maps is again a normed space with pointwise operations and operator norm. Moreover, if Y is complete (i.e. a Banach space) then so is $\mathcal{B}(X, Y)$. In particular, a dual space $X^* \coloneqq \mathcal{B}(X, \mathbb{C})$ of a normed space X is a Banach space.*
4. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be bounded linear maps. Then $\|g \circ f\| \leq \|g\| \cdot \|f\|$.*
5. *The canonical map $\iota\colon X \to X^{**}$, defined by $\iota(x)(\phi) = \phi(x)$, is an isometry. Thus we can regard X as a sub-normed space of X^{**}.*

6. *Let $f\colon X \to Y$ be a bounded linear map. Then its dual $f^*\colon Y^* \to X^*$, defined by $f^*(\phi) = \phi \circ f$, is again bounded linear.* ◁

Proof. 1. See [47, Prop. 2.1.2] or [13, Prop. III.2.1].
2. Immediate by definition.
3. See [47, 2.1.3–4] or [13, Prop. III.2.1, Prop. III.5.4].
4. See [47, 2.1.3].
5. See [47, 2.3.7] or [13, III.§11].
6. See [47, Prop. 2.3.10]. ■

Lemma 2.1.4. *Let X be a locally convex topological vector space. Then there is a canonical bijection $\iota\colon X \overset{\cong}{\to} (X^*, \mathrm{wk}^*)^*$, given by $\iota(x)(\phi) = \phi(x)$.* ◁

Proof. See [13, Thm. V.1.3] or [47, 2.4.4–5]. ■

Notation 2.1.5. Let X be a normed space (or, more generally, a locally convex topological vector space). For $x \in X$ and $\varphi \in X^*$, we write $\langle \varphi, x\rangle \coloneqq \varphi(x)$, emphasizing the duality $\langle\cdot,\cdot\rangle\colon X^* \times X \to \mathbb{C}$. Furthermore, when we have an (often canonical) isomorphism $\iota\colon Y \to X^*$, we also write $\langle y, x\rangle \coloneqq \iota(y)(x)$ and make the isomorphism ι implicit. ◁

Proposition 2.1.6. *Let X and Y be Banach spaces. For every bounded map $f\colon X \to Y$, the dual $f^*\colon Y^* \to X^*$ is weakly* continuous (besides norm-continuous). Conversely, for every weakly* continuous map $h\colon Y^* \to X^*$, there exists a unique bounded map $f\colon X \to Y$ such that $h = f^*$. Hence we establish the following bijective correspondence:*

$$\frac{f\colon X \longrightarrow Y \quad \textit{bounded}}{h\colon Y^* \longrightarrow X^* \quad \textit{weakly* continuous}}$$ ◁

Proof. This is proved in [47, Prop. 2.4.12] except the uniqueness. The uniqueness is showed as follows. Suppose $f, g\colon X \to Y$ are bounded maps with $f^* = g^*$. Then for each $x \in X$, we have $f^*(\varphi)(x) = g^*(\varphi)(x)$ for all $\varphi \in Y^*$, i.e. $\varphi(f(x)) = \varphi(g(x))$ for all $\varphi \in Y^*$. It follows that $f(x) = g(x)$ since Y^* separates the points of Y (see e.g. [47, Cor. 2.3.4]). Hence $f = g$. ■

Corollary 2.1.7. *Let X and Y be Banach spaces. Every weakly* continuous map $Y^* \to X^*$ is bounded.* ◁

Proof. By Prop. 2.1.6 and Prop. 2.1.3.6. ■

Definition 2.1.8. Let $\{V_i\}_{i\in I}$ be a finite family of Banach spaces. We define two different direct sums: an ℓ^∞-*direct sum* $\bigoplus_{i\in I}^\infty V_i$ and an ℓ^1-*direct sum* $\bigoplus_{i\in I}^1 V_i$. These two have the same underlying sets[1]:

$$\bigoplus\nolimits_{i\in I}^{\infty} V_i = \bigoplus\nolimits_{i\in I}^{1} V_i \coloneqq \prod_{i\in I} V_i$$

and pointwise operations, but have the different norms:

$$\|(v_i)_{i\in I}\|_\infty \coloneqq \max_{i\in I}\|v_i\|$$
$$\|(v_i)_{i\in I}\|_1 \coloneqq \sum_{i\in I}\|v_i\| \ ,$$

respectively. ◁

[1] This is because I is finite.

Lemma 2.1.9. *Let $\{V_i\}_{i\in I}$ be a finite family of Banach spaces. There is an isometric isomorphism:*

$$\Big(\bigoplus\nolimits_{i\in I}^{1} V_i\Big)^* \cong \bigoplus\nolimits_{i\in I}^{\infty} V_i^* \ . \qquad \triangleleft$$

Proof. The mapping is given by the following bijection of sets:

$$\mathcal{B}\Big(\bigoplus\nolimits_{i\in I}^{1} V_i, \mathbb{C}\Big) \cong \prod\nolimits_{i\in I} \mathcal{B}(V_i, \mathbb{C}) \ ,$$

which is due to the universality of direct sums as coproduct. Its linearity is easy. We then check it is isometric. Let $(\phi_i)_i \in \bigoplus_i^\infty V_i^*$, which corresponds to $[\phi_i]_i \in (\bigoplus_i^1 V_i)^*$. By definition,

$$\begin{aligned}
\|(\phi_i)_{i\in I}\|_\infty &= \max_{i\in I}\|\phi_i\| \\
&= \max_{i\in I}\big(\sup\{|\phi_i(v)| \mid v \in V_i, \|v\| \le 1\}\big) \\
&= \sup\{|\phi_i(v)| \mid i \in I, v \in V_i, \|v\| \le 1\}
\end{aligned}$$

and

$$\begin{aligned}
\|[\phi_i]_i\| &= \sup\Big\{|[\phi_i]_i((v_i)_i)| \;\Big|\; (v_i)_i \in \bigoplus\nolimits_i^1 V_i, \|(v_i)_i\|_1 \le 1\Big\} \\
&= \sup\Big\{\Big|\sum\nolimits_i \phi_i(v_i)\Big| \;\Big|\; (v_i)_i \in \prod\nolimits_i V_i, \sum\nolimits_i \|v_i\| \le 1\Big\} \ .
\end{aligned}$$

We clearly see $\|(\phi_i)_{i\in I}\|_\infty \le \|[\phi_i]_i\|$, but observe, for $(v_i)_i \in \prod_i V_i$ with $\sum_i \|v_i\| \le 1$,

$$\begin{aligned}
\Big|\sum\nolimits_i \phi_i(v_i)\Big| &\le \sum\nolimits_i |\phi_i(v_i)| \\
&\le \sum\nolimits_i \|\phi_i\|\|v_i\| \\
&\le \max_{i\in I}\|\phi_i\| \\
&= \|(\phi_i)_{i\in I}\|_\infty \ .
\end{aligned}$$

Hence $\|[\phi_i]_i\| \le \|(\phi_i)_{i\in I}\|_\infty$ and then $\|[\phi_i]_i\| = \|(\phi_i)_{i\in I}\|_\infty$. ■

Lemma 2.1.10. *Let $\{V_i\}_{i\in I}$ be a finite family of Banach spaces. There is an isometric isomorphism:*

$$\Big(\bigoplus\nolimits_{i\in I}^{\infty} V_i\Big)^* \cong \bigoplus\nolimits_{i\in I}^{1} V_i^* \ . \qquad \triangleleft$$

Proof. Since algebraic structures are the same as Lem. 2.1.9, it suffices to check the mapping is isometric. Let $(\phi_i)_i \in \bigoplus_i^1 V_i^*$, which corresponds to $[\phi_i]_i \in (\bigoplus_i^\infty V_i)^*$. Then

$$\begin{aligned}
\|(\phi_i)_i\|_1 &= \sum\nolimits_i \|\phi_i\| \\
&= \sum\nolimits_i \sup\{|\phi_i(v)| \mid v \in V_i, \|v\| \le 1\} \\
&= \sup\Big\{\sum\nolimits_i |\phi_i(v_i)| \;\Big|\; (v_i)_i \in \prod\nolimits_i V_i, \forall i \in I.\, \|v_i\| \le 1\Big\} \\
&\stackrel{\star}{=} \sup\Big\{\Big|\sum\nolimits_i \phi_i(v_i)\Big| \;\Big|\; (v_i)_i \in \prod\nolimits_i V_i, \forall i \in I.\, \|v_i\| \le 1\Big\} \\
&= \sup\Big\{\Big|\sum\nolimits_i \phi_i(v_i)\Big| \;\Big|\; (v_i)_i \in \prod\nolimits_i V_i, \max\nolimits_i \|v_i\| \le 1\Big\} \\
&= \sup\Big\{|[\phi_i]_i((v_i)_i)| \;\Big|\; (v_i)_i \in \bigoplus\nolimits_i^\infty V_i, \|(v_i)_i\|_\infty \le 1\Big\} \\
&= \|[\phi_i]_i\| \ .
\end{aligned}$$

The equality $\overset{\star}{=}$ is because we can always reverse the sign of each $\phi_i(v_i)$ by substituting $-v_i$ for v_i, since $\|-v_i\| = \|v_i\|$. ∎

Remark 2.1.11. Lemma 2.1.9 holds even if the index set is infinite, but Lem. 2.1.10 fails in the infinite case. ◁

Definition 2.1.12 ([47, §2.3.6])**.** Let X be a normed space. For a subspace Y of X, the *annihilator* $Y^\perp$ of Y is a subspace of X^* defined by

$$Y^\perp := \{\varphi \in X^* \mid \forall y \in Y.\, \varphi(y) = 0\} \ .$$

For a subspace Z of X^*, the *annihilator* $Z^\perp$ of Z is a subspace of X defined by

$$Z^\perp := \{x \in X \mid \forall \varphi \in Z.\, \varphi(x) = 0\} \ .$$ ◁

Lemma 2.1.13. *Let X be a normed space. For any subspace Z of X^*, the annihilator $Z^\perp$ is norm-closed in X. If Z is weakly* closed in X^*, then $(Z^\perp)^\perp = Z$.* ◁

Proof. The norm-closedness of $Z^\perp$ follows from norm-continuity of functionals $\varphi \in Z$. It is easy to check $Z \subseteq (Z^\perp)^\perp$. To show the converse inclusion when Z is weakly* closed, use [47, Prop. 2.4.10]. ∎

Lemma 2.1.14 ([13, Thm. V.2.2])**.** *Let X be a normed space, $M \subseteq X$ a closed subspace, and $Q\colon X \to X/M$ a quotient map. Then a mapping $f \mapsto f \circ Q$ defines isometric isomorphism $(X/M)^* \overset{\cong}{\to} M^\perp$ of normed spaces. Moreover, the mapping is homeomorphic wrt. the weak* topology on $(X/M)^*$ and the relative topology on $M^\perp$ to the weak* topology on X^*.* ∎

Lemma 2.1.15. *Let X, Y, Z, W be normed spaces and $f\colon X \to Y, g\colon Z \to W$ be dense isometric maps. Let $\oplus$ denote one of the ℓ^∞-direct sum $\oplus^\infty$ and the ℓ^1-direct sum $\oplus^1$. Then $f \oplus g\colon X \oplus Z \to Y \oplus W$ is also dense isometric. Especially, if $\overline{X}$ and $\overline{Y}$ are the completions of X and Y respectively, then $\overline{X} \oplus \overline{Y}$ is the completion of $X \oplus Y$.* ◁

Proof. It is easy to see $f \oplus g$ is isometric. Let $(x_i)_i, (z_j)_j$ be a net in X, Y respectively. Suppose $(f(x_i))_i$ converges to y in Y and $(g(z_j))_j$ converges to w in W. Then $\big((x_i, z_j)\big)_{(i,j)}$ is a net in $X \oplus Z$, and moreover $\big((f \oplus g)(x_i, z_j)\big)_{(i,j)} = \big((f(x_i), g(z_j))\big)_{(i,j)}$ converges to (y, w) in $Y \oplus W$. Hence $f \oplus g$ is dense. ∎

Proposition 2.1.16. *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ denote the sets of bounded operators and trace class operators, respectively. Then*

1. *$\mathcal{T}(\mathcal{H})$ is a Banach space with the trace norm $\|T\|_1 = \mathrm{tr}(|T|)$.*
2. *The dual of $\mathcal{T}(\mathcal{H})$ is isometrically isomorphic to $\mathcal{B}(\mathcal{H})$ by the mapping $\iota\colon \mathcal{B}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})^*$ defined by $\iota(T) = \mathrm{tr}(T(-))$.* ◁

Proof. See [47, Thm. 3.4.13], [14, Thm. 19.2] or [60, §II.1]. ∎

Lemma 2.1.17. *Let $\mathcal{H}$ be a Hilbert space. Recall we have the isometric isomorphism $\iota\colon \mathcal{B}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})^*$ defined by $\iota(T) = \mathrm{tr}(T(-))$. Then a bounded operator $T \in \mathcal{B}(\mathcal{H})$ is positive if and only if $\iota(T)(S) \in \mathbb{R}^+$ for all positive $S \in \mathcal{T}(\mathcal{H})$.* ◁

Proof. (if) Assume that $\iota(T)(S) = \mathrm{tr}(TS) \in \mathbb{R}^+$ for all positive trace class operator S. For any $x \in \mathcal{H}$, $|x\rangle\langle x|$ is obviously positive and trace class. Then

$$\langle x, Tx\rangle = \mathrm{tr}(T|x\rangle\langle x|) \in \mathbb{R}^+ \ .$$

Hence T is positive.

(only if) Let T be a positive operator with $T = A^\dagger A$. Then, for any positive trace class operator S with $S = B^\dagger B$,

$$\mathrm{tr}(TS) = \mathrm{tr}(A^\dagger A B^\dagger B) = \mathrm{tr}(BA^\dagger AB^\dagger) = \mathrm{tr}((AB^\dagger)^\dagger AB^\dagger) \in \mathbb{R}^+. \qquad \blacksquare$$

2.2 Complete partial orders

We here list some elementary definitions and properties. Our references are [1] and [61].

Definition 2.2.1. A preordered set P is *directed* if every finite subset of P has an upper bound, which is equivalent to that P is nonempty and every pair of elements in P has an upper bound. A *directed set* refers to a directed preordered set. $\lhd$

Definition 2.2.2. A poset P is *directed complete* if every directed subset of P has the supremum (i.e. least upper bound, or join) A directed complete poset is abbreviated as a *dcpo.* $\lhd$

Definition 2.2.3. Let P and Q be posets. A function $f\colon P \to Q$ is *Scott-continuous* if it is monotone and preserves directed suprema, that is:

$$f(\bigsqcup D) = \bigsqcup f(D)$$

for any directed subset $D \subseteq P$ with its supremum $\bigsqcup D$. $\lhd$

Remark 2.2.4. If a function $f\colon P \to Q$ between posets preserves directed suprema, then it is automatically monotone and hence Scott-continuous. This is because:

$$x \sqsubseteq y \iff y = x \sqcup y \implies f(y) = f(x) \sqcup f(y) \iff f(x) \sqsubseteq f(y) \ .$$

Nevertheless, it is convenient to assume the monotonicity in advance because $f(D)$ is then a directed subset of Q. $\lhd$

Definition 2.2.5. A poset is *pointed* if it has the least element. A monotone function between pointed posets is *strict* if it preserves the least element. $\lhd$

A notion "monotone net" is useful to describe directed subsets.

Definition 2.2.6. Let P be a poset. A *monotone net* in P is a family $(x_i)_{i\in I}$ in P indexed by a directed set I satisfying

$$i \sqsubseteq j \implies x_i \sqsubseteq x_j$$

for all $i, j \in I$. In other words, the function $x_{(-)}\colon I \to P$ is monotone. $\lhd$

Definition 2.2.7. Let P be a poset, and $(x_i)_{i\in I}$ a monotone net in P. The *supremum* of $(x_i)_{i\in I}$ is defined by:

$$\bigsqcup_{i\in I} x_i \coloneqq \bigsqcup\{x_i \mid i \in I\} \ .$$

Notice that $\{x_i \mid i \in I\}$ is a directed subset of P. $\lhd$

The next proposition is immediate, since every directed subset of a poset can be seen as a monotone net, which is indexed by itself.

Proposition 2.2.8. *Every monotone net in a dcpo has the supremum. Conversely, a poset in which every monotone net has the supremum is directed complete.* ■

We will use a notion that slightly generalizes directed completeness.

Definition 2.2.9. A poset P is *bounded directed complete* if every bounded-above directed subset of P has the supremum. A bounded directed complete poset is abbreviated as a *bdcpo*. ◁

There are weaker notions than directed completeness and Scott continuity.

Definition 2.2.10. An ω-*chain* is a monotone net $(x_n)_{n\in\mathbb{N}}$ indexed by $I = \mathbb{N}$ (with usual order), that is, a monotone sequence. ◁

Definition 2.2.11. A poset is ω-*complete* if every ω-chain has the supremum. An ω-complete poset is abbreviated as an ω*cpo*. ◁

Definition 2.2.12. A function between posets is ω-*continuous* if it is monotone and preserves suprema of ω-chains. ◁

We here present some results we use later.

Proposition 2.2.13. *Let I and J be directed sets, and $(x_{ij})_{(i,j)\in I\times J}$ a monotone net. Note that $(x_{ij})_{j\in J}$ and $(x_{ij})_{i\in I}$ are monotone nets for each $i \in I$ and $j \in J$ respectively. If $\bigsqcup_{j\in J} x_{ij}$ and $\bigsqcup_{i\in I} x_{ij}$ exists for each $i \in I$ and $j \in J$ respectively, then we have*

$$\bigsqcup_{i\in I}\Big(\bigsqcup_{j\in J} x_{ij}\Big) = \bigsqcup_{(i,j)\in I\times J} x_{ij} = \bigsqcup_{j\in J}\Big(\bigsqcup_{i\in I} x_{ij}\Big)$$ ◁

Proof. Apply [1, Prop. 2.1.4.3]. See also [1, Prop. 2.1.12]. ■

Proposition 2.2.14. *Let P, Q, R be posets. A function $f\colon P \times Q \to R$ is Scott-continuous if (and only if) it is separately Scott-continuous.* ◁

Proof. See [1, Lem. 3.2.6]. ■

Lemma 2.2.15. *Let P be a poset and $D \subseteq P$ a directed subset of P. For any element $l \in D$, let $D \cap \uparrow l = \{d \in D \mid l \leq d\}$, which is clearly directed. Then we have*

$$\bigsqcup D = \bigsqcup (D \cap \uparrow l) \ \ ;$$

this means if one side exists, then the other side exists and they are equal. ◁

Proof. Note that for any subset $S \subseteq P$, $u \in P$ is the supremum of S if and only if

$$\forall p \in P. \big(u \sqsubseteq p \iff \forall s \in S.\, s \sqsubseteq p\big) \ .$$

To prove the lemma, therefore, it suffices to show for all $p \in P$,

$$\forall d \in D.\, d \sqsubseteq p \iff \forall x \in D \cap \uparrow l.\, x \sqsubseteq p \ .$$

$\Longrightarrow$ is obvious. We shall show $\Longleftarrow$. Assume $x \sqsubseteq p$ for all $x \in D \cap \uparrow l$. Because D is directed, for any $d \in D$ there exists $e \in D$ such that $d \sqsubseteq e$ and $l \sqsubseteq e$, i.e. $e \in D \cap \uparrow l$. Hence $d \sqsubseteq e \sqsubseteq p$. ■

2.3 Category theory

We assume the reader is familiar with the basics of category theory. The standard textbook of category theory is [37], and a more introductory one is [3]. The article [11] is a good introduction of category theory for physics, putting emphasis on monoidal categories.

We here just fix notations for some categories.

Definition 2.3.1.

1. $\mathbf{Dcppo}_{\perp}$ is the category of pointed dcpos and strict Scott-continuous maps.

2. $\mathbf{Bdcppo}_{\perp}$ is the category of pointed bdcpos and strict Scott-continuous maps.

3. $\boldsymbol{\omega}\mathbf{Cppo}$ is the category of pointed ωcpos and ω-continuous maps. $\triangleleft$

Chapter 3

C^*-algebras

This chapter is devoted to the study of C^*-algebras. We will study basics of C^*-algebras, maps between C^*-algebras, representations of C^*-algebras, matrices of C^*-algebras, completely positive maps, direct sums and tensor products of C^*-algebras. Almost all results in this chapter are well-known, but some results do not seem to be found in the literature. One of such results is the distribution of tensor products over direct sums, which will be proved in §3.7. Finally, in §3.8, we will summarize structures of C^*-algebras from a categorical point of view. The contents of this chapter is a prerequisite for Chap. 4 because W^*-algebras are "special" kind of C^*-algebras.

3.1 Basics of C^*-algebras

We start with the definition of $*$-algebras rather than C^*-algebras.

Definition 3.1.1 ($*$-algebra)**.** A $*$-*algebra* is a complex vector space A with a multiplication $\cdot\colon A \times A \to A$ and an involution $(-)^*\colon A \to A$ that satisfy the following conditions.

- The multiplication is bilinear, i.e.
  - $(x + x') \cdot y = x \cdot y + x' \cdot y$
  - $x \cdot (y + y') = x \cdot y + x \cdot y'$
  - $(tx) \cdot y = t(x \cdot y) = x \cdot (ty)$

  for all $x, x', y, y' \in A$ and $t \in \mathbb{C}$. It is also associative, i.e.
  - $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

  for all $x, y, z \in A$.
- The involution is anti-linear, i.e.
  - $(x + y)^* = x^* + y^*$
  - $(tx)^* = \overline{t}(x^*)$

  for all $x, y \in A$ and $t \in \mathbb{C}$. It also satisfies:
  - $(x^*)^* = x$
  - $(x \cdot y)^* = y^* \cdot x^*$

  for all $x, y \in A$.

Furthermore, in this paper, we require $*$-algebras to be unital. Namely, there exists $1 \in A$ such that $1 \cdot x = x = x \cdot 1$ for all $x \in A$. $\triangleleft$

Note. A multiplication $x \cdot y$ is also written just as xy. Note that we do not symbolically distinguish the multiplication $\cdot \colon A \times A \to A$ from the scalar multiplication $\cdot \colon \mathbb{C} \times A \to A$. Usually the distinction is clear from the context. $\triangleleft$

Definition 3.1.2 (Banach $*$-algebra)**.** A *Banach $*$-algebra* is a $*$-algebra A with a norm $\|\cdot\| \colon A \to \mathbb{R}^+$ which makes A a Banach space and further satisfies:

- $\|xy\| \leq \|x\|\|y\|$
- $\|x^*\| = \|x\|$ $\triangleleft$

Definition 3.1.3 (C^*-algebra)**.** A *C^*-algebra* is a Banach $*$-algebra A satisfying $\|x^*x\| = \|x\|^2$. $\triangleleft$

Remark 3.1.4. Since we have required $*$-algebras to be unital, all C^*-algebras are unital, too, in this paper. Note that it is not usually assumed in the literature. $\triangleleft$

Remark 3.1.5. We may drop the axiom $\|x^*\| = \|x\|$ from the definition of C^*-algebras. This is because $\|x\|^2 = \|x^*x\| \leq \|x^*\|\|x\|$ and then $\|x\| \leq \|x^*\|$, while $\|x^*\| \leq \|x^{**}\| = \|x\|$ by substituting x^* for x. $\triangleleft$

Remark 3.1.6. For a historical reason, the identity $\|x^*x\| = \|x\|^2$ is sometimes called the *B^*-identity*, while the identity $\|x^*x\| = \|x^*\|\|x\|$ may be called the *C^*-identity*. They are obviously equivalent in the presence of $\|x^*\| = \|x\|$. By Remark 3.1.5, the B^*-identity implies the C^*-identity without assuming $\|x^*\| = \|x\|$. The converse in fact holds, though it is highly nontrivial (see e.g. [16, §2 and Thm. 16.1], [52, The last remark in §1.1]). Therefore we do not distinguish two identities, and in what follows we will call the identity $\|x^*x\| = \|x\|^2$ the C^*-identity. $\triangleleft$

Example 3.1.7. Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{B}(\mathcal{H})$, the set of bounded operators on $\mathcal{H}$, is a C^*-algebra. See [13, Example VIII.1.2], [47, 4.3.7]. $\triangleleft$

Definition 3.1.8. Let A be a C^*-algebra. An element $x \in A$ is said to be

1. *invertible* (or *regular*) if $yx = xy = 1$ for some $y \in A$, i.e. the *inverse* x^{-1} exists;
2. *normal* if $x^*x = xx^*$;
3. *unitary* if $x^*x = xx^* = 1$, i.e. $x^* = x^{-1}$;
4. *self-adjoint* (or *hermitian*) if $x^* = x$;
5. *positive* if $x = y^*y$ for some $y \in A$;
6. an *effect* if both x and $1 - x$ are positive;
7. a *projection* if $x^2 = x = x^*$. $\triangleleft$

Notation 3.1.9. Let A be C^*-algebra. We write $\mathcal{SA}(A)$ for the set of self-adjoint elements in A, $\mathcal{Pos}(A)$ or A^+ for the set of positive elements in A, and $\mathcal{Ef}(A)$ for the set of effects in A. ◁

Proposition 3.1.10. *For an element of a C^*-algebra,*

1. *unitary $\implies$ invertible;*
2. *unitary $\implies$ normal;*
3. *self-adjoint $\implies$ normal;*
4. *positive $\implies$ self-adjoint;*
5. *being an effect $\implies$ positive;*
6. *being a projection $\implies$ being an effect.* ◁

Proof. Immediate by definition except 6, which follows from

$$x^*x = x^2 = x$$

and

$$(1-x)^*(1-x) = (1-x)^2 = 1-x \ . \qquad \blacksquare$$

Definition 3.1.11 (Spectrum)**.** Let A be an algebra. A *spectrum* $\mathrm{Sp}_A(x)$ of $x \in A$ is defined by:

$$\mathrm{Sp}_A(x) := \{\lambda \in \mathbb{C} \mid x - \lambda 1 \text{ is not invertible}\} \ . \qquad \triangleleft$$

Definition 3.1.12 (Spectral radius)**.** Let A be an algebra. A *spectral radius* $\|x\|_{\mathrm{sp}}$ of $x \in A$ is defined by:

$$\|x\|_{\mathrm{sp}} := \sup\{|\lambda| \mid \lambda \in \mathrm{Sp}_A(x)\} \ . \qquad \triangleleft$$

Proposition 3.1.13. *Let A be a Banach algebra. Then for any $x \in A$,*

1. ([60, Prop. 2.3]) $\|x\|_{\mathrm{sp}} < \infty$*;*
2. ([60, Prop. 2.4]) $\|x\|_{\mathrm{sp}} = \lim_{n\to\infty} \|x^n\|^{1/n}$. ■

Proposition 3.1.14 ([60, Prop. 4.2])**.** *Let A be a C^*-algebra. Then for a normal element $x \in A$, we have $\|x\| = \|x\|_{\mathrm{sp}}$.* ■

Corollary 3.1.15. *Let A be a C^*-algebra. For any $x \in A$,*

$$\|x\| = \sqrt{\|x^*x\|_{\mathrm{sp}}} \ . \qquad \triangleleft$$

Proof. $\|x\|^2 = \|x^*x\| = \|x^*x\|_{\mathrm{sp}}$. ■

The last result means that a norm of a C^*-algebra is determined by its algebraic structure. The following is an immediate consequence.

Corollary 3.1.16. *For a $*$-algebra A, a norm with which A is a C^*-algebra is unique if exists.* ■

Proposition 3.1.17 ([60, Thm. 6.1])**.** *Let A be a C^*-algebra. Then the set A^+ of positive elements is a closed convex cone in A with $A^+ \cap (-A^+) = \{0\}$. Consequently:*

1. *A^+ is a closed subset of A;*
2. *If x, y is positive and $t, s > 0$, then $tx + sy$ is positive.*
3. *If both x and $-x$ is positive, then $x = 0$.* ■

Definition 3.1.18. Let A be a C^*-algebra. We define a relation $\leq$ on A by

$$x \leq y \overset{\text{def}}{\iff} y - x \text{ is positive } . \qquad \triangleleft$$

Proposition 3.1.19. *The relation $\leq$ defined in Def. 3.1.18 is a partial order.* ◁

Proof. It is reflexive since $x - x = 0$ is positive. To show the transitivity, assume $x \leq y$ and $y \leq z$. Then $z - x = (z - y) + (y - x)$ is positive since a sum of positive elements is positive by Prop. 3.1.17. Hence $x \leq z$. Finally to show the antisymmetry, assume $x \leq y$ and $y \leq x$. It means that both $y - x$ and $x - y = -(y - x)$ are positive. Then $y - x = 0$ by Prop. 3.1.17, thus $x = y$. ■

Note. The ordering on a C^*-algebra defined in Def. 3.1.18 is standard in the literature (e.g. [60, §I.6], [52, §1.4], [14, §1.3]). For matrix algebras, it is called the *Löwner partial order*. It appears in [53, §3.8] and [25, Def. II.2]. ◁

Remark 3.1.20. We may write $x \geq 0$ for "x is positive", which is consistent with the ordering in Def. 3.1.18. ◁

Lemma 3.1.21 ([60, Thm. 6.1])**.** *Let A be a C^*-algebra. A self-adjoint element $x \in A$ is positive if and only if $\mathrm{Sp}_A(x) \subseteq \mathbb{R}^+$.* ◁

Proposition 3.1.22. *Let A be a C^*-algebra and $x \in \mathcal{SA}(A)$ a self-adjoint element. Then*

$$\|x\| \leq M \iff -M1 \leq x \leq M1$$

for any $M \in \mathbb{R}^+$. ◁

Proof. First, note that for each $t \in \mathbb{C}$ we have $\mathrm{Sp}_A(tx) = t \cdot \mathrm{Sp}_A(x)$ and $\mathrm{Sp}_A(x + t1) = \mathrm{Sp}_A(x) + t$. Then

$$\begin{aligned}
&\|x\| \leq M \\
&\iff \|x\|_{\mathrm{sp}} \coloneqq \sup\{|\lambda| \mid \lambda \in \mathrm{Sp}_A(x)\} \leq M && \text{by Prop. 3.1.14} \\
&\iff \forall \lambda \in \mathrm{Sp}_A(x).\, -M \leq \lambda \leq M \\
&\iff \forall \lambda \in \mathrm{Sp}_A(x).\, M - \lambda \geq 0 \text{ and } \lambda + M \geq 0 \\
&\iff \mathrm{Sp}_A(1M - x) \subseteq \mathbb{R}^+ \text{ and } \mathrm{Sp}_A(x + M) \subseteq \mathbb{R}^+ \\
&\iff 1M - x \geq 0 \text{ and } x + 1M \geq 0 && \text{by Lem. 3.1.21} \\
&\iff -M1 \leq x \leq M1
\end{aligned}$$ ■

Corollary 3.1.23. *Let A be a C^*-algebra and $x, y \in A$.*

1. *$x \in \mathcal{SA}(A)$ implies $-\|x\|1 \leq x \leq \|x\|1$.*
2. *$0 \leq x \leq y$ implies $\|x\| \leq \|y\|$.* ■

3.2 Maps between C^*-algebras

We next consider maps between C^*-algebras.

Definition 3.2.1. Let A and B be C^*-algebras. A linear map $f\colon A \to B$ is said to be

1. *multiplicative* if it commutes with the multiplication, i.e. $f(xy) = f(x)f(y)$ for all $x, y \in A$.
2. *involutive* if it commutes with the involution, i.e. $f(x^*) = f(x)^*$ for all $x \in A$.
3. *unital* if it preserves the unit, i.e. $f(1) = 1$.
4. *pre-unital* if it decreases the unit, i.e. $f(1) \le 1$.
5. *positive* if it preserves positive elements, i.e. $f(x) \in B^+$ for all $x \in A^+$.

A multiplicative involutive linear map is called a *$*$-homomorphism.* A bijective $*$-homomorphism is called *$*$-isomorphism.* ◁

Note. Note that we do not assume $*$-homomorphism is unital. Instead, $*$-homomorphism is automatically pre-unital, see Cor. 3.2.12. ◁

Note. The terminology "pre-unital" is not common in the literature. To make matters worse, some authors use "pre-unital" for maps between quantales in the dual way to ours (e.g. [34, Def. 2.5]): a homomorphism $\varphi\colon Q \to Q'$ between quantales is pre-unital if $e' \le \varphi(e)$, where $e \in Q$ and $e' \in Q'$ are the units.

Later it turns out that a positive map between C^*-algebras is pre-unital if and only if contractive. ◁

Note. Since we consider only linear maps, the adjective 'linear' is often omitted. When we say 'positive map', for example, it refers to a positive *linear* map. ◁

The following three propositions are easy consequences from the definition.

Proposition 3.2.2. *If a multiplicative map is surjective, it is unital.* ■

Proposition 3.2.3. *A $*$-homomorphism is positive.* ■

Proposition 3.2.4. *A linear map between C^*-algebras is positive if and only if it is monotone wrt. the orders defined in Def. 3.1.18.* ■

Proposition 3.2.5. *Every positive map between C^*-algebras is involutive and bounded.* ◁

Proof. See [18, Lem. 1]. Note that the unitality is not necessary. For a positive (not necessarily unital) map f, we have an inequality $\|f(x)\| \le 4\|f(1)\|\|x\|$ instead of theirs. ■

We can say more about norm of positive maps although it is nontrivial.

Proposition 3.2.6 ([46, Cor. 2.9])**.** *Let $f\colon A \to B$ be a positive map between C^*-algebras. Then $\|f\| = \|f(1)\|$.* ◁

Remark 3.2.7. For Prop. 3.2.6 to hold, it is crucial that not only A but also B is unital. Even when B does not have a unit, we have $\|f\| \leq 2\|f(1)\|$ ([46, Prop. 2.1]), and $\|f\| = \|f(1)\|$ provided f is completely positive ([46, Prop. 3.6]). ◁

Also nontrivially, $*$-homomorphisms have even more good properties.

Proposition 3.2.8.

1. *Every $*$-homomorphism between C^*-algebras is contractive.*
2. *A $*$-homomorphism between C^*-algebras is isometric if and only if it is injective.* ◁

Proof. 1. See [13, Thm. VIII.4.8, Prop. VIII.1.11.(d)], [17, Prop. 4.67] or [52, Cor. 1.2.6].

2. See [13, Thm. VIII.4.8] or [17, Prop. 4.67]. ■

Corollary 3.2.9. *Every $*$-isomorphism is unital and isometric.* ◁

Proof. By Prop. 3.2.2 and Prop. 3.2.8.2. ■

Lemma 3.2.10. *Let A be a C^*-algebra, and $x \in A$ a positive element. Then $x \leq 1$ if and only if $\|x\| \leq 1$.* ◁

Proof. Note that for $0, 1 \in A$, if $0 = 1$ then $\|1\| = 0$, otherwise $\|1\| = 1$ because $\|1\| = \|1^* \cdot 1\| = \|1\|^2$. Hence we have $\|1\| \leq 1$ in both cases.

Then, using Cor. 3.1.23, $0 \leq x \leq 1$ implies $\|x\| \leq \|1\| \leq 1$, and conversely $\|x\| \leq 1$ implies $x \leq \|x\|1 \leq 1$. ■

Proposition 3.2.11. *Let $f\colon A \to B$ be a positive map between C^*-algebras. Then f is pre-unital if and only if contractive.* ◁

Proof. Note that $f(1)$ is positive. Now

$$\begin{aligned} f(1) \leq 1 &\iff \|f(1)\| \leq 1 && \text{by Lem. 3.2.10} \\ &\iff \|f\| \leq 1 && \text{by Prop. 3.2.6.} \end{aligned}$$ ■

Corollary 3.2.12. *Every $*$-homomorphism between C^*-algebras is pre-unital.* ■

3.3 Representations of C^*-algebras

Definition 3.3.1 (Representation)**.** A *representation* of a C^*-algebra A is a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a Hilbert space and $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism. It is said to be *faithful* if π is injective (that is, $\pi(x) = 0$ implies $x = 0$ for all $x \in A$) and *unital* if π is unital. ◁

The next theorem is fundamental in the theory of C^*-algebras.

Theorem 3.3.2. *Every C^*-algebra admits a faithful unital representation.* ◁

Proof. See one of [60, Thm. I.9.18], [52, Thm. 1.16.6], [13, Thm. 5.17] and [14, Thm. 7.10]. The unitality is not mentioned explicitly, but it is clear from construction. The way constructing the representation is called the Gelfand-Naimark-Segal construction. ■

3.4 Matrices of C^*-algebras and completely positive maps

Definition 3.4.1. Let A be a $*$-algebra. For $n \in \mathbb{N}$, let $\mathcal{M}_n(A)$ denote the set of $n \times n$ matrices with entries from A. Then $\mathcal{M}_n(A)$ is again a $*$-algebra with the following operations:

- The addition and the scalar multiplication are pointwise:

$$[x_{ij}] + [y_{ij}] \coloneqq [x_{ij} + y_{ij}]$$
$$t[x_{ij}] \coloneqq [tx_{ij}]$$

- The multiplication is the matrix multiplication:

$$[x_{ij}][y_{ij}] \coloneqq \left[\sum_k x_{ik} y_{kj}\right]$$

- The involution is the conjugate transpose:

$$[x_{ij}]^* \coloneqq [x_{ji}^*]$$ ◁

Remark 3.4.2. When $n = 1$, $\mathcal{M}_1(A)$ is $*$-isomorphic to A. When $n = 0$, $\mathcal{M}_0(A)$ is the zero space $\{0\}$. ◁

Lemma 3.4.3. *Let $\mathcal{H}$ be a Hilbert space. For each $n \in \mathbb{N}$ there is a $*$-isomorphism*

$$\mathcal{M}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{\oplus n}) \ .$$

Hence $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ is a C^*-algebra. ◁

Proof. See [59, Lem. 1.22] or [46, Exercise 1.2]. ∎

Lemma 3.4.4. *Let A be a C^*-algebra and $(\pi, \mathcal{H})$ a faithful representation of A. Then for each $n \in \mathbb{N}$, $\mathcal{M}_n(\pi)\colon \mathcal{M}_n(A) \to \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ is an injective $*$-homomorphism such that the image $\mathcal{M}_n(\pi)\big(\mathcal{M}_n(A)\big)$ is closed in $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$. Hence $\mathcal{M}_n(A)$ is a C^*-algebra with a faithful representation:*

$$\mathcal{M}_n(A) \xrightarrow{\mathcal{M}_n(\pi)} \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{\oplus n}) \ .$$ ◁

Proof. See [59, Thm. 1.24]. ∎

Remark 3.4.5. [60, §IV.3] gives a slightly different representation of $\mathcal{M}_n(A)$. ◁

Corollary 3.4.6. *Let A be a C^*-algebra. Then $\mathcal{M}_n(A)$ is a C^*-algebra, too.* ◁

Proof. By Thm. 3.3.2 and Lem. 3.4.4. ∎

Definition 3.4.7 (Complete positivity)**.** Every linear map $f\colon A \to B$ between C^*-algebras induces a linear map $\mathcal{M}_n(f)\colon \mathcal{M}_n(A) \to \mathcal{M}_n(B)$ for each $n \in \mathbb{N}$ by

$$\mathcal{M}_n(f)[x_{ij}] \coloneqq [f(x_{ij})] \ .$$

A linear map f is said to be *n-positive* if $\mathcal{M}_n(f)$ is positive, and *completely positive* if for all $n \in \mathbb{N}$, $\mathcal{M}_n(f)$ is positive. ◁

Remark 3.4.8. Every linear map is trivially 0-positive. The 1-positivity is equivalent to ordinary positivity. ◁

Proposition 3.4.9. *Let $f\colon A \to B$ be a linear map between C^*-algebras. Then $\mathcal{M}_n(f)\colon \mathcal{M}_n(A) \to \mathcal{M}_n(B)$ is multiplicative (resp. involutive, unital, pre-unital) for each $n \in N$ if f is multiplicative (resp. involutive, unital, pre-unital).* ◁

Proof. Assume f is multiplicative. Then

$$\begin{aligned}
\mathcal{M}_n(f)([x_{ij}][y_{ij}]) &= \mathcal{M}_n(f)\Big(\Big[\sum\nolimits_k x_{ik}y_{kj}\Big]\Big) \\
&= \Big[\sum\nolimits_k f(x_{ik})f(y_{kj})\Big] \\
&= [f(x_{ij})][f(y_{ij})] \\
&= \mathcal{M}_n(f)([x_{ij}])\mathcal{M}_n(f)([y_{ij}]) \ .
\end{aligned}$$

Assume f is involutive. Then

$$\begin{aligned}
\mathcal{M}_n(f)([x_{ij}]^*) &= \mathcal{M}_n(f)([x_{ji}^*]) \\
&= [f(x_{ji}^*)] \\
&= [f(x_{ji})^*] \\
&= [f(x_{ij})]^* \\
&= \big(\mathcal{M}_n(f)([x_{ij}])\big)^* \ .
\end{aligned}$$

Assume f is unital. Note that the unit of $\mathcal{M}_n(A)$ is $[\delta_{ij}]$, where

$$\delta_{ij} = \begin{cases} 1 \in A & \text{if } i = j \\ 0 \in A & \text{if } i \neq j. \end{cases}$$

Then $\mathcal{M}_n(f)([\delta_{ij}]) = [f(\delta_{ij})]$ is the unit of $\mathcal{M}_n(B)$, because $f(1) = 1$ and $f(0) = 0$.

Assume f is pre-unital. Then $\mathcal{M}_n(f)([\delta_{ij}]) = [f(\delta_{ij})]$, where $f(\delta_{ij}) = 0$ for $i \neq j$ and $f(\delta_{ii}) \leq 1$. It is easy to see $[\delta_{ij}] - [f(\delta_{ij})] = [\delta_{ij} - f(\delta_{ij})]$ is positive, because it is a diagonal matrix such that each diagonal entry is positive. ∎

Corollary 3.4.10. *Let $f\colon A \to B$ be $*$-homomorphism between C^*-algebras. Then $\mathcal{M}_n(f)$ is $*$-homomorphism, too. It follows that f is completely positive (by Prop. 3.2.3).* ∎

The following theorem is useful.

Theorem 3.4.11. *Let $f\colon A \to B$ be a positive map between C^*-algebras. If at least one of A and B is commutative, then f is completely positive.* ◁

Proof. See [60, Cor. IV.3.5 and Prop. IV.3.9]. ∎

3.5 Direct sums of C^*-algebras

Definition 3.5.1. A *direct sum* of a finite family of C^*-algebras $(A_i)_{i\in I}$, denoted by $\bigoplus_{i\in I} A_i$, is defined as follows.

- An underlying set is the product of the underlying sets:

$$\bigoplus_{i\in I} A_i \coloneqq \prod_{i\in I} A_i$$

- Operations are defined pointwisely:
  - $(a_i)_{i\in I} + (b_i)_{i\in I} \coloneqq (a_i + b_i)_{i\in I}$
  - $\alpha \cdot (a_i)_{i\in I} \coloneqq (\alpha \cdot a_i)_{i\in I}$
  - $(a_i)_{i\in I} \cdot (b_i)_{i\in I} \coloneqq (a_i \cdot b_i)_{i\in I}$
  - $(a_i)^*_{i\in I} \coloneqq (a_i^*)_{i\in I}$
- A norm is defined by maximum:

$$\|(a_i)_{i\in I}\| \coloneqq \max_{i\in I} \|a_i\|$$

The disjoint sum is equipped with projections $\pi_i \colon \bigoplus_{i\in I} A_i \to A_i$ and injections $\kappa_i \colon A_i \to \bigoplus_{i\in I} A_i$ defined by

$$\pi_i\big((a_i)_{i\in I}\big) \coloneqq a_i$$

$$\kappa_i(a) \coloneqq (\delta_{ij}(a))_{j\in I} \quad \text{where} \quad \delta_{ij}(a) = \begin{cases} a & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} . \qquad \triangleleft$$

Remark 3.5.2. When $I = \varnothing$, the direct sum $\bigoplus \varnothing$ is the zero space $\{0\}$. $\triangleleft$

Remark 3.5.3. We can define an infinite direct sum, not only a finite one. For an infinite (small) family of C^*-algebras $(A_i)_{i\in I}$, an ℓ^∞*-direct sum* $\oplus^\infty_{i\in I} A_i$ is given by:

- An underlying set: $\displaystyle\bigoplus_{i\in I}^{\infty} A_i \coloneqq \left\{ (a_i)_{i\in I} \in \prod_{i\in I} A_i \;\middle|\; \sup_{i\in I} \|a_i\| < \infty \right\}$
- A norm: $\|(a_i)_{i\in I}\| \coloneqq \sup_{i\in I} \|a_i\|$
- Operations are defined pointwisely.

There is another choice of direct sum. It is called a c_0*-direct sum*, denoted by $\oplus^{c_0}_{i\in I} A_i$. The definition is the same as ℓ^∞-direct sums except that an underlying set is defined by:

$$\bigoplus_{i\in I}^{c_0} A_i \coloneqq \left\{ (a_i)_{i\in I} \in \prod_{i\in I} A_i \;\middle|\; \forall \varepsilon > 0.\, \{ i \in I \mid \|a_i\| \geq \varepsilon \} \text{ is finite} \right\} .$$

Note that if $I = \mathbb{N}$, the condition

$$\forall \varepsilon > 0.\, \{ n \in \mathbb{N} \mid \|a_n\| \geq \varepsilon \} \text{ is finite}$$

is equivalent to $\|a_n\| \to 0$ when $n \to \infty$. $\triangleleft$

Proposition 3.5.4. *Let $\bigoplus_{i\in I} A_i$ be the direct sum of a finite family of C^*-algebras.*

1. *Each projection $\pi_i \colon \bigoplus_{i\in I} A_i \to A_i$ is multiplicative involutive unital (hence completely positive).*
2. *Each injection $\kappa_i \colon A_i \to \bigoplus_{i\in I} A_i$ is multiplicative involutive pre-unital (hence completely positive).* $\triangleleft$

Proof. Straightforward. ∎

Proposition 3.5.5. *Let $\bigoplus_{i\in I} A_i$ be the direct sum of a finite family of C^*-algebras. It is the categorical biproduct as a vector space. That is, it has following universal properties.*

1. *For each C^*-algebra B and for every family of linear maps $\{f_i : B \to A_i\}_{i\in I}$, there is a unique linear map $h : B \to \bigoplus_{i\in I} A_i$ such that $f_i = \pi_i \circ h$ for each $i \in I$.*

   The unique map h is denoted by $\langle f_i\rangle_{i\in I}$, and defined by $\langle f_i\rangle_{i\in I}(x) = (f_i(x))_{i\in I}$.

2. *For each C^*-algebra C and for every family of linear maps $\{g_i : A_i \to C\}_{i\in I}$, there is a unique linear map $k : \bigoplus_{i\in I} A_i \to C$ such that $g_i = k \circ \kappa_i$ for each $i \in I$.*

   The unique map k is denoted by $[g_i]_{i\in I}$, and defined by $[g_i]_{i\in I}((x_i)_{i\in I}) = \sum_{i\in I} g_i(x_i)$.

Pictorially:

$$\begin{tikzcd}
 & B \arrow[d, dashed, "\langle f_i\rangle_i"'] \arrow[dr, "f_i"] & \\
A_j \arrow[r, "\kappa_j"] \arrow[dr, "g_j"'] & \bigoplus_i A_i \arrow[r, "\pi_i"'] \arrow[d, dashed, "{[g_i]_i}"] & A_i \\
 & C &
\end{tikzcd}$$

■

Proposition 3.5.6. *In the setting of Prop. 3.5.5, the following equations hold.*

$$\langle f_i\rangle_i = \sum_i \kappa_i \circ f_i$$
$$[g_i]_i = \sum_i g_i \circ \pi_i$$ ◁

Proof. By direct calculation. ■

Proposition 3.5.7. *Let $(A_i)_{i\in I}$ be a finite family of C^*-algebras, $\{f_i : B \to A_i\}_{i\in I}$ and $\{g_i : A_i \to C\}_{i\in I}$ families of linear maps between C^*-algebras. For the direct sum $\bigoplus_{i\in I} A_i$, and linear maps $\langle f_i\rangle_{i\in I} : B \to \bigoplus_{i\in I} A_i$ and $[g_i]_{i\in I} : \bigoplus_{i\in I} A_i \to C$, the following hold.*

1. *If f_i is multiplicative (resp. involutive, positive, completely positive, unital, pre-unital) for all $i \in I$, then $\langle f_i\rangle_{i\in I}$ is multiplicative (resp. involutive, positive, completely positive, unital, pre-unital), too.*

2. *If g_i is involutive (resp. positive, completely positive) for all $i \in I$, then $[g_i]_{i\in I}$ is involutive (resp. positive, completely positive), too.* ◁

Proof. 1. Assume f_i is multiplicative for all $i \in I$. Then

$$\begin{aligned}
\langle f_i\rangle_i(xy) &= (f_i(xy))_i \\
&= (f_i(x)f_i(y))_i \\
&= (f_i(x))_i \cdot (f_i(y))_i \\
&= \langle f_i\rangle_i(x) \cdot \langle f_i\rangle_i(y) \ .
\end{aligned}$$

Assume f_i is involutive (resp. positive, completely positive) for all $i \in I$. The involutivity (resp. positivity, complete positivity) of $\langle f_i \rangle_i$ follows from Prop. 3.5.6, Prop. 3.5.4.1 and the fact the involutivity (resp. positivity, complete positivity) is preserved by the composition and the addition of maps.

Assume f_i is unital for all $i \in I$. Then

$$\begin{aligned}\langle f_i \rangle_i(1) &= (f_i(1))_i \\ &= (1)_i \ ,\end{aligned}$$

which is the unit of $\oplus_i A_i$.

Assume f_i is pre-unital for all $i \in I$. Then

$$\begin{aligned}\langle f_i \rangle_i(1) &= (f_i(1))_i \\ &\leq (1)_i \ ,\end{aligned}$$

since the order is pointwise.

2. One can follow the same argument as 1. ■

Proposition 3.5.8. *Let A, B be C^*-algebras, and $(\pi_A, \mathcal{H}), (\pi_B, \mathcal{K})$ representations of them respectively. Then we have a representation $(\pi_{A\oplus B}, \mathcal{H} \oplus \mathcal{K})$ of $A \oplus B$, defined by*

$$\pi_{A\oplus B}(a, b)(x, y) = (\pi_A(a)(x), \pi_B(b)(y)) \ .$$

Moreover, the representation $(\mathcal{H}\oplus\mathcal{K}, \pi_{A\oplus B}$ is faithful (resp. unital) if both $(\mathcal{H}, \pi_A)$ and $(\mathcal{K}, \pi_B)$ are faithful (resp. unital). We shall denote the map $\pi_{A\oplus B}$ by $\pi_A \oplus \pi_B$.[1] ◁

Proof. We have just to show $\pi_{A\oplus B}$ is a $*$-homomorphism. In the following, we suppress the subscripts of $\pi_A, \pi_B, \pi_{A\oplus B}$ when it is clear from the context.

$$\begin{aligned}\pi((a, b) + (a', b'))(x, y) &= \pi(a + a', b + b')(x, y) \\ &= (\pi(a + a')(x), \pi(b + b')(y)) \\ &= (\pi(a)(x) + \pi(a')(x), \pi(b)(y) + \pi(b')(y)) \\ &\qquad\qquad \text{since } \pi_A \text{ and } \pi_B \text{ are linear} \\ &= (\pi(a)(x), \pi(b)(y)) + (\pi(a')(x), \pi(b')(y)) \\ &= \pi(a, b)(x, y) + \pi(a', b')(x, y) \\ &= (\pi(a, b) + \pi(a', b'))(x, y)\end{aligned}$$

$$\begin{aligned}\pi(t(a, b))(x, y) &= \pi(ta, tb)(x, y) \\ &= (\pi(ta)(x), \pi(tb)(y)) \\ &= (t\pi(a)(x), t\pi(b)(y)) \\ &\qquad \text{since } \pi_A \text{ and } \pi_B \text{ are linear} \\ &= t(\pi(a)(x), \pi(b)(y)) \\ &= t\pi(a, b)(x, y) \\ &= (t\pi(a, b))(x, y)\end{aligned}$$

[1] In the literature, the notation $\pi \oplus \pi'$ is often reserved for the different meaning. See [60, Def. I.9.15]

$$
\begin{aligned}
\pi((a,b)\cdot(a',b'))(x,y) &= \pi(aa',bb')(x,y) \\
&= (\pi(aa')(x), \pi(bb')(y)) \\
&= (\pi(a)(\pi(a')(x)), \pi(b)(\pi(b')(y))) \\
&\qquad\qquad \text{since } \pi_A \text{ and } \pi_B \text{ are multiplicative} \\
&= \pi(a,b)(\pi(a')(x), \pi(b')(y)) \\
&= \pi(a,b)(\pi(a',b')(x,y)) \\
&= (\pi(a,b)\circ\pi(a',b'))(x,y)
\end{aligned}
$$

$$
\begin{aligned}
\pi((a,b)^*)(x,y) &= \pi(a^*,b^*)(x,y) \\
&= (\pi(a^*)(x), \pi(b^*)(y)) \\
&= (\pi(a)^\dagger(x), \pi(b)^\dagger(y)) \ ,
\end{aligned}
$$

where the last equality is since π_A and π_B are involutive. We here need to show

$$
\pi(a,b)^\dagger(x,y) = (\pi(a)^\dagger(x), \pi(b)^\dagger(y)) \ .
$$

It is easily seen by:

$$
\begin{aligned}
\langle (x,y), \pi(a,b)(x',y') \rangle &= \langle (x,y), (\pi(a)(x'), \pi(b)(y')) \rangle \\
&= \langle x, \pi(a)(x') \rangle + \langle y, \pi(b)(y') \rangle \\
&= \langle \pi(a)^\dagger(x), x' \rangle + \langle \pi(b)^\dagger(y), y' \rangle \\
&= \langle (\pi(a)^\dagger(x), \pi(b)^\dagger(y)), (x',y') \rangle
\end{aligned}
$$

We assume that representations $(\pi_A, \mathcal{H})$ and $(\pi_B, \mathcal{K})$ are faithful, i.e. π_A and π_B are injective. Then

$$
\begin{aligned}
&\pi(a,b) = 0 \\
&\iff \pi(a,b)(x,y) = 0 \text{ for all } (x,y) \in \mathcal{H}\oplus\mathcal{K} \\
&\iff (\pi(a)(x), \pi(b)(y)) = 0 \text{ for all } (x,y) \in \mathcal{H}\oplus\mathcal{K} \\
&\iff \pi(a)(x) = 0, \pi(b)(y) = 0 \text{ for all } x \in \mathcal{H}, y \in \mathcal{K} \\
&\iff \pi(a) = 0, \pi(b) = 0 \\
&\iff a = 0, b = 0 \\
&\iff (a,b) = 0 \ .
\end{aligned}
$$

Hence $\pi_{A\oplus B}$ is injective, i.e. the representation $(\pi_{A\oplus B}, \mathcal{H}\oplus\mathcal{K})$ is faithful.

We assume that representations $(\pi_A, \mathcal{H})$ and $(\pi_B, \mathcal{K})$ are unital, i.e. π_A and π_B are unital. Then

$$
\pi(1,1)(x,y) = (\pi(1)(x), \pi(1)(y)) = (x,y) \ .
$$

Hence $\pi_{A\oplus B}$ is unital, i.e. the representation $(\pi_{A\oplus B}, \mathcal{H}\oplus\mathcal{K})$ is unital. ∎

3.6 Tensor products of C^*-algebras

Definition 3.6.1. Let A, B be C^*-algebras. The algebraic tensor product $A \odot B$ (as vector spaces) is a $*$-algebra with the following operations. The operations

$$
\begin{aligned}
(x\otimes y)(z\otimes w) &:= xz \otimes yw \\
(x\otimes y)^* &:= x^* \otimes y^*
\end{aligned}
$$

extend to a bilinear map $\cdot\colon (A\odot B)^2 \to A\odot B$ and an antilinear map $(-)^*\colon A\odot B\to A\odot B$. Explicitly, they are given by

$$\Big(\sum_i x_i\otimes y_i\Big)\Big(\sum_j z_j\otimes w_j\Big) \coloneqq \sum_i\sum_j x_iz_j\otimes y_iw_j$$
$$\Big(\sum_i x_i\otimes y_i\Big)^* \coloneqq \sum_i x_i^*\otimes y_i^* \ . \qquad \triangleleft$$

Definition 3.6.2. Let A, B be C^*-algebras and $(\mathcal{H}, \pi_A), (\mathcal{K}, \pi_B)$ representations of them, respectively. Then we obtain a representation $(\mathcal{H}\otimes\mathcal{K}, \pi)$ of $A\odot B$ defined by:

$$\pi(x\otimes y) \coloneqq \pi_A(x)\otimes\pi_B(y) \ ,$$

or, more explicitly,

$$\pi\Big(\sum_i x_i\otimes y_i\Big) \coloneqq \sum_i \pi_A(x_i)\otimes\pi_B(y_i) \ .$$

We shall denote the map π by $\pi_A\odot\pi_B$. If both $(\mathcal{H}, \pi_A)$ and $(\mathcal{K}, \pi_B)$ are faithful, then $(\mathcal{H}\otimes\mathcal{K}, \pi_A\odot\pi_B)$ is faithful (see [10, Prop. 3.1.12 and Lem. 3.3.9]). $\triangleleft$

Lemma 3.6.3. *Let A, B be C^*-algebras and $(\mathcal{H}, \pi_A), (\mathcal{K}, \pi_B)$ be faithful representations of them, respectively. Note that we have a faithful representation $(\mathcal{H}\otimes\mathcal{K}, \pi_A\odot\pi_B)$ of $A\odot B$. Then we define a norm on $A\odot B$ by*

$$\|x\| \coloneqq \|(\pi_A\odot\pi_B)(x)\|$$

for $x\in A\odot B$. Here, the following hold.

1. *The norm on $A\odot B$ is a C^*-cross-norm; it satisfies, besides usual conditions for a norm,*
   - $\|xy\| \le \|x\|\|y\|$
   - $\|x^*x\| = \|x\|^2$

   for $x, y\in A\odot B$, and
   - $\|a\otimes b\| = \|a\|\|b\|$

   for $a\in A, b\in B$.
2. *The norm does not depend on a choice of faithful representations of A and B.* $\triangleleft$

Proof. See [60, Def. IV.4.8, Thm. IV.4.9, etc.] or [10, Def. 3.3.4, Prop. 3.3.11, etc.]. ∎

Definition 3.6.4 (Tensor product of C^*-algebras)**.** A *(spatial) tensor product* of C^*-algebras A and B, denoted by $A\otimes B$, is a completion of $A\odot B$ wrt. the norm defined in Lem. 3.6.3. $\triangleleft$

Proposition 3.6.5. *Let A, B be C^*-algebras and $(\mathcal{H}, \pi_A), (\mathcal{K}, \pi_B)$ representations of them respectively. Then the representation $(\mathcal{H}\otimes\mathcal{K}, \pi_A\odot\pi_B)$ of $A\odot B$ extends to a representation $(\mathcal{H}\otimes\mathcal{K}, \pi_A\otimes\pi_B)$ of $A\otimes B$.*

$$\begin{tikzcd}
A\otimes B \arrow[dr, "\pi_A\otimes\pi_B"] & \\
A\odot B \arrow[u, hook, "dense"] \arrow[r, "\pi_A\odot\pi_B"'] & \mathcal{B}(\mathcal{H}\otimes\mathcal{K})
\end{tikzcd}$$

If both $(\mathcal{H}, \pi_A)$ and $(\mathcal{K}, \pi_B)$ are faithful, then $(\mathcal{H}\otimes\mathcal{K}, \pi_A\otimes\pi_B)$ is faithful. $\triangleleft$

Proof. See [60, Def. IV.4.8, Thm. IV.4.9.(iii)] or [52, Prop. 1.22.9]. ■

Remark 3.6.6. For C^*-algebras A and B, a norm on $A \odot B$ is not necessarily canonically determined. In other words, it is possible for $A \odot B$ to have C^*-cross-norms other than the one defined in Lem. 3.6.3. In the literature, the norm defined in Lem. 3.6.3 is called a *spatial* (or *injective*, *minimal*) C^*-norm, which is in fact the least C^*-norm. A tensor product defined in Def. 3.6.4 is called a *spatial* (or *injective*, *minimal*) tensor product. Another important C^*-cross-norm is the greatest one, which is called a *projective* (or *maximal*) C^*-norm in the literature. For further details, see [60, §IV.4], [10, §3.3] or [52, §1.22]. ◁

Definition 3.6.7. Let A, A', B, B' be $*$-algebras, and $f\colon A \to A', g\colon B \to B'$ linear maps. We define $f \odot g\colon A \odot B \to A' \odot B'$ by

$$(f \odot g)\bigl(\textstyle\sum_i a_i \otimes b_i\bigr) := \textstyle\sum_i f(a_i) \otimes g(b_i) \ . \qquad ◁$$

The C^*-tensor product of maps is not easy to handle as seen in the following fact.

Fact 3.6.8 ([10, Prop. 3.5.2])**.** *There exists a positive unital isometry $f\colon A \to A$ such that $f \odot \mathrm{id}_A\colon A \odot A \to A \odot A$ is unbounded (i.e. not norm-continuous).* ◁

However, if maps are $*$-homomorphism, the situation becomes simple.

Proposition 3.6.9 ([60, Prop. 4.22])**.** *Let A, A', B, B' be C^*-algebras, and $f\colon A \to A', g\colon B \to B'$ be $*$-homomorphisms. It is easy to see $f \odot g\colon A \odot B \to A' \odot B'$ is again a $*$-homomorphism. Moreover, it extends to a $*$-homomorphism $f \otimes g\colon A \otimes B \to A' \otimes B'$.* ■

More interestingly and importantly, tensor products of completely positive maps work well.

Theorem 3.6.10. *Let A, A', B, B' be C^*-algebras, and $f\colon A \to A', g\colon B \to B'$ completely positive maps. Then $f \odot g$ extends to a completely positive map $f \otimes g\colon A \otimes B \to A' \otimes B'$.* ◁

Proof. See [60, Prop. IV.4.23.(i)] or [10, Thm. 3.5.3]. ■

Remark 3.6.11. In Prop. 3.6.9 and Thm. 3.6.10, if both f and g are unital, then $f \otimes g$ is unital, too, since $f \odot g$ is clearly unital, and $A \odot B, A' \odot B'$ are (norm-dense) unital $*$-subalgebras of $A \otimes B, A' \otimes B'$ respectively. Moreover, if both f and g are pre-unital, then $f \otimes g$ is pre-unital. This is because

$$\begin{aligned} 1 \otimes 1 - f(1) \otimes g(1) &= 1 \otimes 1 - f(1) \otimes 1 + f(1) \otimes 1 - f(1) \otimes g(1) \\ &= (1 - f(1)) \otimes 1 + f(1) \otimes (1 - g(1)) \\ &\geq 0 \ . \end{aligned}$$

Note here that we use the positivity of f. ◁

3.7 Tensor products distribute over direct sums

In this section, we will show that tensor products distribute over direct sums. The author cannot find this result in the literature, so that we give a rather detailed proof.

Proposition 3.7.1 ($\otimes$ distributes over $\oplus$)**.** *Let A, B, C be C^*-algebras. Then we have a $*$-isomorphism:*

$$(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C) \ .$$

The mapping is given as the extension of

$$\sum\nolimits_i (a_i, b_i) \otimes c_i \ \longmapsto \ \sum\nolimits_i (a_i \otimes c_i, b_i \otimes c_i) \ . \qquad \triangleleft$$

Proof. Note firstly the following $*$-isomorphism:

$$(A \oplus B) \odot C \cong (A \odot C) \oplus (B \odot C) \ . \tag{3.1}$$

The mappings are given by

$$\sum\nolimits_i (a_i, b_i) \otimes c_i \ \longmapsto \ \sum\nolimits_i (a_i \otimes c_i, b_i \otimes c_i)$$

and

$$\sum\nolimits_i (a_i, 0) \otimes c_i + \sum\nolimits_j (0, b_i) \otimes c'_j \ \longleftarrow\!\!\shortmid \ \left(\sum\nolimits_i a_i \otimes c_i, \sum\nolimits_j b_j \otimes c'_j\right) \ .$$

Let

$$\iota_A \colon A \odot C \hookrightarrow A \otimes C$$
$$\iota_B \colon B \odot C \hookrightarrow B \otimes C$$

be dense inclusions. Then we have an injective $*$-homomorphism:

$$\iota_A \oplus \iota_B \colon (A \odot C) \oplus (B \odot C) \longrightarrow (A \otimes C) \oplus (B \otimes C) \ ,$$

which is also dense by Lem. 2.1.15. Because of the uniqueness of completion, it suffices to show (3.1) is isometric wrt. spatial C^*-norms.

Let $(\mathcal{H}_A, \pi_A), (\mathcal{H}_B, \pi_B), (\mathcal{H}_C, \pi_C)$ be faithful representations of A, B, C respectively. Notice that the following diagram commutes:

$$\begin{tikzcd}
(A \odot C) \oplus (B \odot C) \arrow[d, "\iota_A \oplus \iota_B"'] \arrow[dr, "(\pi_A \odot \pi_C) \oplus (\pi_B \odot \pi_C)"] & \\
(A \otimes C) \oplus (B \otimes C) \arrow[r, "(\pi_A \otimes \pi_C) \oplus (\pi_B \otimes \pi_C)"'] & \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H}_C) \oplus (\mathcal{H}_B \otimes \mathcal{H}_C))
\end{tikzcd} \ ,$$

and hence $(\pi_A \odot \pi_C) \oplus (\pi_B \odot \pi_C)$ is isometric. It is known that we have the following isomorphism of Hilbert spaces:

$$(\mathcal{H}_A \oplus \mathcal{H}_B) \otimes \mathcal{H}_C \cong (\mathcal{H}_A \otimes \mathcal{H}_C) \oplus (\mathcal{H}_B \otimes \mathcal{H}_C) \ ,$$

hence we have

$$\mathcal{B}((\mathcal{H}_A \oplus \mathcal{H}_B) \otimes \mathcal{H}_C) \cong \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H}_C) \oplus (\mathcal{H}_B \otimes \mathcal{H}_C)) \ .$$

By Prop. 3.5.8, $(\mathcal{H}_A \oplus \mathcal{H}_B, \pi_A \oplus \pi_B)$ is a faithful representation of $A \oplus B$. To show (3.1) is isometric, it suffices to show the following diagram commutes:

$$\begin{tikzcd} (A \oplus B) \odot C \arrow[r, "(\pi_A \oplus \pi_B) \odot \pi_C"] \arrow[d, "\cong"] & \mathcal{B}((\mathcal{H}_A \oplus \mathcal{H}_B) \otimes \mathcal{H}_C) \arrow[d, "\cong"] \\ (A \odot C) \oplus (B \odot C) \arrow[r, "(\pi_A \odot \pi_C) \oplus (\pi_B \odot \pi_C)"'] & \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H}_C) \oplus (\mathcal{H}_B \otimes \mathcal{H}_C)) \end{tikzcd} \ .$$

It follows from the commutativity of the following diagram:

$$\begin{tikzcd} (\mathcal{H}_A \oplus \mathcal{H}_B) \otimes \mathcal{H}_C \arrow[r, "\cong"] \arrow[d, "(\pi_A(a) \oplus \pi_B(b)) \otimes \pi_C(c)"'] & (\mathcal{H}_A \otimes \mathcal{H}_C) \oplus (\mathcal{H}_B \otimes \mathcal{H}_C) \arrow[d, "(\pi_A(a) \otimes \pi_C(c)) \oplus (\pi_B(b) \otimes \pi_C(c))"] \\ (\mathcal{H}_A \oplus \mathcal{H}_B) \otimes \mathcal{H}_C \arrow[r, "\cong"] & (\mathcal{H}_A \otimes \mathcal{H}_C) \oplus (\mathcal{H}_B \otimes \mathcal{H}_C) \end{tikzcd}$$

for each $a \in A, b \in B, c \in C$. ■

We also have a result for the nullary direct sum, i.e. the zero space. The proof is trivial.

Proposition 3.7.2. *Let A be a C^*-algebra, and let 0 denotes the zero space $\{0\}$, which is a C^*-algebra in trivial way. Then*

$$0 \otimes A \cong 0 \ .$$ ■

3.8 Categories of C^*-algebras

In this section, we shall summarize structures of C^*-algebras from a categorical point of view. First of all, notice that we have many choices of maps between C^*-algebras.

Definition 3.8.1. We define categories $\mathbf{Cstar}_{\mathrm{X}}$ of C^*-algebras and maps between them of the kind denoted by the subscript X. Specifically:

1. $\mathbf{Cstar}_{\text{M-I}}$ is the category of C^*-algebras and multiplicative involutive maps (i.e. $*$-homomorphisms).
2. $\mathbf{Cstar}_{\text{M-I-U}}$ is the category of C^*-algebras and multiplicative involutive unital maps (i.e. unital $*$-homomorphisms).
3. $\mathbf{Cstar}_{\text{CP}}$ is the category of C^*-algebras and completely positive maps.
4. $\mathbf{Cstar}_{\text{CP-PU}}$ is the category of C^*-algebras and completely positive pre-unital maps.
5. $\mathbf{Cstar}_{\text{CP-U}}$ is the category of C^*-algebras and completely positive unital maps.

Notice that all these categories have the same collection of objects. Note also that all C^*-algebras are assumed to be unital, and maps are linear. ◁

Proposition 3.8.2. *There are the following inclusions of categories.*

$$\begin{tikzcd} \mathbf{Cstar}_{\text{M-I-U}} \arrow[r, hook] \arrow[d, hook] & \mathbf{Cstar}_{\text{M-I}} \arrow[d, hook] & \\ \mathbf{Cstar}_{\text{CP-U}} \arrow[r, hook] & \mathbf{Cstar}_{\text{CP-PU}} \arrow[r, hook] & \mathbf{Cstar}_{\text{CP}} \end{tikzcd}$$ ◁

Proof. Note that multiplicative involutive maps ($*$-homomorphisms) are completely positive (Cor. 3.4.10) and pre-unital (Cor. 3.2.12). ∎

Proposition 3.8.3. *Finite direct sums of C^*-algebras form categorical finite products in all of the five categories defined in Def. 3.8.1.* ◁

Proof. Finite direct sums form categorical products as vector spaces (Prop. 3.5.5.1). Hence it suffices to check the projections satisfy the conditions of maps, and the tupling $\langle \cdot, \cdot \rangle$ of maps preserves the structures of maps. We have already done it in Prop. 3.5.4.1 and Prop. 3.5.7.1. ∎

Remark 3.8.4. In fact, ℓ^∞-direct sums are infinite categorical products. ◁

Proposition 3.8.5. *Finite direct sums of C^*-algebras form categorical finite coproducts in* $\mathbf{Cstar}_{\mathrm{CP}}$*. Hence, combined with Prop. 3.8.3,* $\mathbf{Cstar}_{\mathrm{CP}}$ *has finite biproducts.* ◁

Proof. Use Prop. 3.5.5.2, Prop. 3.5.4.2 and Prop. 3.5.7.2. ∎

Remark 3.8.6. The nullary direct sum, i.e. the zero space 0, is the initial object in the categories $\mathbf{Cstar}_{\text{M-I}}$, $\mathbf{Cstar}_{\text{CP-PU}}$ (and $\mathbf{Cstar}_{\mathrm{CP}}$, of course). Hence they have the zero object. ◁

Proposition 3.8.7. *The (spatial) C^*-tensor products of C^*-algebras makes all of five categories defined in Def. 3.8.1 symmetric monoidal categories.* ◁

Proof. The functoriality of the C^*-tensor products wrt. each category follows from Prop. 3.6.9, Thm. 3.6.10 and Remark 3.6.11. Then, it suffices to show that $\mathbf{Cstar}_{\text{M-I}}$ with the C^*-tensor products is symmetric monoidal, i.e. there exist isomorphisms that satisfy the axioms for a symmetric monoidal category. This is because isomorphisms in $\mathbf{Cstar}_{\text{M-I}}$ are isomorphism in the other categories. We will skip the detail since it is straightforward and well-known (see e.g. [40, §2.3], [65, Scholium 6.19]). ∎

Remark 3.8.8. By Fact 3.6.8, the C^*-tensor products fail to be functorial in the category of C^*-algebras and just positive maps. ◁

The distribution of the C^*-tensor product over direct sum (Prop. 3.7.1, Prop. 3.7.2) means the preservation of (categorical) finite products by the functor $(-) \otimes A$ for each C^*-algebra A.

Chapter 4

W^*-algebras

This chapter is devoted to the study of W^*-algebras. As the previous chapter, we will study basics of W^*-algebras, representations of W^*-algebras, matrices of W^*-algebras, direct sums and tensor products of W^*-algebras. Direct sums of W^*-algebras are the same as C^*-algebras, while tensor products of W^*-algebras are more complicated than C^*-algebras. The distribution of tensor products over direct sums of W^*-algebras seems missing again in the literature, hence we show it in §4.6. Then we summarize structures of W^*-algebras from a categorical point of view in §4.7.

We will successively study the unique structure of W^*-algebras. W^*-algebras have a pretty good order structure, which is one of features of W^*-algebras that makes them different from C^*-algebras. We review such "monotone closedness" of W^*-algebras in §4.8. Finally, in §4.9, we show this order structure of W^*-algebras is "lifted" into categories of W^*-algebras. What we show is, Specifically, that the category $\mathbf{Wstar}_{\mathrm{CP}}$ is a $\mathbf{Bdcppo}_{\perp}$-enriched category and the category $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$ is a $\mathbf{Dcppo}_{\perp}$-enriched category. This result is, to the author's knowledge, not previously observed.

4.1 Basics of W^*-algebras

W^*-algebras are "special" kind of C^*-algebras. We can define them in several equivalent ways, but we here adopt the following definition.

Definition 4.1.1 (W^*-algebra)**.** A W^*-*algebra* is a C^*-algebra that is a dual space of some Banach space. Specifically, a C^*-algebra A is a W^*-algebra if there exists a Banach space X and an isometric isomorphism $\iota\colon A \xrightarrow{\cong} X^*$ of Banach spaces. The Banach space X is called a *predual* of A. ◁

Proposition 4.1.2. *A predual of a W^*-algebra is unique up to (unique) isometric isomorphisms. For a W^*-algebra M, we shall denote the predual of M by M_*.* ◁

Proof. See [52, Cor. 1.13.3] or [60, Cor. III.3.9]. ■

Definition 4.1.3. Let M be a W^*-algebra. Using the predual M_*, We can equip M with the weak* topology $\sigma(M, M_*)$, which is the coarsest (weakest) topology that makes functions $M \to \mathbb{C}, x \mapsto \langle x, \varphi\rangle$ continuous for each $\varphi \in M_*$. We call this topology the *ultraweak* (or *σ-weak*) topology on M. ◁

Example 4.1.4. Recall that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra for a Hilbert space $\mathcal{H}$ (Example 3.1.7). By Prop. 2.1.16, $\mathcal{B}(\mathcal{H})$ has the predual $\mathcal{T}(\mathcal{H})$, the set of trace class operators on $\mathcal{H}$. Hence $\mathcal{B}(\mathcal{H})$ is a W^*-algebra. ◁

Definition 4.1.5 (Normal map)**.** A linear map between W^*-algebras is *normal* if it is continuous wrt. the ultraweak topologies. ◁

Proposition 4.1.6. *Let $f\colon M \to N$ be a linear map between W^*-algebras. Then f is normal if and only if there exists unique bounded map $f_*\colon N_* \to M_*$ such that the following diagram commutes.*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\cong"] & N \arrow[d, "\cong"] \\ (M_*)^* \arrow[r, "(f_*)^*"] & (N_*)^* \end{tikzcd}$$

◁

Proof. This is a consequence of Prop. 2.1.6. ■

Proposition 4.1.7 (W^*-subalgebra)**.** *Let M be a W^*-algebra. Suppose that $N \subseteq M$ be a ultraweakly closed $*$-subalgebra of M. Then N is a W^*-algebra. Moreover, the ultraweak topology of N coincides with a relative topology to the ultraweak topology of M.* ◁

Proof. Note that ultraweak closedness implies norm-closedness, so that N is a C^*-algebra. Hence we have only to show N has the predual. By Lem. 2.1.13, $N^\perp$ is norm-closed in M_* and $(N^\perp)^\perp = N$. By applying Lem. 2.1.14, we obtain an isometric isomorphism:

$$(M_*/N^\perp) \xrightarrow{\cong} (N^\perp)^\perp = N \ .$$

Hence $M_*/N^\perp$ is the predual of N. The latter half follows from the latter half of Lem. 2.1.14.

See also [52, Def. 1.1.4]. ■

Remark 4.1.8. In contrast to C^*-algebras, W^*-algebras are necessarily unital. In other words, we can show that if a (not necessarily unital) C^*-algebra has the predual, then it has the unit. See [51, Appendix], and also [52, §1.6–7]. ◁

Remark 4.1.9. Every finite dimensional C^*-algebra A is a W^*-algebra, because A is canonically isometrically isomorphic to its double dual A^{**}. Hence A^* is the predual of A. For the same reason, every (necessarily bounded) linear map between finite dimensional W^*-algebras is normal. ◁

4.2 Representations of W^*-algebras

Definition 4.2.1 (Representation)**.** A *representation* of a W^*-algebra M is a representation $(\mathcal{H}, \pi)$ of M as a C^*-algebra (see Def. 3.3.1). It is said to be *normal* if π is normal. ◁

Theorem 4.2.2. *Every W^*-algebra admits a normal faithful unital representation.* ◁

Proof. See [52, Thm. 1.16.7] or [60, Thm. III.3.5]. ■

For a normal $*$-homomorphism $f\colon M \to N$, its image $f(M)$ is ultraweakly closed in N ([52, Prop. 1.16.2]), so that $f(M)$ is a W^*-subalgebra of N by Prop. 4.1.7. Therefore, Thm. 4.2.2 states that W^*-algebras are characterized as ultraweakly closed unital $*$-subalgebras of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, which are called *von Neumann algebras* on $\mathcal{H}$ and studied very well. Thanks to Thm. 4.2.2, many of results for von Neumann algebras can be applied equally to W^*-algebras.

4.3 Matrices of W^*-algebras

Lemma 4.3.1 ([60, §II.2])**.** *Let $\mathcal{H}$ be a Hilbert space. A net $(T_i)_{i\in I}$ converges ultraweakly to T in $\mathcal{B}(\mathcal{H})$ if and only if for every sequences $(\xi_n)_{n=1}^{\infty}$ and $(\eta_n)_{n=1}^{\infty}$ in $\mathcal{H}$ with*

$$\sum_{n=1}^{\infty}\|\xi_n\|^2 < \infty \ , \quad \sum_{n=1}^{\infty}\|\eta_n\|^2 < \infty \ ,$$

we have:

$$\left|\sum_{n=1}^{\infty}\langle \xi_n, (T - T_i)\eta_n\rangle\right| \to 0 \ . \qquad \blacksquare$$

Proposition 4.3.2. *Let $\mathcal{H}$ be a Hilbert space. Note the isometric isomorphism: $\mathcal{M}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{\oplus n})$ (see Lem. 3.4.3). A net $(T_i)_{i\in I} = ([T_{kli}]_{lk})_{i\in I}$ converges ultraweakly to $T = [T_{kl}]$ in $\mathcal{B}(\mathcal{H}^{\oplus n})$ if and only if a net $(T_{kli})_{i\in I}$ converges ultraweakly to T_{kl} in $\mathcal{B}(\mathcal{H})$ for each $k, l \in \{1, \ldots, n\}$.* $\triangleleft$

Proof. By Lem. 4.3.1, a net $(T_i)_{i\in I} = ([T_{kli}]_{kl})_{i\in I}$ converges ultraweakly to $T = [T_{kl}]_{kl}$ in $\mathcal{B}(\mathcal{H}^{\oplus n})$ if and only if for every sequences $(\xi_m)_{m=1}^{\infty} = ((\xi_{km})_k)_{m=1}^{\infty}$ and $(\eta_m)_{m=1}^{\infty} = ((\eta_{lm})_l)_{m=1}^{\infty}$ in $\mathcal{H}^{\oplus n}$ with

$$\sum_{n=1}^{\infty}\|\xi_n\|^2 < \infty \ , \quad \sum_{n=1}^{\infty}\|\eta_n\|^2 < \infty \ , \tag{4.1}$$

we have:

$$\left|\sum_{m=1}^{\infty}\langle \xi_m, (T - T_i)\eta_m\rangle\right| \to 0 \ .$$

Here the following holds:

$$\sum_{m=1}^{\infty}\|\xi_m\|^2 = \sum_{m=1}^{\infty}\sum_{k=1}^{n}\|\xi_{km}\|^2 = \sum_{k=1}^{n}\sum_{m=1}^{\infty}\|\xi_{km}\|^2 < \infty \ , \tag{4.2}$$

$$\sum_{m=1}^{\infty}\|\eta_m\|^2 = \sum_{m=1}^{\infty}\sum_{l=1}^{n}\|\eta_{lm}\|^2 = \sum_{l=1}^{n}\sum_{m=1}^{\infty}\|\eta_{lm}\|^2 < \infty \ , \tag{4.3}$$

and

$$\begin{aligned}
\left|\sum_{m=1}^{\infty}\langle \xi_m, (T - T_i)\eta_m\rangle\right| &= \left|\sum_{m=1}^{\infty}\langle (\xi_{km})_k, ([T_{kl}]_{kl} - [T_{kli}]_{kl})(\eta_{lm})_l\rangle\right| \\
&= \left|\sum_{m=1}^{\infty}\langle (\xi_{km})_k, ([T_{kl} - T_{kli}]_{kl})(\eta_{lm})_l\rangle\right| \\
&= \left|\sum_{m=1}^{\infty}\Big\langle (\xi_{km})_k, \Big(\sum_{l=1}^{n}(T_{kl} - T_{kli})\eta_{lm}\Big)_k\Big\rangle\right| \\
&= \left|\sum_{m=1}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{n}\langle \xi_{km}, (T_{kl} - T_{kli})\eta_{lm}\rangle\right| \\
&= \left|\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{\infty}\langle \xi_{km}, (T_{kl} - T_{kli})\eta_{lm}\rangle\right| \ .
\end{aligned}$$

Note that each series converges absolutely.

Assume for each $k, l \in \{1, \ldots, n\}$, a net $(T_{kli})_{i\in I}$ converges ultraweakly to T_{kl} in $\mathcal{B}(\mathcal{H})$. Namely, for every sequences $(\xi_{km})_{m=1}^{\infty}$ and $(\eta_{lm})_{m=1}^{\infty}$ in $\mathcal{H}$ with

$$\sum_{m=1}^{\infty} \|\xi_{km}\|^2 < \infty \ , \quad \sum_{m=1}^{\infty} \|\eta_{lm}\|^2 < \infty \ , \tag{4.4}$$

we have:

$$\left| \sum_{m=1}^{\infty} \langle \xi_{km}, (T_{kl} - T_{kli})\eta_{lm} \rangle \right| \to 0 \ .$$

Now, for every sequences $(\xi_m)_{m=1}^{\infty} = ((\xi_{km})_k)_{m=1}^{\infty}$ and $(\eta_m)_{m=1}^{\infty} = ((\eta_{lm})_l)_{m=1}^{\infty}$ in $\mathcal{H}^{\oplus n}$, (4.1) ($\Longleftrightarrow$ (4.2), (4.3)) implies (4.4) for each $l, k \in \{1, \ldots, n\}$. Hence

$$\begin{aligned}
\left| \sum_{m=1}^{\infty} \langle \xi_m, (T - T_i)\eta_m \rangle \right| &= \left| \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{\infty} \langle \xi_{km}, (T_{kl} - T_{kli})\eta_{lm} \rangle \right| \\
&\leq \sum_{k=1}^{n} \sum_{l=1}^{n} \left| \sum_{m=1}^{\infty} \langle \xi_{km}, (T_{kl} - T_{kli})\eta_{lm} \rangle \right| \\
&\to 0 \ ,
\end{aligned}$$

because

$$\left| \sum_{m=1}^{\infty} \langle \xi_{km}, (T_{kl} - T_{kli})\eta_{lm} \rangle \right| \to 0$$

for each $k, l \in \{1, \ldots, n\}$.

Conversely, assume a net $(T_i)_{i\in I} = ([T_{kli}]_{lk})_{i\in I}$ converges ultraweakly to $T = [T_{kl}]$. For each $k', l' \in \{1, \ldots, n\}$ and for every sequences $(\xi'_m)_{m=1}^{\infty}$ and $(\eta'_m)_{m=1}^{\infty}$ in $\mathcal{H}$ with

$$\sum_{m=1}^{\infty} \|\xi'_m\|^2 < \infty \ , \quad \sum_{m=1}^{\infty} \|\eta'_m\|^2 < \infty \ ,$$

we define sequences $(\xi_m)_{m=1}^{\infty} = ((\xi_{km})_k)_{m=1}^{\infty}$ and $(\eta_m)_{m=1}^{\infty} = ((\eta_{lm})_l)_{m=1}^{\infty}$ in $\mathcal{H}^{\oplus n}$ by

$$\xi_{km} = \begin{cases} \xi'_m & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases} \ , \qquad \eta_{lm} = \begin{cases} \eta'_m & \text{if } l = l' \\ 0 & \text{otherwise} \end{cases} \ .$$

Then they satisfy the conditions (4.1). Hence

$$\begin{aligned}
\left| \sum_{m=1}^{\infty} \langle \xi_m, (T - T_i)\eta_m \rangle \right| &= \left| \sum_{m=1}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} \langle \xi_{km}, (T_{kl} - T_{kli})\eta_{lm} \rangle \right| \\
&= \left| \sum_{m=1}^{\infty} \langle \xi_{k'm}, (T_{k'l'} - T_{k'l'i})\eta_{l'm} \rangle \right| \\
&= \left| \sum_{m=1}^{\infty} \langle \xi'_m, (T_{k'l'} - T_{k'l'i})\eta'_m \rangle \right| \\
&\to 0 \ .
\end{aligned}$$

It follows that a net $(T_{k'l'i})_{i\in I}$ converges ultraweakly to $T_{k'l'}$. ∎

Proposition 4.3.3. *Let M be a W^*-algebra Then $\mathcal{M}_n(M)$ is a W^*-algebra. Moreover, a net $(x_i)_{i\in I} = ([x_{kli}]_{lk})_{i\in I}$ converges ultraweakly to $x = [x_{kl}]$ in $\mathcal{M}_n(M)$ if and only if a net $(x_{kli})_{i\in I}$ converges ultraweakly to x_{kl} in M for each $k, l \in \{1, \ldots, n\}$.* ◁

Proof. Take a normal faithful representation $(\mathcal{H}, \pi)$ of M. Then we have a faithful representation $(\mathcal{H}^{\oplus n}, \mathcal{M}_n(\pi))$ of $\mathcal{M}_n(M)$ by Lem. 3.4.4. Since the image $\pi(M)$ is ultraweakly closed in $\mathcal{B}(\mathcal{H})$, the image $\mathcal{M}_n(\pi)(\mathcal{M}_n(M)) = \mathcal{M}_n(\pi(M))$ is ultraweakly closed in $\mathcal{B}(\mathcal{H}^{\oplus n})$, too, by Prop. 4.3.2. Then $\mathcal{M}_n(\pi)(\mathcal{M}_n(M))$ is a sub-W^*-algebra of $\mathcal{B}(\mathcal{H}^{\oplus n})$. Because $\mathcal{M}_n(M)$ is $*$-isomorphic to the image $\mathcal{M}_n(\pi)(\mathcal{M}_n(M))$, $\mathcal{M}_n(M)$ is a W^*-algebra, too.

For the latter half,

a net $(x_i)_{i\in I} = ([x_{kli}]_{lk})_{i\in I}$ converges ultraweakly to $x = [x_{kl}]$ in $\mathcal{M}_n(M)$

$\iff$ (by $\mathcal{M}_n(M) \cong \mathcal{M}_n(\pi)(\mathcal{M}_n(M))$)

a net $(\mathcal{M}_n(\pi)(x_i))_{i\in I} = ([\pi(x_{kli})]_{lk})_{i\in I}$ converges ultraweakly to $\mathcal{M}_n(\pi)(x) = [\pi(x_{kl})]$ in $\mathcal{M}_n(\pi)(\mathcal{M}_n(M)) = \mathcal{M}_n(\pi(M))$

$\iff$ (since $\mathcal{M}_n(\pi)(\mathcal{M}_n(M))$ is a ultraweakly closed subspace of $\mathcal{B}(\mathcal{H}^{\oplus n})$)

a net $(\mathcal{M}_n(\pi)(x_i))_{i\in I} = ([\pi(x_{kli})]_{lk})_{i\in I}$ converges ultraweakly to $\mathcal{M}_n(\pi)(x) = [\pi(x_{kl})]$ in $\mathcal{B}(\mathcal{H}^{\oplus n})$

$\iff$ (by Prop. 4.3.2)

a net $(\pi(x_{kli}))_{i\in I}$ converges ultraweakly to $\pi(x_{kl})$ in $\mathcal{B}(\mathcal{H})$ for each $k, l \in \{1, \dots, n\}$

$\iff$ (since $\pi(M)$ is a ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$)

a net $(\pi(x_{kli}))_{i\in I}$ converges ultraweakly to $\pi(x_{kl})$ in $\pi(M)$ for each $k, l \in \{1, \dots, n\}$

$\iff$ (by $M \cong \pi(M)$)

a net $(x_{kli})_{i\in I}$ converges ultraweakly to x_{kl} in M for each $k, l \in \{1, \dots, n\}$. ■

Proposition 4.3.4. *Let $f\colon M \to N$ be a normal map between W^*-algebras. Then $\mathcal{M}_n(f)\colon \mathcal{M}_n(M) \to \mathcal{M}_n(N)$ is normal, too.* ◁

Proof. Use the latter half of Prop. 4.3.3. ■

4.4 Direct sums of W^*-algebras

A direct sum of W^*-algebras as C^*-algebras is again a W^*-algebra. In fact, we can give its predual explicitly.

Proposition 4.4.1. *Let $\{M_i\}_{i\in I}$ be a finite family of W^*-algebras. Let $\bigoplus_{i\in I} M_i$ be a direct sum of $\{M_i\}_{i\in I}$ as C^*-algebras. Then $\bigoplus^1_{i\in I} M_{i*}$ is the predual of $\bigoplus_{i\in I} M_i$, where M_{i*} is the predual of M_i. Hence $\bigoplus_{i\in I} M_i$ is a W^*-algebra.* ◁

Proof. Let $\{M_i\}_{i\in I}$ be a finite family of W^*-algebras with isometric isomorphisms $\iota_i\colon (M_{i*})^* \to M_i$. Then

$$\bigoplus\nolimits_i \iota_i\colon \bigoplus\nolimits_i^\infty (M_{i*})^* \longrightarrow \bigoplus\nolimits_i M_i$$

is an isometric isomorphism because $\bigoplus_i \iota_i^{-1}$ is clearly the inverse and

$$\begin{aligned}\left\|\left(\bigoplus\nolimits_i \iota_i\right)((\phi_i)_i)\right\| &= \left\|\left(\iota_i(\phi_i)\right)_i\right\| \\ &= \max_{i\in I}\|\iota_i(\phi_i)\| \\ &= \max_{i\in I}\|\phi_i\| \\ &= \|(\phi_i)_i\|_\infty \ .\end{aligned}$$

By Lem. 2.1.9, we now see that $\bigoplus_i^1 M_{i*}$ is the predual of $\bigoplus_i M_i$. ■

Note. See also [52, Def. 1.1.5]. ◁

As expected, structures associated with a direct sum respect normality.

Proposition 4.4.2. *Let $\{M_i\}_{i\in I}$ be a finite family of W^*-algebras, and L, K W^*-algebras. Let $\{f_i\colon L \to M_i\}_{i\in I}$ and $\{g_i\colon M_i \to K\}_{i\in I}$ be families of normal maps. Then the following maps are normal.*

$$\begin{aligned}
\pi_i &\colon \bigoplus\nolimits_i M_i \longrightarrow M_i\\
\kappa_i &\colon M_i \longrightarrow \bigoplus\nolimits_i M_i\\
\langle f_i\rangle_i &\colon L \longrightarrow \bigoplus\nolimits_i M_i\\
[g_i]_i &\colon \bigoplus\nolimits_i M_i \longrightarrow K
\end{aligned}$$ ◁

Proof. We can explicitly give their predual maps. Recall that $\bigoplus_i^1 M_{i*}$ is the predual of $\bigoplus_i M_i$, and suppose that f_{i*} and g_{i*} are the predual maps of f_i and g_i respectively. Then, the predual maps of $\pi_i, \kappa_i, \langle f_i\rangle_i, [g_i]_i$ are respectively the following maps:

$$\begin{aligned}
\kappa_i' &\colon M_{i*} \longrightarrow \bigoplus\nolimits_i^1 M_{i*}\\
\pi_i' &\colon \bigoplus\nolimits_i^1 M_{i*} \longrightarrow M_{i*}\\
[f_{i*}]_i &\colon \bigoplus\nolimits_i^1 M_{i*} \longrightarrow L_*\\
\langle g_{i*}\rangle_i &\colon K_* \longrightarrow \bigoplus\nolimits_i^1 M_{i*}
\end{aligned}$$

It is straightforward to check the following diagrams commute.

$$\begin{tikzcd}
\bigoplus_i M_i \arrow[r, "\pi_i"] \arrow[d, "\cong"] & M_i \arrow[d, "\cong"] \\
(\bigoplus_i^1 M_{i*})^* \arrow[r, "\kappa_i'^*"] & (M_{i_*})^*
\end{tikzcd}
\qquad
\begin{tikzcd}
M_i \arrow[r, "\kappa_i"] \arrow[d, "\cong"] & \bigoplus_i M_i \arrow[d, "\cong"] \\
(M_{i_*})^* \arrow[r, "\pi_i'^*"] & (\bigoplus_i^1 M_{i*})^*
\end{tikzcd}$$

$$\begin{tikzcd}
L \arrow[r, "\langle f_i\rangle_i"] \arrow[d, "\cong"] & \bigoplus_i M_i \arrow[d, "\cong"] \\
(L_*)^* \arrow[r, "{[f_{i*}]_i^*}"] & (\bigoplus_i^1 M_{i*})^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigoplus_i M_i \arrow[r, "{[g_i]_i}"] \arrow[d, "\cong"] & K \arrow[d, "\cong"] \\
(\bigoplus_i^1 M_{i*})^* \arrow[r, "\langle g_{i*}\rangle_i^*"] & (K_*)^*
\end{tikzcd}$$ ■

Remark 4.4.3. Another way to show that $\langle f_i\rangle_i$ and $[g_i]_i$ are normal is to use the following equations.

$$\begin{aligned}
\langle f_i\rangle_i &= \sum_i \kappa_i \circ f_i\\
[g_i]_i &= \sum_i g_i \circ \pi_i
\end{aligned}$$ ◁

4.5 Tensor products of W^*-algebras

Let M and N be W^*-algebras. Note that we have the embeddings $M_* \hookrightarrow M^*, N_* \hookrightarrow N^*$. Then there exists canonical embeddings:

$$M_* \odot N_* \hookrightarrow M^* \odot N^* \hookrightarrow (M \odot N)^* \cong (M \otimes N)^* \ .$$

Hence we can equip $M_* \odot N_*$ with the dual norm of the spatial C^*-norm on $M \odot N$. Let $M_* \otimes N_*$ denote the completion of $M_* \odot N_*$ wrt. this norm. Then $M_* \otimes N_*$ can be seen as a closed subspace of $(M \otimes N)^*$. Let $I \subseteq (M \otimes N)^{**}$ be the annihilator of $M_* \otimes N_*$, i.e.

$$I = (M_* \otimes N_*)^\perp = \{\varphi \in (M \otimes N)^{**} \mid \forall x \in M_* \otimes N_*. \langle x, \varphi \rangle = 0\} \ .$$

Then $(M_* \otimes N_*)^* \cong (M \otimes N)^{**}/I$ is a C^*-algebra, and hence a W^*-algebra with the predual $M_* \otimes N_*$. Note that I is a two-sided ideal of $(M \otimes N)^{**}$ (and see [13, Thm. III.10.1, Thm. V.2.3] [52, Thm. 1.17.2, Cor. 1.17.3]). The C^*-tensor product $M \otimes N$ is embedded into $(M_* \otimes N_*)^*$ by

$$M \otimes N \xrightarrow{\cong} (M \otimes N + I)/I \hookrightarrow (M \otimes N)^{**}/I \xrightarrow{\cong} (M_* \otimes N_*)^* \ ,$$

and moreover this embedding is ultraweakly dense. We now define tensor products of W^*-algebras as follows (for further details, see [52, §1.22], [60, §IV.5] or [50]):

Definition 4.5.1. A *(spatial) tensor product* of W^*-algebras M and N is defined to be $(M_* \otimes N_*)^*$ $(\cong (M_* \odot N_*)^*)$, and denoted by $M \mathbin{\overline{\otimes}} N$. By definition, its predual is $(M \mathbin{\overline{\otimes}} N)_* = M_* \otimes N_*$. ◁

This tensor product is “spatial” in the following sense.

Theorem 4.5.2. *Let M, N be W^*-algebras and $(\mathcal{H}, \pi_M), (\mathcal{K}, \pi_N)$ be a normal representations. Then the representation $(\mathcal{H} \otimes \mathcal{K}, \pi_M \odot \pi_N)$ of $M \odot N$ extends to a normal representation $(\mathcal{H} \otimes \mathcal{K}, \pi_M \mathbin{\overline{\otimes}} \pi_N)$ of $M \mathbin{\overline{\otimes}} N$. Moreover, if both $(\mathcal{H}, \pi_M)$ and $(\mathcal{K}, \pi_N)$ is faithful (resp. unital), then $(\mathcal{H} \otimes \mathcal{K}, \pi_M \mathbin{\overline{\otimes}} \pi_N)$ is faithful (resp. unital). If both $(\mathcal{H}, \pi_M)$ and $(\mathcal{K}, \pi_N)$ are faithful and unital, then the image $(\pi_M \mathbin{\overline{\otimes}} \pi_N)(M \mathbin{\overline{\otimes}} N)$ coincides with the tensor product of von Neumann algebras $\pi_M(M)$ and $\pi_N(N)$.* ◁

Proof. See [52, Prop. 1.22.11] and [60, Thm. IV.5.2]. ■

Proposition 4.5.3. *Let M, N be W^*-algebras. Consider a chain of the canonical embeddings:*

$$M \odot N \xhookrightarrow{\iota} M \otimes N \xhookrightarrow{\theta} M \mathbin{\overline{\otimes}} N$$

Then $M \odot N$ is ultraweakly dense in $M \mathbin{\overline{\otimes}} N$. ◁

Proof. Note that ι is a norm-dense embedding into a Banach space and θ is an isometry. Hence the norm closure $\mathrm{cl}(M \odot N)$ of $M \odot N$ in $M \mathbin{\overline{\otimes}} N$ coincides with $M \otimes N$ when seen as subspaces of $M \mathbin{\overline{\otimes}} N$. Now recall that θ is a ultraweakly dense embedding. Hence

$$M \mathbin{\overline{\otimes}} N = \text{uw-cl}(M \otimes N) = \text{uw-cl}(\mathrm{cl}(M \odot N)) = \text{uw-cl}(M \odot N) \ ,$$

where the last equality holds because the ultraweak topology is coarser than the norm topology. ■

Let $f\colon M \to M'$ and $g\colon N \to N'$ be normal maps between W^*-algebras. By their normality, we have the bounded maps between their preduals: $f_*\colon M'_* \to M_*, g_*\colon N'_* \to N_*$. Then, a map $f_* \odot g_*\colon M'_* \odot N'_* \to M_* \odot N_*$ is bounded if

$f \odot g \colon M \odot N \to M' \odot N'$ is bounded, because $f_* \odot g_*$ can be considered as a restriction of $(f \odot g)^*$ as in the diagram:

$$\begin{tikzcd}
M'_* \odot N'_* \arrow[r, "f_* \odot g_*"] \arrow[d] & M_* \odot N_* \arrow[d] \\
M'^* \odot N'^* \arrow[r, "f^* \odot g^*"] \arrow[d] & M^* \odot N^* \arrow[d] \\
(M' \odot N')^* \arrow[r, "(f \odot g)^*"] & (M \odot N)^*
\end{tikzcd} \quad ,$$

where vertical arrows are canonical embeddings. On that occasion we can extend $f_* \odot g_*$ to a bounded map $f_* \otimes g_* \colon M'_* \otimes N'_* \to M_* \otimes N_*$, and hence we have a normal map $(f_* \otimes g_*)^* \colon (M_* \otimes N_*)^* \to (M'_* \otimes N'_*)^*$. We denote this map by $f \mathbin{\overline{\otimes}} g \colon M \mathbin{\overline{\otimes}} N \to M' \mathbin{\overline{\otimes}} N'$, which is extension of $f \otimes g$, hence of $f \odot g$, in the sense that the following diagram commutes:

$$\begin{tikzcd}
M \odot N \arrow[r, "f \odot g"] \arrow[d] & M' \otimes N' \arrow[d] \\
M \otimes N \arrow[r, "f \otimes g"] \arrow[d] & M' \otimes N' \arrow[d] \\
M \mathbin{\overline{\otimes}} N \arrow[r, "f \mathbin{\overline{\otimes}} g"] & M' \mathbin{\overline{\otimes}} N'
\end{tikzcd} \quad ,$$

where vertical diagrams are canonical embeddings.

Lemma 4.5.4. *Let M, N be W^*-algebras and $A \subseteq M$ a ultraweakly dense $*$-subalgebra of M. Let $f \colon M \to N$ be a normal map and $f|_A \colon A \to N$ a restriction of f to A. Then, if $f|_A$ is multiplicative (resp. involutive, unital, pre-unital), then f is multiplicative (resp. involutive, unital, pre-unital), too.* ◁

Proof. Assume $f|_A$ is multiplicative (resp. involutive). Then the multiplicativity (resp. involutivity) of f follows from ultraweak continuity of each operation. Note that the multiplication is only separately ultraweakly continuous, but it is sufficient as follows. Let $(x_i)_i$ and $(y_j)_j$ be nets in A with $\text{uw-lim}_i x_i = x$ and $\text{uw-lim}_j y_j = y$. Then

$$\begin{aligned}
f(xy) &= f((\text{uw-lim}_i x_i)(\text{uw-lim}_j y_j)) \\
&= f(\text{uw-lim}_i (x_i \,\text{uw-lim}_j y_j)) \\
&= f(\text{uw-lim}_i \,\text{uw-lim}_j x_i y_j) \\
&= \text{uw-lim}_i \,\text{uw-lim}_j f(x_i y_j) \\
&= \text{uw-lim}_i \,\text{uw-lim}_j f(x_i) f(y_j) \\
&= \text{uw-lim}_i (f(x_i) \,\text{uw-lim}_j f(y_j)) \\
&= (\text{uw-lim}_i f(x_i))(\text{uw-lim}_j f(y_j)) \\
&= f(\text{uw-lim}_i x_i) f(\text{uw-lim}_j y_j) \\
&= f(x) f(y) \quad .
\end{aligned}$$

Note that if $1 \in A$ is the unit in A, then 1 is also the unit in M. This is because for all $x \in M$,

$$\begin{aligned}
1 \cdot x &= 1 \cdot \text{uw-lim}_i x_i \\
&= \text{uw-lim}_i (1 \cdot x_i) \\
&= \text{uw-lim}_i x_i \\
&= x \quad ,
\end{aligned}$$

and similarly $x \cdot 1 = 1$. Hence if $f|_A$ is unital (resp. pre-unital) then f is unital (resp. pre-unital). ■

The next proposition is immediate from Prop. 3.6.9 and the arguments so far.

Proposition 4.5.5. *Let M, M', N, N' be W^*-algebras, and $f\colon M \to M', g\colon N \to N'$ normal $*$-homomorphisms. Then $f \odot g$ extends to normal $*$-homomorphisms $f \mathbin{\overline{\otimes}} g\colon M \mathbin{\overline{\otimes}} N \to M' \mathbin{\overline{\otimes}} N'$.* ■

Moreover, we have a similar result to Thm. 3.6.10 for the tensor products of W^*-algebras.

Theorem 4.5.6. *Let M, M', N, N' be W^*-algebras, and $f\colon M \to M', g\colon N \to N'$ normal completely positive maps. Then $f \odot g$ extends to a normal completely positive map $f \mathbin{\overline{\otimes}} g\colon M \mathbin{\overline{\otimes}} N \to M' \mathbin{\overline{\otimes}} N'$.* ◁

Proof. See [60, Prop. IV.5.13]. This is in fact a proposition for von Neumann algebras, but we can apply it to W^*-algebras via Thm. 4.5.2. ■

Remark 4.5.7. In Prop. 4.5.5 and Thm. 4.5.6, it is easy to see, by Lem. 4.5.4, if both f and g are unital (resp. pre-unital), then $f \mathbin{\overline{\otimes}} g$ is unital (resp. pre-unital), too. ◁

4.6 Tensor products distribute over direct sums

In this section we show that tensor products of W^*-algebras distribute over direct sums. This result seems missing in the literature.

Proposition 4.6.1 ($\overline{\otimes}$ distributes over $\oplus$)**.** *Let M, N, L be W^*-algebras. Then we have*

$$(M \oplus N) \mathbin{\overline{\otimes}} L \cong (M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L) \ . \qquad ◁$$

Proof. Note that their preduals are explicitly given:

$$\begin{aligned} ((M \oplus N) \mathbin{\overline{\otimes}} L)_* &= ((M \oplus N)_* \otimes L_*) = (M_* \oplus^1 N_*) \otimes L_* \\ ((M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L))_* &= (M \mathbin{\overline{\otimes}} L)_* \oplus^1 (N \mathbin{\overline{\otimes}} L)_* = (M_* \otimes L_*) \oplus^1 (N_* \otimes L_*) \end{aligned}$$

Hence we firstly show the preduals are isometrically isomorphic. As usual, there is an algebraic isomorphism:

$$(M_* \oplus N_*) \odot L_* \cong (M_* \odot L_*) \oplus (N_* \odot L_*) \ ,$$

given by

$$(x, y) \otimes z \ \longmapsto\ (x \otimes z, y \otimes z) \ .$$

The norms on them are given by embeddings:

$$\begin{gathered} (M_* \oplus^1 N_*) \odot L_* \hookrightarrow (M \oplus N)^* \odot L^* \hookrightarrow ((M \oplus N) \odot L)^* \cong ((M \oplus N) \otimes L)^* \\ M_* \odot L_* \hookrightarrow M^* \odot L^* \hookrightarrow (M \odot L)^* \cong (M \otimes L)^* \\ N_* \odot L_* \hookrightarrow N^* \odot L^* \hookrightarrow (N \odot L)^* \cong (N \otimes L)^* \ . \end{gathered}$$

Notice that the following diagram commutes:

$$\begin{tikzcd}
(M_* \oplus^1 N_*) \odot L_* \arrow[rr] \arrow[d, "\cong"] & & ((M \oplus N) \otimes L)^* \arrow[d, "\cong"', "\text{Prop. 3.7.1}"] \\
(M_* \odot L_*) \oplus^1 (N_* \odot L_*) \arrow[r] & (M \otimes L)^* \oplus^1 (N \otimes L)^* \arrow[r, "\cong", "\text{Lem. 2.1.10}"'] & ((M \otimes L) \oplus (N \otimes L))^*
\end{tikzcd} \ ,$$

so that the norms on $(M_* \oplus^1 N_*) \odot L_*$ and $(M_* \odot L_*) \oplus^1 (N_* \odot L_*)$ coincide. By Lem. 2.1.15,

$$(M_* \odot L_*) \oplus^1 (N_* \odot L_*) \longrightarrow (M_* \otimes L_*) \oplus^1 (N_* \otimes L_*)$$

is a dense isometry. Hence

$$(M_* \oplus^1 N_*) \otimes L_* \cong (M_* \otimes L_*) \oplus^1 (N_* \otimes L_*)$$

by the uniqueness of completion. Finally, we have to show that the isomorphism is a $*$-isomorphism. By Lem. 4.5.4, it suffices to show that the following diagram commutes:

$$\begin{tikzcd}
(M \oplus N) \otimes L \arrow[r, "\cong"] \arrow[dd] & (M \otimes L) \oplus (N \otimes L) \arrow[d] \\
& (M_* \otimes N_*)^* \oplus (N_* \otimes L_*)^* \arrow[d, "\cong"] \\
((M_* \oplus^1 N_*) \otimes L_*)^* \arrow[r, "\Phi^*", "\cong"'] \arrow[d, equal] & ((M_* \otimes N_*) \oplus^1 (N_* \otimes L_*))^* \arrow[d, equal] \\
(M \oplus N) \mathbin{\overline{\otimes}} L & (M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L)
\end{tikzcd} \ ,$$

where Φ^* is the dual map of

$$\Phi \colon (M_* \otimes L_*) \oplus^1 (N_* \otimes L_*) \xrightarrow{\cong} (M_* \oplus^1 N_*) \otimes L_* \ .$$

To check the commutativity is straightforward. ■

Proposition 4.6.2. *Let M be a W^*-algebra, and let 0 denotes the zero space $\{0\}$, which is a W^*-algebra with the predual 0. Then*

$$0 \mathbin{\overline{\otimes}} M \cong 0 \ .$$

◁

Proof. $0 \mathbin{\overline{\otimes}} M \coloneqq (0 \otimes M_*)^* \cong 0^* \cong 0$ ■

Proposition 4.6.3. *Let M, N, L be W^*-algebras. Then the isomorphism in Prop. 4.6.1:*

$$(M \oplus N) \mathbin{\overline{\otimes}} L \cong (M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L)$$

respects projections and injections of the direct sum. Namely, the following diagrams commute.

$$\begin{tikzcd}
(M \oplus N) \mathbin{\overline{\otimes}} L \arrow[d, "\cong"'] \arrow[dr, "\pi_1 \mathbin{\overline{\otimes}} \mathrm{id}_L"] & \\
(M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L) \arrow[r, "\pi_1"'] & M \mathbin{\overline{\otimes}} L
\end{tikzcd}
\qquad
\begin{tikzcd}
(M \oplus N) \mathbin{\overline{\otimes}} L \arrow[d, "\cong"'] \arrow[dr, "\pi_2 \mathbin{\overline{\otimes}} \mathrm{id}_L"] & \\
(M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L) \arrow[r, "\pi_2"'] & N \mathbin{\overline{\otimes}} L
\end{tikzcd}$$

$$\begin{tikzcd}
& (M \oplus N) \mathbin{\overline{\otimes}} L \arrow[d, "\cong"] \\
M \mathbin{\overline{\otimes}} L \arrow[ur, "\kappa_1 \mathbin{\overline{\otimes}} \mathrm{id}_L"] \arrow[r, "\kappa_1"'] & (M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L)
\end{tikzcd}
\qquad
\begin{tikzcd}
& (M \oplus N) \mathbin{\overline{\otimes}} L \arrow[d, "\cong"] \\
N \mathbin{\overline{\otimes}} L \arrow[ur, "\kappa_2 \mathbin{\overline{\otimes}} \mathrm{id}_L"] \arrow[r, "\kappa_2"'] & (M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L)
\end{tikzcd}$$

◁

Proof. For the first diagram to commute, it suffices to show that the following composites coincide:

$$\begin{tikzcd}
(M \oplus N) \odot L \arrow[r, hook] & (M \oplus N) \mathbin{\overline{\otimes}} L \arrow[d, "\cong"'] \arrow[dr, "\pi_1 \mathbin{\overline{\otimes}} \mathrm{id}_L"] & \\
& (M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L) \arrow[r, "\pi_1"'] & M \mathbin{\overline{\otimes}} L
\end{tikzcd}$$

because of the ultraweak denseness of $(M \oplus N) \odot L$ in $(M \oplus N) \mathbin{\overline{\otimes}} L$. It is shown via the commutativity of the diagram:

$$\begin{tikzcd}
(M \oplus N) \odot L \arrow[d, "\cong"'] \arrow[dr, "\pi_1 \odot \mathrm{id}_L"] & \\
(M \odot L) \oplus (N \odot L) \arrow[r, "\pi_1"'] & M \odot L
\end{tikzcd}$$

The second diagram commutes in the similar way. For the third one, it suffices to show the following composed arrows are equal.

$$\begin{tikzcd}
& & (M \oplus N) \mathbin{\overline{\otimes}} L \arrow[d, "\cong"] \\
M \odot L \arrow[r, hook] & M \mathbin{\overline{\otimes}} L \arrow[ur, "\kappa_1 \mathbin{\overline{\otimes}} \mathrm{id}_L"] \arrow[r, "\kappa_1"'] & (M \mathbin{\overline{\otimes}} L) \oplus (N \mathbin{\overline{\otimes}} L)
\end{tikzcd}$$

It follows via the commutativity of the diagram:

$$\begin{tikzcd}
& (M \oplus N) \odot L \arrow[d, "\cong"] \\
M \odot L \arrow[ur, "\kappa_1 \odot \mathrm{id}_L"] \arrow[r, "\kappa_1"'] & (M \odot L) \oplus (N \odot L)
\end{tikzcd}$$

The fourth diagram commute in the same way. ■

4.7 Categories of W^*-algebras

Definition 4.7.1. We define categories $\mathbf{Wstar}_{\mathrm{X}}$ of W^*-algebras and maps between them of kind denoted by the subscript X. Specifically:

1. $\mathbf{Wstar}_{\text{M-I}}$ is the category of W^*-algebras and normal multiplicative involutive maps (i.e. normal $*$-homomorphisms).

2. $\mathbf{Wstar}_{\text{M-I-U}}$ is the category of W^*-algebras and normal multiplicative involutive unital maps (i.e. normal unital $*$-homomorphisms).

3. $\mathbf{Wstar}_{\text{CP}}$ is the category of W^*-algebras and normal completely positive maps.

4. $\mathbf{Wstar}_{\text{CP-PU}}$ is the category of W^*-algebras and normal completely positive pre-unital maps.

5. $\mathbf{Wstar}_{\text{CP-U}}$ is the category of W^*-algebras and normal completely positive unital maps.

Notice that all categories have the same collection of objects. ◁

Proposition 4.7.2. *There are the following inclusions of categories.*

$$\begin{tikzcd}
\mathbf{Wstar}_{\text{M-I-U}} \arrow[r, hook] \arrow[d, hook] & \mathbf{Wstar}_{\text{M-I}} \arrow[d, hook] & \\
\mathbf{Wstar}_{\text{CP-U}} \arrow[r, hook] & \mathbf{Wstar}_{\text{CP-PU}} \arrow[r, hook] & \mathbf{Wstar}_{\text{CP}}
\end{tikzcd}$$

■

Proposition 4.7.3. *Finite direct sums of W^*-algebras form categorical finite products in all of five categories defined in Def. 4.7.1.* ◁

Proof. By Prop. 3.5.5.1, Prop. 3.5.4.1, Prop. 3.5.7.1 and Prop. 4.4.2. ■

Proposition 4.7.4. *Finite direct sums of W^*-algebras form categorical finite coproducts in* $\mathbf{Wstar}_{\text{CP}}$*. Hence, combined with Prop. 4.7.3,* $\mathbf{Wstar}_{\text{CP}}$ *has finite biproducts.* ◁

Proof. By Prop. 3.5.5.2, Prop. 3.5.4.2, Prop. 3.5.7.2 and Prop. 4.4.2. ■

Remark 4.7.5. The nullary direct sum, i.e. the zero space 0, is the initial object in the categories $\mathbf{Wstar}_{\text{M-I}}$, $\mathbf{Wstar}_{\text{CP-PU}}$ (and $\mathbf{Wstar}_{\text{CP}}$, of course). Hence they have the zero object. ◁

Proposition 4.7.6. *The tensor products of W^*-algebras makes all of five categories defined in Def. 4.7.1 symmetric monoidal categories.* ◁

Proof. The functoriality of tensor products wrt. each category follows from Prop. 4.5.5, Thm. 4.5.6 and Remark 4.5.7. Then it suffices to show the category $\mathbf{Wstar}_{\text{M-I}}$ is symmetric monoidal, since isomorphisms in $\mathbf{Wstar}_{\text{M-I}}$ are isomorphisms in other categories. We will skip the detail because it is routine and known fact (see e.g. [4, §2], [32, Prop. 7.2]) ■

Remark 4.7.7. In an unpublished paper [32], it is claimed that in the category $\mathbf{Wstar}_{\text{M-I-U}}$, for each W^*-algebra M the functor $(-)\mathbin{\overline{\otimes}} M$ has a left adjoint. In other words, the opposite category of $\mathbf{Wstar}_{\text{M-I-U}}$ is *closed* symmetric monoidal. I have not checked the detail yet, but if it is the case then it follows, by the theorem in category theory [37, Thm. V.5.1], that $(-)\mathbin{\overline{\otimes}} M$ preserves any limits in $\mathbf{Wstar}_{\text{M-I-U}}$. Therefore Prop. 4.6.1 and Prop. 4.6.2 are abstractly implied. ◁

4.8 Complete partial orders in W^*-algebras

Monotone closedness is one of the unique properties of W^*-algebras; C^*-algebras do not possess this property in general (Prop. 4.8.7).

Definition 4.8.1 (Monotone closed)**.** A C^*-algebra A is *monotone closed* if every norm-bounded directed subset of $\mathcal{SA}(A)$ has the supremum in $\mathcal{SA}(A)$. ◁

Theorem 4.8.2. *Every W^*-algebra M is monotone closed. Moreover, every norm-bounded monotone net $(x_i)_i$ in $\mathcal{SA}(M)$ converges ultraweakly to $\sup_i x_i$.* ◁

Proof. See [52, Lem. 1.7.4], [60, Thm. III.3.16], or [14, Prop. 43.1]. ■

Lemma 4.8.3. *Let A be a C^*-algebra and $S \subseteq \mathcal{SA}(A)$ be a set of self-adjoint elements of A. Then S is norm-bounded if and only if S is order-theoretically bounded in $\mathcal{SA}(A)$.* ◁

Proof. Assume S is norm-bounded, that is, there exists $m \in \mathbb{R}^+$ such that $\|x\| \leq m$ for all $x \in S$. Then, by Prop. 3.1.22, we have $-m1 \leq x \leq m1$ for all $x \in S$. Hence S is order-theoretically bounded in $\mathcal{SA}(A)$.

Conversely, assume S is order-theoretically bounded in $\mathcal{SA}(A)$, that is, there exists $l, u \in \mathcal{SA}(A)$ such that $l \leq x \leq u$ for all $x \in S$. By Prop. 3.1.22, we have $-\|l\|1 \leq l$ and $u \leq \|u\|1$. Hence for all $x \in S$,

$$-m1 \leq l \leq x \leq u \leq m1 \ ,$$

where $m := \max(\|l\|, \|u\|)$. It follows by Prop. 3.1.22 that $\|x\| \leq m$ for all $x \in S$, i.e. S is norm-bounded. ∎

Proposition 4.8.4. *Let A be a C^*-algebra. The following are equivalent.*

1. *A is monotone closed.*
2. *$\mathcal{SA}(A)$ is a bdcpo.*
3. *A^+ is a bdcpo.*
4. *$\mathcal{E}f(A) = [0,1]_A$ is a dcpo.* ◁

Proof. $2 \implies 3$ and $3 \implies 4$ are trivial.

$2 \implies 1$ follows from Lem. 4.8.3, while $1 \implies 2$ also follows from Lem. 4.8.3 with Lem. 2.2.15.

$3 \implies 1$: Let $D \subseteq \mathcal{SA}(A)$ be a norm-bounded directed subset, and by Lem. 4.8.3 assume $l \leq d \leq u$ for all $d \in D$, for some $l, d \in \mathcal{SA}(A)$. Let $D' := D - l = \{d - l \mid d \in D\}$. Then D' is a directed subset of A^+, bounded above by $u - l$. Hence we have $\sup D'$. It is easy to see $l + \sup D'$ is the supremum of D.

$4 \implies 3$: For a given directed subset $D \subseteq A^+$ bounded above by $u \geq 0$, let $D' := \|u\|^{-1} D = \{\|u\|^{-1} d \mid d \in D\}$. Note that we may assume $u \neq 0$. Because $u \leq \|u\|1$ for any $u \in A^+$, D' is a directed subset of $[0,1]_A$. Hence we have $\sup D'$. It is easy to see $\|u\| \sup D'$ is the supremum of D. ∎

Corollary 4.8.5. *Let M be a W^*-algebra. Then the following hold.*

1. *$\mathcal{SA}(M)$ is a bdcpo.*
2. *M^+ is a (pointed) bdcpo.*
3. *$\mathcal{E}f(M) = [0,1]_M$ is a (pointed) dcpo.* ◁

Proof. By Thm. 4.8.2 and Prop. 4.8.4. ∎

Normality (i.e. ultraweak continuity) of maps between W^*-algebras can be order-theoretically characterized.

Theorem 4.8.6. *Let $f\colon M \to N$ be a positive map between W^*-algebra. The following are equivalent.*

1. *f is normal, i.e. ultraweakly continuous.*
2. *f preserves the supremum of every norm-bounded directed subset of $\mathcal{SA}(M)$.*
3. *The restriction of f to $\mathcal{SA}(M)$, i.e. $f|_{\mathcal{SA}(M)}\colon \mathcal{SA}(M) \to \mathcal{SA}(N)$, is Scott-continuous.*

4. *The restriction of f to M^+, i.e. $f|_{M^+}: M^+ \to N^+$, is Scott-continuous.* ◁

Proof. 1 $\iff$ 2 is shown in [14, Cor. 46.5] or [7, Prop. III.2.2.2]. The equivalence of 2, 3 and 4 is straightforward. ■

In general, C^*-algebras are not monotone closed. We will give a counterexample.

Proposition 4.8.7. *There exists a C^*-algebra A such that $\mathcal{E}f(A)$ is not an ωcpo, and hence A is not monotone closed.* ◁

Proof. The (real, closed) unit interval $[0,1]$ is a compact Hausdorff space. Then a space $C([0,1])$ of complex-valued continuous functions on the unit interval is a (commutative) C^*-algebra. The positivity in $C([0,1])$ as a C^*-algebra is the pointwise positivity, and the order is the pointwise order. The unit is the constant function on 1. Define a monotone sequence $(f_n)_{n=1}^\infty$ in $\mathcal{E}f(C([0,1]))$ by:

$$f_n(x) = \begin{cases} 0 & \left(0 \le x < \dfrac{1}{2}\right) \\ 2nx - n & \left(\dfrac{1}{2} \le x < \dfrac{1}{2} + \dfrac{1}{2n}\right) \\ 1 & \left(\dfrac{1}{2} + \dfrac{1}{2n} \le x \le 1\right) \end{cases}$$

Assume the monotone sequence $(f_n)_{n=1}^\infty$ has the supremum $g \in C([0,1])$. Then an easy calculation shows $g(x) = 0$ for $[0,1/2)$ and $g(x) = 1$ for $(1/2,1]$, so that g is never continuous. Hence the assumption is false: $(f_n)_{n=1}^\infty$ do not have the supremum. ■

4.9 Enrichment of categories of W^*-algebras

In the previous section, we have showed that W^*-algebras have the (b)dcpo structure. In the present section, we will show such structures are "lifted" to the set of morphisms. In other words, several categories of W^*-algebras are enriched over suitable cpo structures.

Proposition 4.9.1. *Let M be a W^*-algebra. Let $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ be a bounded monotone net in $\mathcal{SA}(M)$. Then the following hold.*

1. *If $x_i \le y_i$ for all $i \in I$, then $\sup_{i\in I} x_i \le \sup_{i\in I} y_i$.*
2. $\sup_{i\in I}(x_i + y_i) = \sup_{i\in I} x_i + \sup_{i\in I} y_i$ ◁

Proof. 1. Assume $x_i \le y_i$ for all $i \in I$. Then $x_i \le y_i \le \sup_i y_i$ for all $i \in I$. It follows that $\sup_i x_i \le \sup_i y_i$.

2. Because $x_i \le \sup_i x_i$ and $y_i \le \sup_i y_i$, we have $x_i + y_i \le \sup_i x_i + \sup_i y_i$ for each $i \in I$. Hence $\sup_i(x_i + y_i) \le \sup_i x_i + \sup_i y_i$. Conversely, because $x_i + y_i \le \sup_i x_i + y_i$, we have $\sup_i(x_i + y_i) \le \sup_i(\sup_i x_i + y_i) = \sup_i x_i + \sup_i y_i$. Therefore $\sup_i(x_i + y_i) = \sup_i x_i + \sup_i y_i$. ■

Proposition 4.9.2. *Let M be a W^*-algebra. The addition $+: M \times M \to M$ and the scalar multiplication $\cdot: \mathbb{C} \times M \to M$ are continuous wrt. the ultraweak topology.* ◁

Proof. Recall that the ultraweak topology is defined as weak* topology. The claim is general property of weak* topology. See e.g. [13, Chap. IV–V] or [47, §2.4]. ■

Lemma 4.9.3 ([52, Lem. 1.7.1]). *Let M be a W^*-algebra. Then both $\mathcal{SA}(M)$ and M^+ are ultraweakly closed.* ■

Definition 4.9.4. Let M, N be W^*-algebras. Recall $\mathbf{Wstar}_{\mathrm{CP}}(M, N)$ is the set of normal completely positive maps from M to N. We define a relation $\sqsubseteq \subseteq (\mathbf{Wstar}_{\mathrm{CP}}(M, N))^2$ by

$$\begin{aligned} f \sqsubseteq g &\overset{\text{def}}{\iff} g - f \text{ is completely positive} \\ &\iff \mathcal{M}_n(g - f) = \mathcal{M}_n(g) - \mathcal{M}_n(f) \text{ is positive for all } n \in \mathbb{N} \\ &\iff \mathcal{M}_n(f)(x) \leq \mathcal{M}_n(g)(x) \text{ for all } n \in \mathbb{N} \text{ and } x \in (\mathcal{M}_n(M))^+ \ . \end{aligned}$$

It is clear that $\sqsubseteq$ is a partial order. ◁

Proposition 4.9.5. *Let M, N be W^*-algebras and $(f_i)_{i \in I}$ a norm-bounded monotone net in $\mathbf{Wstar}_{\mathrm{CP}}(M, N)$ with the order $\sqsubseteq$. We define a map $f \colon M^+ \to N^+$ by*

$$f(x) \coloneqq \sup_{i \in I} f_i(x) = \operatorname*{uw-lim}_{i \in I} f_i(x)$$

for $x \in M^+$. Note here that $(f_i(x))_{i \in I}$ is a norm-bounded monotone net in N^+, and hence $\sup_{i \in I} f_i(x)$ exists by Thm. 4.8.2. Then

1. *f is $\mathbb{R}^+$-linear, hence it extends to ($\mathbb{C}$-)linear map $f \colon M \to N$.*
2. *f is normal.* ◁

Proof. 1. For $x, y \in M^+$,

$$\begin{aligned} f(x + y) &= \text{uw-lim}_i (f_i(x + y)) \\ &= \text{uw-lim}_i (f_i(x) + f_i(y)) \\ &= \text{uw-lim}_i f_i(x) + \text{uw-lim}_i f_i(y) \\ &\qquad \text{by ultraweak continuity of addition} \\ &= f(x) + f(y) \ . \end{aligned}$$

For $x \in M^+$ and $t \geq 0$,

$$\begin{aligned} f(tx) &= \text{uw-lim}_i (f_i(tx)) \\ &= \text{uw-lim}_i (t f_i(x)) \\ &= t \, \text{uw-lim}_i f_i(x) \\ &\qquad \text{by ultraweak continuity of scalar multiplication} \\ &= t f(x) \ . \end{aligned}$$

2. Let $(x_j)_{j \in J}$ be a bounded monotone net in M^+. Then

$$\begin{aligned} f\Big(\sup_{j \in J} x_j\Big) &= \sup_{i \in I} f_i\Big(\sup_{j \in J} x_j\Big) \\ &= \sup_{i \in I}\Big(\sup_{j \in J} f_i(x_j)\Big) \ , \end{aligned}$$

where the last equality follows from the normality of each f_i. Note that $(f_i(x_j))_{(i,j)\in I\times J}$ is also a bounded monotone net, because for each $i \le i'$ and $j \le j'$, $f_i(x_j) \le f_i(x_{j'}) \le f_{i'}(x_{j'})$. Hence, by Prop. 2.2.13,

$$\begin{aligned}\sup_{i\in I}\Big(\sup_{j\in J} f_i(x_j)\Big) &= \sup_{(i,j)\in I\times J} f_i(x_j)\\ &= \sup_{j\in J}\Big(\sup_{i\in I} f_i(x_j)\Big)\\ &= \sup_{j\in J} f(x_j) \ .\end{aligned}$$

Therefore f is normal by Thm. 4.8.6. ■

Lemma 4.9.6. *In the setting of Prop. 4.9.5, for any $x \in M$, a (not necessary monotone) net $(f_i(x))_{i\in I}$ converges ultraweakly to $f(x)$.* ◁

Proof. Note that any $x \in M$ can be decomposed as

$$x = x_1 - x_2 + ix_3 - ix_4 \ ,$$

where $x_1, x_2, x_3, x_4 \in M^+$ (see e.g. [52, Def. 1.4.3]). Then for each x_k ($k \in \{1,2,3,4\}$), $f(x_k) = \sup_j f_j(x_k) = \text{uw-lim}_j f_j(x_k)$. By ultraweak continuity of the addition and the scalar multiplication,

$$\begin{aligned}&f(x)\\ &= f(x_1) - f(x_2) + if(x_3) - if(x_4)\\ &= \text{uw-lim}_j f_j(x_1) - \text{uw-lim}_j f_j(x_2) + i\,\text{uw-lim}_j f_j(x_3) - i\,\text{uw-lim}_j f_j(x_4)\\ &= \text{uw-lim}_j (f_j(x_1) - f_j(x_2) + if_j(x_3) - if_j(x_4))\\ &= \text{uw-lim}_j f_j(x) \ .\end{aligned}$$

■

Theorem 4.9.7. *In the setting of Prop. 4.9.5, $f\colon M \to N$ is completely positive.* ◁

Proof. Consider $\mathcal{M}_n(f)\colon \mathcal{M}_n(M) \to \mathcal{M}_n(N)$. For $[x_{kl}]_{kl} \in \mathcal{M}_n(M)^+$,

$$\begin{aligned}\mathcal{M}_n(f)([x_{kl}]_{kl}) &= [f(x_{kl})]_{kl}\\ &= [\text{uw-}\lim_i f_i(x_{kl})]_{kl} && \text{by Lem. 4.9.6}\\ &= \text{uw-}\lim_i [f_i(x_{kl})]_{kl} && \text{by Prop. 4.3.3}\\ &= \text{uw-}\lim_i \mathcal{M}_n(f_i)([x_{kl}]_{kl})\\ &\ge 0 \ ,\end{aligned}$$

where the last inequality holds because each f_i is completely positive, hence each $\mathcal{M}_n(f_i)([x_{kl}]_{kl})$ is positive and by Lem. 4.9.3. ■

Theorem 4.9.8. *In the setting of Prop. 4.9.5, f is the supremum of $(f_i)_i$ in* $\mathbf{Wstar}_{\mathrm{CP}}(M,N)$*:*

$$f = \bigsqcup_{i\in I} f_i \ .$$

◁

Proof. First of all, $f \in \mathbf{Wstar}_{\mathrm{CP}}(M, N)$ by Prop. 4.9.5 and Prop. 4.9.7. Recall that, in the proof of Prop. 4.9.7, for each $[x_{kl}]_{kl} \in (\mathcal{M}_n(M))^+$

$$\begin{aligned}\mathcal{M}_n(f)([x_{kl}]_{kl}) &= \text{uw-lim}_i\, \mathcal{M}_n(f_i)([x_{kl}]_{kl}) \\ &= \sup_i \mathcal{M}_n(f_i)([x_{kl}]_{kl}) \ ,\end{aligned}$$

because a net $(\mathcal{M}_n(f_i)([x_{kl}]_{kl}))_i$ is monotone. Hence $\mathcal{M}_n(f_i)([x_{kl}]_{kl}) \leq \mathcal{M}_n(f)([x_{kl}]_{kl})$, so that $f_i \sqsubseteq f$ for all $i \in I$. Now it is easy to see f is the supremum of $(f_i)_{i \in I}$ because f is defined as the pointwise supremum. ■

Corollary 4.9.9. *Let M and N be W^*-algebras.*

1. $\mathbf{Wstar}_{\mathrm{CP}}(M, N)$ *is a pointed bdcpo.*
2. $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}(M, N)$ *is a pointed dcpo.* ◁

Proof. 1. Suppose $(f_i)_i$ is a monotone net in $\mathbf{Wstar}_{\mathrm{CP}}(M, N)$ bounded above, say by g. Then $(f_i)_i$ is norm-bounded since $0 \leq f_i(1) \leq g(1)$ implies $\|f_i\| = \|f_i(1)\| \leq \|g(1)\|$ by Prop. 3.2.6 and Cor. 3.1.23. Therefore $\bigsqcup_i f_i$ exists by Thm. 4.9.8. It is pointed because the zero map $0 \in \mathbf{Wstar}_{\mathrm{CP}}(M, N)$ is the least element.

2. Suppose $(f_i)_i$ is a monotone net in $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}(M, N)$. It is norm-bounded by Prop. 3.2.11, hence $\bigsqcup_i f_i$ exists in $\mathbf{Wstar}_{\mathrm{CP}}(M, N)$. Then $(\bigsqcup_i f_i)(1) = \sup_i f_i(1) \leq 1$ since $f_i(1) \leq 1$ for each $i \in I$. Therefore $\bigsqcup_i f_i \in \mathbf{Wstar}_{\mathrm{CP\text{-}PU}}(M, N)$. It is pointed because $0 \in \mathbf{Wstar}_{\mathrm{CP\text{-}PU}}(M, N)$. ■

In what follows, we will show that $\mathbf{Wstar}_{\mathrm{CP}}$ and $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$ are enriched over $\mathbf{Bdcppo}_\perp$ and $\mathbf{Dcppo}_\perp$ respectively, and their categorical structures such as categorical and monoidal products are also suitably enriched.

Proposition 4.9.10. *Let M, N, L be W^*-algebras. Let $f, f' \in \mathbf{Wstar}_{\mathrm{CP}}(M, N)$ and $g, g' \in \mathbf{Wstar}_{\mathrm{CP}}(N, L)$.*

$$f \sqsubseteq f' \implies g \circ f \sqsubseteq g \circ f' \tag{4.5}$$

$$g \sqsubseteq g' \implies g \circ f \sqsubseteq g' \circ f \ . \tag{4.6}$$

Moreover, for norm-bounded monotone nets $(f_i)_i$ in $\mathbf{Wstar}_{\mathrm{CP}}(M, N)$ and $(g_i)_i$ in $\mathbf{Wstar}_{\mathrm{CP}}(N, L)$, we have

$$\Big(\bigsqcup\nolimits_i g_i\Big) \circ f = \bigsqcup\nolimits_i (g_i \circ f) \tag{4.7}$$

$$g \circ \Big(\bigsqcup\nolimits_i f_i\Big) = \bigsqcup\nolimits_i (g \circ f_i) \ . \tag{4.8}$$

◁

Proof. (4.5): For each $n \in \mathbb{N}$, let $x \in \mathcal{M}_n(M)^+$ be a positive element. Then $0 \leq \mathcal{M}_n(f)(x) \leq \mathcal{M}_n(f')(x)$ by $f \sqsubseteq f'$ and the complete positivity of f, f'. Since g is completely positive, too, we have $\mathcal{M}_n(g)(\mathcal{M}_n(f)(x)) \leq \mathcal{M}_n(g)(\mathcal{M}_n(f')(x))$, i.e. $\mathcal{M}_n(g \circ f)(x) \leq \mathcal{M}_n(g \circ f')(x)$.

(4.6): Firstly $\mathcal{M}_n(f)(x)$ is positive because f is completely positive. By $g \sqsubseteq g'$, we have $\mathcal{M}_n(g)(\mathcal{M}_n(f)(x)) \leq \mathcal{M}_n(g')(\mathcal{M}_n(f)(x))$, i.e. $\mathcal{M}_n(g \circ f)(x) \leq \mathcal{M}_n(g' \circ f)(x)$.

(4.7): Recall that $\bigsqcup_i f_i$ is defined by pointwise supremum $(\bigsqcup_i f_i)(x) = \sup_i f_i(x)$ for positive x. Then

$$\begin{aligned}\Big(\Big(\bigsqcup\nolimits_i g_i\Big) \circ f\Big)(x) &= \Big(\bigsqcup\nolimits_i g_i\Big)(f(x)) \\ &= \sup\nolimits_i g_i(f(x)) \\ &= \sup\nolimits_i (g_i \circ f)(x) \\ &= \Big(\bigsqcup\nolimits_i (g_i \circ f)\Big)(x)\end{aligned}$$

for all $x \in M^+$.

(4.8):

$$\begin{aligned}\Big(g \circ \Big(\bigsqcup\nolimits_i f_i\Big)\Big)(x) &= g(\sup\nolimits_i f_i(x)) \\ &= \sup\nolimits_i g(f_i(x)) && \text{since } g \text{ is normal} \\ &= \sup\nolimits_i (g \circ f_i)(x) \\ &= \Big(\bigsqcup\nolimits_i (g \circ f_i)\Big)(x)\end{aligned}$$

for all $x \in M^+$. ∎

Corollary 4.9.11. *Let M, N, L be W^*-algebras. The composition maps*

$$\circ\colon \mathbf{Wstar}_{\mathrm{CP}}(N, L) \times \mathbf{Wstar}_{\mathrm{CP}}(M, N) \longrightarrow \mathbf{Wstar}_{\mathrm{CP}}(M, L)$$

and

$$\circ\colon \mathbf{Wstar}_{\text{CP-PU}}(N, L) \times \mathbf{Wstar}_{\text{CP-PU}}(M, N) \longrightarrow \mathbf{Wstar}_{\text{CP-PU}}(M, L)$$

are strict and Scott-continuous. Hence $\mathbf{Wstar}_{\mathrm{CP}}$ is $\mathbf{Bdcppo}_{\perp}$-enriched and $\mathbf{Wstar}_{\text{CP-PU}}$ is $\mathbf{Dcppo}_{\perp}$-enriched. ◁

Proof. Proposition 4.9.10 shows the Scott-continuity (via Prop. 2.2.14). The strictness is immediate. ∎

Proposition 4.9.12. *Let M, N be W^*-algebras. Let $f, f', g, g' \in \mathbf{Wstar}_{\mathrm{CP}}(M, N)$. Then*

$$\begin{aligned} f \sqsubseteq f' &\implies f + g \sqsubseteq f' + g \\ g \sqsubseteq g' &\implies f + g \sqsubseteq f + g' \ . \end{aligned} \tag{4.9}$$

Moreover, for norm-bounded monotone nets $(f_i)_i$ and $(g_i)_i$ in $\mathbf{Wstar}_{\mathrm{CP}}(M, N)$, we have

$$\begin{aligned} \bigsqcup\nolimits_i (f_i + g) &= \Big(\bigsqcup\nolimits_i f_i\Big) + g \\ \bigsqcup\nolimits_i (f + g_i) &= f + \Big(\bigsqcup\nolimits_i g_i\Big) \ . \end{aligned} \tag{4.10}$$

◁

Proof. We show only (4.9) and (4.10) because the addition is symmetric.

(4.9): For any $n \in \mathbb{N}$ and $x \in (\mathcal{M}_n(M))^+$,

$$\begin{aligned}\mathcal{M}_n(f+g)(x) &= \mathcal{M}_n(f)(x) + \mathcal{M}_n(g)(x) \\ &\leq \mathcal{M}_n(f')(x) + \mathcal{M}_n(g)(x) && \text{by } f \sqsubseteq f' \\ &= \mathcal{M}_n(f'+g)(x) \ .\end{aligned}$$

(4.10): For each $x \in M$,

$$\begin{aligned}
(\bigsqcup_i (f_i + g))(x) &= \text{uw-lim}_i (f_i(x) + g(x)) \\
&= (\text{uw-lim}_i\, f_i(x)) + g(x) \\
&\qquad \text{since the addition is ultraweakly continuous} \\
&= (\bigsqcup_i f_i)(x) + g(x) \\
&= ((\bigsqcup_i f_i) + g)(x) \ .
\end{aligned}$$

■

Proposition 4.9.13. *Let M, N be W^*-algebras. and $f, f' \in \mathbf{Wstar}_{\mathrm{CP}}(M, N)$. Then for each non-negative real $t \in \mathbb{R}^+$,*

$$f \sqsubseteq f' \implies tf \sqsubseteq tf' \ .$$

Moreover, for a norm-bounded monotone net $(f_i)_i$ in $\mathbf{Wstar}_{\mathrm{CP}}(M, N)$, we have

$$\bigsqcup_i (tf_i) = t(\bigsqcup_i f_i) \ .$$

◁

Proof. For any $n \in \mathbb{N}$ and $x \in (\mathcal{M}_n(M))^+$,

$$\begin{aligned}
\mathcal{M}_n(tf)(x) &= t\mathcal{M}_n(f)(x) \\
&\le t\mathcal{M}_n(f')(x) && \text{by } f \sqsubseteq f' \\
&= \mathcal{M}_n(tf')(x) \ .
\end{aligned}$$

Hence $f \sqsubseteq f'$ implies $tf \sqsubseteq tf'$.

For each $x \in M$,

$$\begin{aligned}
(\bigsqcup_i (tf_i))(x) &= \text{uw-lim}_i (tf_i(x)) \\
&= t(\text{uw-lim}_i\, f_i(x)) \\
&\qquad \text{since the scalar multiplication is ultraweakly continuous} \\
&= t(\bigsqcup_i f_i)(x) \\
&= (t(\bigsqcup_i f_i))(x) \ .
\end{aligned}$$

Hence $\bigsqcup_i (tf_i) = t(\bigsqcup_i f_i)$. ■

Proposition 4.9.14. *Let M, N, L be W^*-algebras and $f, f' \in \mathbf{Wstar}_{\mathrm{CP}}(L, M)$, $g, g' \in \mathbf{Wstar}_{\mathrm{CP}}(L, N)$.*

$$\begin{aligned}
f \sqsubseteq f' &\implies \langle f, g\rangle \sqsubseteq \langle f', g\rangle \\
g \sqsubseteq g' &\implies \langle f, g\rangle \sqsubseteq \langle f, g'\rangle \ .
\end{aligned}$$

Moreover, for a bounded monotone net $(f_i)_i$ in $\mathbf{Wstar}_{\mathrm{CP}}(L, M)$ and $(g_i)_i$ in $\mathbf{Wstar}_{\mathrm{CP}}(L, N)$ we have

$$\begin{aligned}
\bigsqcup_i \langle f_i, g\rangle &= \langle \bigsqcup_i f_i, g\rangle \\
\bigsqcup_i \langle f, g_i\rangle &= \langle f, \bigsqcup_i g_i\rangle \ .
\end{aligned}$$

◁

Proof. Assume $f \sqsubseteq f'$. Note that $\langle f, g\rangle = \kappa_1 \circ f + \kappa_2 \circ g$, and the composition and addition of maps preserves the order $\sqsubseteq$. Hence

$$\langle f, g\rangle = \kappa_1 \circ f + \kappa_2 \circ g \sqsubseteq \kappa_1 \circ f' + \kappa_2 \circ g = \langle f', g\rangle \ .$$

Similarly $g \sqsubseteq g'$ implies $\langle f, g\rangle \sqsubseteq \langle f, g'\rangle$.

We will show the latter part. Let $(f_i \colon L \to M)_i$ be a bounded monotone net. Via the equation $\langle f, g\rangle = \kappa_1 \circ f + \kappa_2 \circ g$ again,

$$\begin{aligned}\langle \textstyle\bigsqcup_i f_i, g\rangle &= \kappa_1 \circ (\textstyle\bigsqcup_i f_i) + \kappa_2 \circ g \\ &= \textstyle\bigsqcup_i (\kappa_1 \circ f_i) + \kappa_2 \circ g \\ &= \textstyle\bigsqcup_i (\kappa_1 \circ f_i + \kappa_2 \circ g) \\ &= \textstyle\bigsqcup_i \langle f_i, g\rangle \ .\end{aligned}$$

In a similar way $\bigsqcup_i \langle f, g_i\rangle = \langle f, \bigsqcup_i g_i\rangle$ is showed. ■

Corollary 4.9.15. *Let M, N, L be W^*-algebras. The canonical isomorphisms*

$$\mathbf{Wstar}_{\mathrm{CP}}(L, M) \times \mathbf{Wstar}_{\mathrm{CP}}(L, N) \cong \mathbf{Wstar}_{\mathrm{CP}}(L, M \oplus N)$$

and

$$\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}(L, M) \times \mathbf{Wstar}_{\mathrm{CP\text{-}PU}}(L, N) \cong \mathbf{Wstar}_{\mathrm{CP\text{-}PU}}(L, M \oplus N)$$

are strict and Scott-continuous in both directions. Consequently, $\mathbf{Wstar}_{\mathrm{CP}}$ and $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$ have $\mathbf{Bdcppo}_\perp$-enriched and $\mathbf{Dcppo}_\perp$-enriched binary products respectively. The nullary product, i.e. the terminal object, is trivially enriched in both categories, so that they have enriched finite products. ◁

Proof. Proposition 4.9.14 shows the direction $\mathbf{Wstar}_{\mathrm{CP}}(L, M) \times \mathbf{Wstar}_{\mathrm{CP}}(L, N) \to \mathbf{Wstar}_{\mathrm{CP}}(L, M \oplus N)$ is Scott-continuous, while the other direction is obtained by composing the projections, so that it is also Scott-continuous. The strictness is easy. The case of $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$ is showed for the same reason. ■

Proposition 4.9.16. *Let M, M', N, N' be W^*-algebras, and $f, f' \in \mathbf{Wstar}_{\mathrm{CP}}(M, M')$, $g, g' \in \mathbf{Wstar}_{\mathrm{CP}}(N, N')$. Then*

$$\begin{aligned} f \sqsubseteq f' &\implies f \mathbin{\overline{\otimes}} g \sqsubseteq f' \mathbin{\overline{\otimes}} g \\ g \sqsubseteq g' &\implies f \mathbin{\overline{\otimes}} g \sqsubseteq f \mathbin{\overline{\otimes}} g' \ . \end{aligned}$$ ◁

Proof. Assume $f \sqsubseteq f'$. By definition $f' - f$ is completely positive. Then $(f' - f) \mathbin{\overline{\otimes}} g = f' \mathbin{\overline{\otimes}} g - f \mathbin{\overline{\otimes}} g$ is completely positive, too, by Thm. 4.5.6. Hence $f \mathbin{\overline{\otimes}} g \sqsubseteq f' \mathbin{\overline{\otimes}} g$. The other one is showed in a similar way. ■

Theorem 4.9.17. *Let M, M', N, N' be W^*-algebras. Then we have*

$$f \mathbin{\overline{\otimes}} \Big(\textstyle\bigsqcup_i g_i\Big) = \textstyle\bigsqcup_i (f \mathbin{\overline{\otimes}} g_i)$$

for $f \in \mathbf{Wstar}_{\mathrm{CP}}(M, M')$ and a norm-bounded monotone net $(g_i)_i$ in $\mathbf{Wstar}_{\mathrm{CP}}(N, N')$, and

$$\Big(\textstyle\bigsqcup_i f_i\Big) \mathbin{\overline{\otimes}} g = \textstyle\bigsqcup_i (f_i \mathbin{\overline{\otimes}} g)$$

for a norm-bounded monotone net $(f_i)_i$ in $\mathbf{Wstar}_{\mathrm{CP}}(M, M')$ and $g \in \mathbf{Wstar}_{\mathrm{CP}}(N, N')$.

◁

Proof. Because the tensor product is symmetric, we show only the first equation. Since both $f \mathbin{\overline{\otimes}} \left(\bigsqcup_i g_i\right)$ and $\bigsqcup_i (f \mathbin{\overline{\otimes}} g_i)$ are normal, i.e. ultraweakly continuous, it suffices to show they coincides on ultraweakly dense subset $M \odot N$ of the domain $M \mathbin{\overline{\otimes}} N$ (by Prop. 4.5.3). For this, it is sufficient to show $(f \mathbin{\overline{\otimes}} \left(\bigsqcup_i g_i\right))(x \otimes y) = (\bigsqcup_i (f \mathbin{\overline{\otimes}} g_i))(x \otimes y)$ for all $x \in M$ and $y \in N$. By Lem. 4.9.6, what we will show is:

$$f(x) \mathbin{\overline{\otimes}} (\operatorname*{uw-lim}_i g_i(y)) = \operatorname*{uw-lim}_i (f(x) \mathbin{\overline{\otimes}} g_i(y)) \ .$$

Let $b := \text{uw-lim}_i\, g_i(y)$. Below, we assume M'_* and N'_* are embedded into M'^* and N'^* respectively. For $\varphi \otimes \psi \in M'_* \odot N'_* \subseteq M'_* \otimes N'_* = (M' \mathbin{\overline{\otimes}} N')_*$,

$$\begin{aligned}(\varphi \otimes \psi)(f(x) \otimes g_i(y)) &= \varphi(f(x)) \cdot \psi(g_i(y)) \\ &\to \varphi(f(x)) \cdot \psi(b) \\ &= (\varphi \otimes \psi)(f(x) \otimes b) \ ,\end{aligned}$$

because $\psi(g_i(y)) \to \psi(b)$. Hence for all $\chi \in M'_* \odot N'_*$, we have $\chi(f(x) \otimes g_i(y)) \to \chi(f(x) \otimes b)$. Let $(\chi_j)_j$ be a net in $M'_* \odot N'_*$ (norm-)convergent to $\xi \in M'_* \otimes N'_*$. Note that the norm comes from the dual norm of the spatial C^*-norm:

$$M'_* \odot N'_* \subseteq M'^* \odot N'^* \subseteq (M' \odot N')^* \cong (M' \otimes N')^* \ .$$

Then

$$\begin{aligned}
&|\xi(f(x) \otimes b) - \xi(f(x) \otimes g_i(y))| \\
&\le |\xi(f(x) \otimes b) - \chi_j(f(x) \otimes b)| \\
&\qquad + |\chi_j(f(x) \otimes b) - \chi_j(f(x) \otimes g_i(y))| \\
&\qquad\qquad + |\chi_j(f(x) \otimes g_i(y)) - \xi(f(x) \otimes g_i(y))| \\
&\le \|\xi - \chi_j\| \|f(x) \otimes b\| \\
&\qquad + |\chi_j(f(x) \otimes b) - \chi_j(f(x) \otimes g_i(y))| \\
&\qquad\qquad + \|\xi - \chi_j\| \|f(x) \otimes g_i(y)\| \\
&= \|\xi - \chi_j\| \|f(x)\| \|b\| \\
&\qquad + |\chi_j(f(x) \otimes b) - \chi_j(f(x) \otimes g_i(y))| \\
&\qquad\qquad + \|\xi - \chi_j\| \|f(x)\| \|g_i(y)\| \\
&= |\chi_j(f(x) \otimes b) - \chi_j(f(x) \otimes g_i(y))| + \|\xi - \chi_j\| \|f(x)\| (\|b\| + \|g_i(y)\|) \ .
\end{aligned}$$

Because $\sup_i \|g_i(y)\| < \infty$ and $\chi_j \to \xi$, for large enough j we have

$$|\xi(f(x) \otimes b) - \xi(f(x) \otimes g_i(y))| < |\chi_j(f(x) \otimes b) - \chi_j(f(x) \otimes g_i(y))| + \varepsilon$$

for arbitrary $\varepsilon > 0$. Then, since $\chi_j(f(x) \otimes g_i(y)) \to \chi_j(f(x) \otimes b)$, we have

$$|\xi(f(x) \otimes b) - \xi(f(x) \otimes g_i(y))| < 2\varepsilon$$

for sufficiently large i. It follows that

$$\operatorname*{uw-lim}_i (f(x) \mathbin{\overline{\otimes}} g_i(y)) = f(x) \mathbin{\overline{\otimes}} b \ .$$

■

Corollary 4.9.18. *Let M, M', N, N' be W^*-algebras. The maps*

$$\begin{aligned}
\overline{\otimes}\colon \mathbf{Wstar}_{\mathrm{CP}}(M, M') \times \mathbf{Wstar}_{\mathrm{CP}}(N, N') &\longrightarrow \mathbf{Wstar}_{\mathrm{CP}}(M \mathbin{\overline{\otimes}} N, M' \mathbin{\overline{\otimes}} N') \\
\overline{\otimes}\colon \mathbf{Wstar}_{\text{CP-PU}}(M, M') \times \mathbf{Wstar}_{\text{CP-PU}}(N, N') &\longrightarrow \mathbf{Wstar}_{\text{CP-PU}}(M \mathbin{\overline{\otimes}} N, M' \mathbin{\overline{\otimes}} N')
\end{aligned}$$

are strict and Scott-continuous. Consequently, the functors

$$\overline{\otimes}\colon \mathbf{Wstar}_{\text{CP}} \times \mathbf{Wstar}_{\text{CP}} \longrightarrow \mathbf{Wstar}_{\text{CP}}$$
$$\overline{\otimes}\colon \mathbf{Wstar}_{\text{CP-PU}} \times \mathbf{Wstar}_{\text{CP-PU}} \longrightarrow \mathbf{Wstar}_{\text{CP-PU}}$$

are $\mathbf{Bdcppo}_\perp$*-enriched and* $\mathbf{Dcppo}_\perp$*-enriched respectively.* ■

Proposition 4.9.19. *There exist C^*-algebras A, B and a bounded-above monotone sequence $(f_n)_n$ in $\mathbf{Cstar}_{\text{CP-PU}}(A,B)$ such that $(f_n)_n$ does not have the supremum. Hence $\mathbf{Cstar}_{\text{CP-PU}}(A,B)$ is not an ωcpo and $\mathbf{Cstar}_{\text{CP}}(A,B)$ is not a $b\omega$cpo.* ◁

Proof. Take $A \coloneqq \mathbb{C}$. Notice that there is a bijective correspondence between linear maps $f\colon \mathbb{C} \to B$ and elements $b \in B$; the mapping is given by $f \mapsto f(1)$. Moreover, f is (completely) positive and pre-unital if and only if $f(1) \in \mathcal{E}f(B)$. For $f, g \in \mathbf{Cstar}_{\text{CP-PU}}(\mathbb{C}, B)$,

$$\begin{aligned} f \sqsubseteq g &\overset{\text{def}}{\iff} g - f \text{ is completely positive} \\ &\iff g - f \text{ is positive} && \text{by Thm. 3.4.11} \\ &\iff (g-f)(1) \geq 0 \\ &\iff f(1) \leq g(1) \ . \end{aligned}$$

Hence there is an order-isomorphism $\mathbf{Cstar}_{\text{CP-PU}}(\mathbb{C}, B) \cong \mathcal{E}f(B)$. By Prop. 4.8.7, there exists a C^*-algebra B such that $\mathcal{E}f(B)$ is not an ωcpo (take $B \coloneqq C([0,1])$ for example). Hence the claim follows. ■

We can summarize the results in this section as follows. The category $\mathbf{Wstar}_{\text{CP}}$ (resp. $\mathbf{Wstar}_{\text{CP-PU}}$) is a symmetric monoidal $\mathbf{Bdcppo}_\perp$-enriched (resp. $\mathbf{Dcppo}_\perp$-enriched) category with (enriched) finite products.

Chapter 5

Semantics for a quantum programming language

In this chapter, we will give a denotational semantics for a quantum programming language by operator algebras. We first review an elementary quantum mechanics and, in particular, a notion of *quantum operation*. We then briefly explain Selinger's quantum programming language QPL and his original denotational semantics. Next, we show the category $\mathbf{Wstar}_{\text{CP-PU}}$ is traced monoidal, and the opposite category of $\mathbf{Wstar}_{\text{CP-PU}}$ is an elementary quantum flow chart category. Therefore it gives a denotational semantics for the language QPL. Then we compare Selinger's and our model. It turns out that Selinger's category $\mathbf{Q}$ can be (contravariantly) embedded into our category $\mathbf{Wstar}_{\text{CP-PU}}$, and moreover $\mathbf{Q}$ coincides with (up to equivalence of categories) the finite dimensional part of $\mathbf{Wstar}_{\text{CP-PU}}$. Hence our model can be thought of as an infinite dimensional extension of his model. Finally, we discuss infinite types in QPL. Especially, we show our model can accommodate any (countable) classical types and function between them.

5.1 Quantum operation

According to the quantum mechanics, a quantum system is associated with a Hilbert space $\mathcal{H}$. A (mixed, subnormalized) *quantum state* of the system is represented by a positive trace class operator ρ on $\mathcal{H}$ with $\mathrm{tr}(\rho) \leq 1$. We denote the set of quantum states by $\mathcal{S}t(\mathcal{H})$. A quantum operation, which transforms one quantum state to another, is defined mathematically as follows.

Definition 5.1.1. A *quantum operation* from a system associated with a Hilbert space $\mathcal{H}$ to another with $\mathcal{K}$ is a trace-decreasing completely positive linear map $\mathcal{E}\colon \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ between the sets of trace class operators. Here, "trace-decreasing" means $\mathrm{tr}(\mathcal{E}(T)) \leq \mathrm{tr}(T)$ for all positive trace class operators $T \in \mathcal{T}(\mathcal{H})$. ◁

It is easy to see that a quantum operation $\mathcal{E}\colon \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ can be restricted to a map $\mathcal{E}\colon \mathcal{S}t(\mathcal{H}) \to \mathcal{S}t(\mathcal{K})$, that is, it indeed maps quantum states to quantum states. If Hilbert spaces are finite dimensional, say $\mathcal{H} = \mathbb{C}^n$ and $\mathcal{K} = \mathbb{C}^m$, then the set $\mathcal{T}(\mathbb{C}^n)$ of trace class operators coincides with the set of all linear operators. Hence it is identified with the set $\mathcal{M}_n \coloneqq \mathcal{M}_n(\mathbb{C}) = \mathbb{C}^{n\times n}$ of complex $n \times n$-matrices. Therefore, a quantum operation is a trace-decreasing completely positive map $\mathcal{E}\colon \mathcal{M}_n \to \mathcal{M}_m$ between the sets of matrices.

The class of quantum operations contains fundamental operations such as unitary transformations, measurements and preparations of quantum states. Conversely, every quantum operation can be realized by a combination of such operations. For further details, see e.g. [44, §8.2], [26, Chap. 4].

5.2 Selinger's QPL

In [53], Selinger proposed a quantum programming language QPL and its denotational semantics. The language QPL is a first-order *functional* language with loop and recursion. Then an interpretation of a program $P\colon \Gamma \to \Gamma'$ will be given as a certain function $[\![P]\!]\colon [\![\Gamma]\!] \to [\![\Gamma']\!]$. However, the function $[\![P]\!]$ cannot be just a quantum operation since the program involves not only quantum but also classical data. To give denotation of programs, Selinger constructed the category $\mathbf{Q}$ as follows.

Definition 5.2.1. The category $\mathbf{CPM}_s$ is defined as follows.

- An object is a natural number.
- An arrow $f\colon n \to m$ is a completely positive linear map $f\colon \mathcal{M}_n \to \mathcal{M}_m$. ◁

Definition 5.2.2. The category $\mathbf{CPM}$ is the finite biproduct completion of $\mathbf{CPM}_s$. Specifically:

- An object is a sequence $\vec{n} = (n_1, \dots, n_k)$ of natural numbers.
- An arrow $f\colon \vec{n} \to \vec{m}$ is a matrix (f_{ij}) of arrows $f_{ij}\colon n_j \to m_i$ in $\mathbf{CPM}_s$. ◁

Definition 5.2.3. The category $\mathbf{Q}$ is a subcategory of $\mathbf{CPM}$ such that

- Objects are the same as $\mathbf{CPM}$.
- An arrow is $f\colon \vec{n} \to \vec{m}$ in $\mathbf{CPM}$ which is trace-decreasing, i.e.
$$\sum_i \sum_j \mathrm{tr}(f_{ij}(A_j)) \le \sum_j \mathrm{tr}(A_j)$$
for all positive $A_j \in \mathcal{M}_{n_j}$. ◁

Note. The category $\mathbf{CPM}$ appears originally in [53] as $\mathbf{W}$ with an explicit definition, but later in [38, Def. 2.3] and [45, §2.3] as the finite biproduct completion of $\mathbf{CPM}_s$. See [54, §5] for finite biproduct completion. ◁

Selinger's idea in interpreting a program with classical and quantum data is the identification of classical data with (classical) controls. Each entry of a sequence $\vec{n} = (n_1, \dots, n_k)$ represents a control. For example, having one bit is identified with having two choices, hence an interpretation of a type **bit** is $[\![\mathbf{bit}]\!] = (1,1) \in \mathbf{Q}$, while an interpretation of a type **qbit** is $[\![\mathbf{qbit}]\!] = (2) \in \mathbf{Q}$.

For the category $\mathbf{Q}$, we have the following facts.

- $\mathbf{Q}$ has finite coproducts $(\oplus, 0)$.
- $\mathbf{Q}$ has a symmetric monoidal structure $(\otimes, I)$, which distribute over coproducts $\oplus$.
- $\mathbf{Q}$ is $\boldsymbol{\omega}\mathbf{Cppo}$-enriched. The finite coproducts and the tensor products are suitably enriched.

- The $\boldsymbol{\omega}$**Cppo**-enriched structure with coproducts $(\oplus, 0)$ induces the monoidal trace wrt. $(\oplus, 0)$.

Consequently, the category $\mathbf{Q}$ is structured enough to interpret the language QPL. Furthermore, Selinger has axiomatized a category which can interpret QPL.

Definition 5.2.4 ([53, §6.6])**.** An *elementary quantum flow chart category* is a symmetric monoidal category $(\mathbf{C}, \otimes, I)$ such that

- $\mathbf{C}$ has finite coproducts $(\oplus, 0)$.
- $\mathbf{C}$ has a monoidal trace wrt. finite coproducts $(\oplus, 0)$.
- For each $A \in \mathbf{C}$, $A \otimes (-)$ is a traced monoidal functor.
- $\mathbf{C}$ has a distinguished object **qbit** with arrows $\iota\colon I \oplus I \to \mathbf{qbit}$ and $p\colon \mathbf{qbit} \to I \oplus I$ such that $p \circ \iota = \mathrm{id}$. $\lhd$

Given an elementary quantum flow chart category $\mathbb{C}$ and an assignment of built-in unitary operators, we can interpret QPL programs without recursion as arrows in $\mathbb{C}$. If an elementary quantum flow chart category $\mathbb{C}$ is $\boldsymbol{\omega}$**Cppo**-enriched, we can interpret QPL programs with recursion.

In the remain of this chapter, we will show that the opposite category of $\mathbf{Wstar}_{\text{CP-PU}}$ is an elementary quantum flow chart category, hence we can give the denotation of the language QPL by W^*-algebras.

5.3 Monoidal trace

The monoidal trace of the category $\mathbf{Q}$ is induced by its $\boldsymbol{\omega}$**Cppo**-enriched structure with finite coproducts. The construction is valid for any $\boldsymbol{\omega}$**Cppo**-enriched category with finite coproducts [53, §6.4]. A rigorous discussion for this point is found in Appendix A. As we showed (Cor. 4.9.11, Cor. 4.9.15), the category $\mathbf{Wstar}_{\text{CP-PU}}$ is a $\mathbf{Dcppo}_{\perp}$-enriched category with finite products. Applying the construction to the opposite category to $\mathbf{Wstar}_{\text{CP-PU}}$, we obtained a monoidal trace wrt. coproducts in the opposite category, and hence a monoidal trace wrt. products in $\mathbf{Wstar}_{\text{CP-PU}}$.

Here we will sketch the (dual) construction explicitly. Let M, N, L be W^*-algebras, and $f\colon M \oplus L \to N \oplus L$ a normal completely positive pre-unital map. We shall define a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathbf{Wstar}_{\text{CP-PU}}(M, N \oplus L)$ inductively by

$$\begin{aligned} f_0 &\coloneqq \perp \\ f_{n+1} &\coloneqq f \circ \langle \mathrm{id}_M, \pi_2 \circ f_n \rangle \ . \end{aligned}$$

For the second formula, see the diagram:

$$\begin{tikzcd} & M & \\ M \arrow[ur, "\mathrm{id}_M"] \arrow[r, dashed] \arrow[d, "f_n"'] & M \oplus L \arrow[u] \arrow[r, "f"] \arrow[d] & N \oplus L \\ N \oplus L \arrow[r, "\pi_2"] & L & \end{tikzcd}$$

Then clearly $\perp = f_0 \sqsubseteq f_1$, and $f_n \sqsubseteq f_{n+1}$ implies $f \circ \langle \mathrm{id}_M, \pi_2 \circ f_n \rangle \sqsubseteq f \circ \langle \mathrm{id}_M, \pi_2 \circ f_{n+1} \rangle$, i.e. $f_{n+1} \sqsubseteq f_{n+2}$. Hence $f_n \sqsubseteq f_{n+1}$ for all $n \in \mathbb{N}$, that is, $(f_n)_{n \in \mathbb{N}}$ is an ω-chain, and so a monotone net. Therefore we obtain

$$\bigsqcup_{n \in \mathbb{N}} f_n \colon M \longrightarrow N \oplus L \ ,$$

and define

$$\mathrm{Tr}(f) \coloneqq \pi_1 \circ \Big(\bigsqcup_{n\in\mathbb{N}} f_n\Big) \colon M \longrightarrow N \enspace .$$

We have now defined a family of functions

$$\mathrm{Tr}\colon \mathbf{Wstar}_{\text{CP-PU}}(M \oplus L, N \oplus L) \longrightarrow \mathbf{Wstar}_{\text{CP-PU}}(M, N)$$

for $M, N, L \in \mathbf{Wstar}_{\text{CP-PU}}$, which indeed satisfies axioms for a monoidal trace.

Theorem 5.3.1. *The category* $\mathbf{Wstar}_{\text{CP-PU}}$ *has a monoidal trace wrt. finite products* $(\oplus, 0)$*.* ◁

Proof. We can apply Thm. A.0.1. By Cor. 4.9.11 and Cor. 4.9.15, $\mathbf{Wstar}_{\text{CP-PU}}$ is a $\mathbf{Dcppo}_\perp$-enriched (hence $\boldsymbol{\omega}\mathbf{Cppo}$-enriched) cartesian category. It is easy to see that the composition is left-strict. ■

Moreover, the monoidal trace is nicely related to tensor products.

Theorem 5.3.2. *For each* $M \in \mathbf{Wstar}_{\text{CP-PU}}$*, a functor* $(-)\mathbin{\overline{\otimes}} M \colon \mathbf{Wstar}_{\text{CP-PU}} \to \mathbf{Wstar}_{\text{CP-PU}}$ *is a traced monoidal functor.* ◁

Proof. We can apply Thm. A.0.2. The functor $(-)\mathbin{\overline{\otimes}} M$ preserves finite products because of distribution of tensor products over finite products (Prop. 4.6.1, Prop. 4.6.2 and Prop. 4.6.3). Moreover it is $\mathbf{Dcppo}_\perp$-enriched by Cor. 4.9.18. It is easy to see $\perp \mathbin{\overline{\otimes}} \mathrm{id}_M = \perp$. ■

5.4 Semantics by W^*-algebras

We finally show that the opposite category of $\mathbf{Wstar}_{\text{CP-PU}}$ is an elementary quantum flow chart category. A distinguished object **qbit** in Def. 5.2.4 is $\mathcal{M}_2$, i.e. the algebra of 2×2-matrices. We define two maps

$$\iota\colon \mathcal{M}_2 \longrightarrow \mathbb{C} \oplus \mathbb{C}$$
$$p\colon \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathcal{M}_2$$

by

$$\iota\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = (x, w)$$

and

$$p(x, y) = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

It is straightforward to see the two maps are positive, hence completely positive by Thm. 3.4.11 (notice that $\mathbb{C}\oplus\mathbb{C}$ is commutative). They are also normal because the dimensions are finite. Moreover they are clearly unital. Therefore ι and p are arrows in $\mathbf{Wstar}_{\text{CP-PU}}$. It is clear that $\iota \circ p = \mathrm{id}$, hence $p \circ \iota = \mathrm{id}$ in $(\mathbf{Wstar}_{\text{CP-PU}})^{\mathrm{op}}$.

Theorem 5.4.1. *The opposite category of* $\mathbf{Wstar}_{\text{CP-PU}}$ *is an elementary quantum flow chart category with a distinguished object* $\mathbf{qbit} = \mathcal{M}_2$ *and arrows* ι *and* p *defined above.* ■

5.5 Schrödinger vs. Heisenberg picture

Recall that for a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ is a W^*-algebra with the predual $\mathcal{T}(\mathcal{H})$ (Example 4.1.4). By Prop. 4.1.6, for every normal map $\mathcal{E}\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$, there exists a corresponding bounded map $\mathcal{E}_*\colon \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$ between preduals. They are related in the following way:

$$\operatorname{tr}(\mathcal{E}(S)\cdot T) = \operatorname{tr}(S\cdot \mathcal{E}_*(T))$$

for all $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{T}(\mathcal{K})$. Furthermore the following hold.

Proposition 5.5.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Suppose a normal map $\mathcal{E}\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ and a bounded map $\mathcal{E}_*\colon \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$ correspond in the sense of Prop. 4.1.6. Then*

1. *$\mathcal{E}$ is completely positive if and only if $\mathcal{E}_*$ is completely positive.*
2. *$\mathcal{E}$ is unital if and only if $\mathcal{E}_*$ preserves trace.*
3. *$\mathcal{E}$ is pre-unital if and only if $\mathcal{E}_*$ decreases trace.* ◁

Proof. See [26, §4.1.2]. ■

Hence, a normal completely positive pre-unital map $\mathcal{E}\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$, i.e. an arrow in $\mathbf{Wstar}_{\text{CP-PU}}$, corresponds to a completely positive trace-decreasing map $\mathcal{E}_*\colon \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$, i.e. a quantum operation defined in Def. 5.1.1. This is the well-known duality between the Heisenberg and Schrödinger pictures: one transforms observables (or effects), whereas another transforms states.

In other words, our semantics of the language QPL in $\mathbf{Wstar}_{\text{CP-PU}}$ is the Heisenberg picture, while Selinger's semantics in $\mathbf{Q}$ is the Schrödinger picture. In the words of [15], our semantics can be thought of as the *weakest precondition* semantics. This is because a positive pre-unital map $\mathcal{E}\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ can be restricted to a map $\mathcal{E}\colon \mathcal{E}f(\mathcal{H}) \to \mathcal{E}f(\mathcal{K})$ between their effects, where $\mathcal{E}f(\mathcal{H}) \coloneqq \mathcal{E}f(\mathcal{B}(\mathcal{H}))$ is the set of effects on $\mathcal{H}$, and coincides with the set of *predicates* in [15].

5.6 Embedding $\mathbf{Q}$ into $\mathbf{Wstar}_{\text{CP-PU}}$

As seen in the previous section, the two semantics of the language QPL in $\mathbf{Wstar}_{\text{CP-PU}}$ and $\mathbf{Q}$ are different viewpoints for essentially the same thing. In fact, the category $\mathbf{Q}$ can be contravariantly embedded into $\mathbf{Wstar}_{\text{CP-PU}}$.

First, we shall embed $\mathbf{CPM}$ into $\mathbf{Wstar}_{\text{CP}}$. Notice that the following bijective correspondence.

$$\begin{array}{c}
f\colon (n_1,\dots,n_k) \longrightarrow (m_1,\dots,m_l) \quad \text{in } \mathbf{CPM} \\
\hline\hline
f_{ij}\colon n_j \longrightarrow m_i \quad \text{in } \mathbf{CPM}_s\text{, for each } i,j \\
\hline\hline
f_{ij}\colon \mathcal{M}_{n_j} \longrightarrow \mathcal{M}_{m_i} \quad \text{completely positive, for each } i,j \\
\hline\hline
(f_{ij})^*\colon \mathcal{M}_{m_i} \longrightarrow \mathcal{M}_{n_j} \quad \text{in } \mathbf{Wstar}_{\text{CP}}\text{, for each } i,j \\
\hline\hline
I(f)\colon \bigoplus_{i=1}^{l} \mathcal{M}_{m_i} \longrightarrow \bigoplus_{j=1}^{k} \mathcal{M}_{n_j} \quad \text{in } \mathbf{Wstar}_{\text{CP}}
\end{array}$$

Here we use the self-duality:

$$\mathcal{M}_{n_j} \cong \mathcal{B}(\mathbb{C}^{n_j}) \cong \mathcal{T}(\mathbb{C}^{n_j})^* \cong (\mathcal{M}_{n_j})^* \ .$$

by the finite dimensional case of Prop. 2.1.16. Then the mapping $I(n_1, \dots, n_k) = \bigoplus_{j=1}^{k} \mathcal{M}_{n_j}$ defines a contravariant functor $I\colon \mathbf{CPM} \to (\mathbf{Wstar}_{\mathrm{CP}})^{\mathrm{op}}$, which is full and faithful by definition, and clearly injective on objects.

Theorem 5.6.1. *There is a (full) embedding $I\colon \mathbf{CPM} \to (\mathbf{Wstar}_{\mathrm{CP}})^{\mathrm{op}}$.* ■

Finally, we will show the functor restricts to a full and faithful functor $I'\colon \mathbf{Q} \to (\mathbf{Wstar}_{\mathrm{CP\text{-}PU}})^{\mathrm{op}}$ as follows.

$$\begin{aligned}
& I(f)\colon \bigoplus_{i=1}^{l} \mathcal{M}_{m_i} \longrightarrow \bigoplus_{j=1}^{k} \mathcal{M}_{n_j} \text{ is pre-unital} \\
\iff & I(f)((1)_{i=1}^{l}) \le (1)_{j=1}^{k} \\
\iff & \sum_{i=1}^{l} (f_{ij})^*(1) \le 1 \text{ for each j} \\
\iff & 1 - \sum_{i=1}^{l} (f_{ij})^*(1) \ge 0 \text{ for each j} \\
\overset{\star}{\iff} & (\mathrm{tr} - \sum_{i=1}^{l} \mathrm{tr} \circ f_{ij})(A) \ge 0 \text{ for each positive } A \in \mathcal{M}_{n_j}, \text{ for each j} \\
\iff & \sum_{j=1}^{k} (\mathrm{tr} - \sum_{i=1}^{l} \mathrm{tr} \circ f_{ij})(A_j) \ge 0 \text{ for each positive } A_j \in \mathcal{M}_{n_j} \\
\iff & \sum_{i=1}^{l} \sum_{j=1}^{k} \mathrm{tr}(f_{ij}(A_j)) \le \sum_{j=1}^{k} \mathrm{tr}(A_j) \\
\iff & f\colon \vec{n} \to \vec{m} \text{ is trace-decreasing (Def. 5.2.3)}
\end{aligned}$$

For the equivalence $\overset{\star}{\iff}$, we use Lem. 2.1.17. Hence we have shown the next theorem.

Theorem 5.6.2. *There is a (full) embedding $I'\colon \mathbf{Q} \to (\mathbf{Wstar}_{\mathrm{CP\text{-}PU}})^{\mathrm{op}}$.* ■

In fact, we can say more about the embedding. Notice that $I(\vec{n})$ is a finite dimensional W^*-algebra for each $\vec{n} \in \mathbf{CPM}$. Hence the embedding gives a functor $I\colon \mathbf{CPM} \to (\mathbf{FdWstar}_{\mathrm{CP}})^{\mathrm{op}}$, where $\mathbf{FdWstar}_{\mathrm{CP}}$ denotes the category of finite dimensional W^*-algebras and normal completely positive maps.

Theorem 5.6.3. *The embeddings*

$$\begin{aligned}
I&\colon \mathbf{CPM} \to (\mathbf{FdWstar}_{\mathrm{CP}})^{\mathrm{op}} \\
I'&\colon \mathbf{Q} \to (\mathbf{FdWstar}_{\mathrm{CP\text{-}PU}})^{\mathrm{op}}
\end{aligned}$$

give equivalences of categories:

$$\begin{aligned}
\mathbf{CPM} &\simeq (\mathbf{FdWstar}_{\mathrm{CP}})^{\mathrm{op}} \\
\mathbf{Q} &\simeq (\mathbf{FdWstar}_{\mathrm{CP\text{-}PU}})^{\mathrm{op}}
\end{aligned}$$ ◁

Proof. By Lem. 5.6.4 below, I and I' are essentially surjective. By the theorem in category theory [37, Thm. IV.4.1], a full, faithful and essentially surjective functor is a part of equivalence. ■

Lemma 5.6.4 ([60, Thm. I.11.2]). *Let A be a finite dimensional C^*-algebra (hence W^*-algebra, see Rem. 4.1.9). There exists a (unique up to permutations) sequence $(n_1, \ldots, n_k)$ of positive integers and a $*$-isomorphism:*

$$A \cong \bigoplus_{j=1}^{k} \mathcal{M}_{n_j} \quad .$$

■

Remark 5.6.5. By Rem. 4.1.9, $\mathbf{FdWstar}_{\mathrm{CP}} = \mathbf{FdCstar}_{\mathrm{CP}}$ and $\mathbf{FdWstar}_{\text{CP-PU}} = \mathbf{FdCstar}_{\text{CP-PU}}$. ◁

5.7 QPL with infinite types

We have defined the category $\mathbf{Wstar}_{\text{CP-PU}}$ to give the denotational semantics of the language QPL. Because Selinger's category $\mathbf{Q}$ is contravariantly embedded into $\mathbf{Wstar}_{\text{CP-PU}}$, the category $\mathbf{Wstar}_{\text{CP-PU}}$ can be thought of as an infinite dimensional extension of $\mathbf{Q}$. Working in the category $\mathbf{Wstar}_{\text{CP-PU}}$ rather than $\mathbf{Q}$ enables us to handle infinite types. For example, as Selinger suggested in [53, §7.3], a type **int** should be interpreted as $[\![\mathbf{int}]\!] = \ell^\infty(\mathbb{N}) = \bigoplus_{n\in\mathbb{N}} \mathbb{C}$, which is indeed in $\mathbf{Wstar}_{\text{CP-PU}}$, but not in $\mathbf{Q}$.

We will present slightly more general statements.

Definition 5.7.1. Let S be an at most countable set. We define:

$$\begin{aligned} \ell^1(S) &\coloneqq \Big\{\varphi\colon S \to \mathbb{C} \;\Big|\; \sum_{s\in S} |\varphi(s)| < \infty\Big\} \\ \ell^2(S) &\coloneqq \Big\{\varphi\colon S \to \mathbb{C} \;\Big|\; \sum_{s\in S} |\varphi(s)|^2 < \infty\Big\} \\ \ell^\infty(S) &\coloneqq \Big\{\varphi\colon S \to \mathbb{C} \;\Big|\; \sup_{s\in S} |\varphi(s)| < \infty\Big\} \quad . \end{aligned}$$

◁

Proposition 5.7.2. *Let S and T be at most countable sets.*

1. *$\ell^2(S)$ is a Hilbert space.*
2. *$\ell^\infty(S)$ is a W^*-algebra with the predual $\ell^1(S)$. Moreover it has a canonical normal unital faithful representation $\pi\colon \ell^\infty(S) \to \mathcal{B}(\ell^2(S))$ by $\pi(\varphi)(\psi) = \varphi\psi$ (pointwise multiplication).*
3. *There is a $*$-isomorphism: $\ell^\infty(S) \mathbin{\overline{\otimes}} \ell^\infty(T) \cong \ell^\infty(S \times T)$*
4. *Any function $f\colon S \to T$ induces a normal unital $*$-homomorphism $\ell^\infty(f)\colon \ell^\infty(T) \to \ell^\infty(S)$ by $\ell^\infty(f)(\varphi) = \varphi \circ f$.* ◁

Proof. We will just sketch the proof of 3. By Thm. 4.5.2, we identify $\overline{\otimes}$ as the tensor product of von Neumann algebras. Note the isomorphism $\ell^2(S) \otimes \ell^2(T) \cong \ell^2(S \times T)$ of Hilbert spaces. By the identification $\mathcal{B}(\ell^2(S) \otimes \ell^2(T)) \cong \mathcal{B}(\ell^2(S \times T))$, we have inclusion $\ell^\infty(S) \odot \ell^\infty(T) \subseteq \ell^\infty(S \times T)$. The (ultra)weak denseness of the inclusion proves $\ell^\infty(S) \mathbin{\overline{\otimes}} \ell^\infty(T) \cong \ell^\infty(S \times T)$. ■

These statements mean that at most countable sets $S_1, \ldots, S_n, T$ and a function $f\colon S_1 \times \cdots \times S_n \to T$ inhabit in $\mathbf{Wstar}_{\text{M-I}}$ (hence in $\mathbf{Wstar}_{\text{CP-PU}}$) as:

$$\ell^\infty(f)\colon \ell^\infty(T) \longrightarrow \ell^\infty(S_1 \times \cdots \times S_n) \cong \ell^\infty(S_1) \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} \ell^\infty(S_n) \quad .$$

In other words, we can interpret any (countable) classical data type and function between them in $\mathbf{Wstar}_{\text{CP-PU}}$. For example, when we interpret the type **nat** by $[\![\mathbf{nat}]\!] = \ell^\infty(\mathbb{N})$, we can build in any function $f \colon \mathbb{N}^n \to \mathbb{N}$ into the language.

The inhabitancy of classical data in W^*-algebras is in fact described more generally. It is known that there is the following dual equivalence of categories

$$\mathbf{CWstar}_{\text{M-I-U}} \simeq \mathbf{LocMeas}^{\text{op}}$$

between the category of commutative W^*-algebras and normal unital $*$-homomorphisms and the category of localizable measurable spaces and measurable functions [33, §5.8] (see also [52, Prop. 1.18.1] and [60, Thm. III.1.18]). Then, Prop. 5.7.2 should be generalized for localizable measurable spaces and measurable functions (cf. [7, Example III.1.5.6]). This will be future work.

Chapter 6

Conclusions and future work

6.1 Conclusions

We can summarize the results of the thesis as follows. As a mathematical result, we have shown that the category $\mathbf{Wstar}_{\mathrm{CP}}$ is $\mathbf{Bdcppo}_{\perp}$-enriched and $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$ is $\mathbf{Dcppo}_{\perp}$-enriched. Moreover, both categories have suitably enriched finite products. Tensor products of W^*-algebras make both categories symmetric monoidal in a compatible way with enriched structure. As a contribution to quantum computation, we have shown the opposite category of $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$ is an elementary quantum flow chart category. Hence the category $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$ gives rise to a denotational semantics for the quantum programming language QPL. This semantics can be seen as an extension of Selinger's original semantics, and can accommodate infinite data and classical data well. We hope our results demonstrate that operator algebras are useful and meaningful in the area of quantum computation as well as other areas where operator algebras are successfully used.

6.2 Future work

An unpublished paper [32] claims that the functor $(-)\overline{\otimes}M$ on the category $\mathbf{Wstar}_{\mathrm{M\text{-}I\text{-}U}}$ has a left adjoint for each W^*-algebra M. It means that the opposite category of $\mathbf{Wstar}_{\mathrm{M\text{-}I\text{-}U}}$ is a *closed* symmetric monoidal category. The fact suggest that W^*-algebras model a higher-order functional quantum programming language (or a quantum lambda calculus), not only a first-order language. A recent work [45] gives a denotational semantics for a "full" quantum lambda calculus by extending the category **CPM** into the category that can accommodate infinite structures. It also suggest that W^*-algebras can also model a "full" quantum lambda calculus because they can accommodate infinite structures. An investigation of this line is one of future works.

Another future work is to make use of the dual equivalence of the category of commutative W^*-algebras and the category of localizable measurable space. Following the line of §5.7, we want to show that via the dual equivalence, any morphism between localizable measurable space (i.e. a certain "classical" function) can be used as a build-in function in the language QPL.

Moreover, I conjecture that Furber and Jacobs' work [18] about the Gelfand duality holds for commutative W^*-algebras and localizable measurable space. Specifically, my conjecture is that the Kleisli category of the Giry monad on the category of localizable measurable space is dual to the category of commutative W^*-algebras and normal (completely) positive unital maps. It follows that any probabilistic function between localizable measurable spaces can be embedded into $\mathbf{Wstar}_{\text{CP-PU}}$. Then we will be able to use any probabilistic function as a built-in function in the language QPL.

Yet another direction is to examine the effect logic (see e.g. [28, 29]) of W^*-algebras. For example, Furber and Jacobs [18] established the following "state-and-effect" triangles about commutative C^*-algebras.

$$\begin{tikzcd}
\mathbf{EMod}^{\mathrm{op}} \arrow[rr, bend left=10] \arrow[rr, phantom, "\top"] & & \mathcal{EM}(\mathcal{R}) \arrow[ll, bend left=10] \\
& (\mathbf{CCstar}_{\text{P-U}})^{\mathrm{op}} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Here, states are described by the Eilenberg-Moore category $\mathcal{EM}(\mathcal{R})$ of the Radon monad $\mathcal{R}$, and effects (predicates) by the category **EMod** of effect modules. The adjunction between them describes the duality between states and effects. It seems that a similar result can be obtained for W^*-algebras.

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Appendix A

Monoidal traces on $\boldsymbol{\omega}$Cppo-enriched cartesian categories

The goal of this appendix is to establish the following two theorems, which are stated informally in [53, §6.4][1].

Theorem A.0.1. *Every $\boldsymbol{\omega}\mathbf{Cppo}$-enriched cartesian category with left-strict composition (i.e. $\perp \circ f = \perp$) is traced. For $f\colon A \times X \to B \times X$, the trace $\mathrm{Tr}(f)\colon A \to B$ is given by*

$$\mathrm{Tr}(f) \coloneqq \pi_1 \circ \bigsqcup_{n \in \mathbb{N}} \mathrm{Tr}^{(n)}(f) \ , \tag{A.1}$$

where $\mathrm{Tr}^{(n)}(f)\colon A \to B \times X$ is defined by

$$\begin{aligned} \mathrm{Tr}^{(0)}(f) &= \perp \\ \mathrm{Tr}^{(n+1)}(f) &= f \circ \langle \mathrm{id}_A, \pi_2 \circ \mathrm{Tr}^{(n)}(f) \rangle \ . \end{aligned}$$ ◁

Theorem A.0.2. *Let $\mathbf{C}$ and $\mathbf{D}$ be $\boldsymbol{\omega}\mathbf{Cppo}$-enriched cartesian categories, which are traced by Thm. A.0.1. Then, every $\boldsymbol{\omega}\mathbf{Cppo}$-enriched cartesian functor between $\mathbf{C}$ and $\mathbf{D}$ satisfying $F\perp = \perp$ is traced.* ◁

Let us clarify terminology. A *cartesian category* refers to a (symmetric) monoidal category whose monoidal structure is given by finite products. In other words, it is a category with a *choice* of a terminal object and binary products. For an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched cartesian category, note that its cartesian product functor is required to be $\boldsymbol{\omega}\mathbf{Cppo}$-enriched. It is equivalent to saying that the tupling $\langle \cdot, \cdot \rangle$ is ω-continuous. A functor between cartesian categories is said to be *cartesian* if it is strong monoidal (equivalently, if it preserves finite products).

A.1 Proof of Theorem A.0.1

Theorem A.0.1 is showed via the following two theorems.

Theorem A.1.1 (Hyland/Hasegawa)**.** *A cartesian category is traced if and only if it has a Conway operator. Moreover, a trace operator* Tr *and a Conway operator* Fix *are related bijectively as follows:*

$$\mathrm{Tr}(f) = \pi_1 \circ f \circ \langle \mathrm{id}_A, \mathrm{Fix}(\pi_2 \circ f) \rangle \tag{A.2}$$

for $f\colon A \times X \to B \times X$, and

$$\mathrm{Fix}(g) = \mathrm{Tr}(\Delta_X \circ g)$$

for $g\colon A \times X \to X$, where $\Delta_X = \langle \mathrm{id}_X, \mathrm{id}_X \rangle$ is the diagonal map. ◁

[1] In fact, their dual are stated there.

Proof. See [24, Thm. 7.1.1]. ∎

Theorem A.1.2. *Every* $\boldsymbol{\omega}$**Cppo***-enriched cartesian category with left-strict composition (i.e.* $\bot \circ f = \bot$*) has a Conway operator* Fix*. For* $g\colon A \times X \to X$*,* $\mathrm{Fix}(g)\colon A \to X$ *is given by*

$$\mathrm{Fix}(g) \coloneqq \bigsqcup_{n \in \mathbb{N}} \mathrm{Fix}^{(n)}(g) \ , \tag{A.3}$$

where $\mathrm{Fix}^{(n)}(g)\colon A \to X$ *is defined by*

$$\begin{aligned} \mathrm{Fix}^{(0)}(g) &= \bot \\ \mathrm{Fix}^{(n+1)}(g) &= g \circ \langle \mathrm{id}_A, \mathrm{Fix}^{(n)}(g) \rangle \ . \end{aligned}$$ ◁

Proof. See [27, Lem. A.1 in Appendix]. ∎

Now we will prove Thm A.0.1.

Proof of Theorem A.0.1. Let $\mathbf{C}$ be an $\boldsymbol{\omega}$**Cppo**-enriched cartesian category with left-strict composition. Then, by Thm. A.1.1 and Thm. A.1.2, $\mathbf{C}$ is traced. We have just to check the equation (A.1). For $f\colon A \times X \to B \times X$,

$$\begin{aligned} \mathrm{Tr}(f) &= \pi_1 \circ f \circ \langle \mathrm{id}_A, \mathrm{Fix}(\pi_2 \circ f) \rangle && \text{by (A.2)} \\ &= \pi_1 \circ f \circ \Big\langle \mathrm{id}_A, \bigsqcup_{n \in \mathbb{N}} \mathrm{Fix}^{(n)}(\pi_2 \circ f) \Big\rangle && \text{by (A.3)} \\ &= \pi_1 \circ \bigsqcup_{n \in \mathbb{N}} \Big(f \circ \langle \mathrm{id}_A, \mathrm{Fix}^{(n)}(\pi_2 \circ f) \rangle \Big) \ . \end{aligned}$$

It is easy to see, by induction on n,

$$\mathrm{Tr}^{(n)}(f) \le f \circ \langle \mathrm{id}_A, \mathrm{Fix}^{(n)}(\pi_2 \circ f) \rangle \le \mathrm{Tr}^{(n+1)}(f)$$

for all $n \in \mathbb{N}$. It follows that

$$\bigsqcup_{n \in \mathbb{N}} \Big(f \circ \langle \mathrm{id}_A, \mathrm{Fix}^{(n)}(\pi_2 \circ f) \rangle \Big) = \bigsqcup_{n \in \mathbb{N}} \mathrm{Tr}^{(n)}(f) \ .$$

Hence we have

$$\mathrm{Tr}(f) = \pi_1 \circ \bigsqcup_{n \in \mathbb{N}} \mathrm{Tr}^{(n)}(f) \ .$$ ∎

A.2 Proof of Theorem A.0.2

In this section, we will prove Thm. A.0.2.

Proof of Theorem A.0.2. What we have to show is:

$$F\,\mathrm{Tr}(f) = \mathrm{Tr}(\phi_B \circ Ff \circ \phi_A^{-1})$$

for $f\colon A \times X \to B \times X$ in $\mathbf{C}$, where

$$\begin{aligned} \phi_A &\colon F(A \times X) \xrightarrow{\cong} FA \times FX \\ \phi_B &\colon F(B \times X) \xrightarrow{\cong} FB \times FX \end{aligned}$$

are canonical (coherence) isomorphisms. First note that ϕ_A makes the following diagrams commute.

$$
\begin{tikzcd}
 & F(A\times X) \arrow[dl, "F\pi_1"'] \arrow[d, "\phi_A", "\cong"'] \arrow[dr, "F\pi_2"] & \\
FA & FA\times FX \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & FX
\end{tikzcd}
\qquad
\begin{tikzcd}
FW \arrow[d, "F\langle k,l\rangle"'] \arrow[dr, "\langle Fk,Fl\rangle"] & \\
F(A\times X) \arrow[r, "\cong", "\phi_A"'] & FA\times FX
\end{tikzcd}
$$

In the right diagram, $k\colon W\to A$ and $l\colon W\to X$ are arbitrary arrows in $\mathbf{C}$. Similar diagrams commute for ϕ_B. Now,

$$
\begin{aligned}
F\operatorname{Tr}(f) &= F\Big(\pi_1\circ\bigsqcup_{n\in\mathbb{N}}\operatorname{Tr}^{(n)}(f)\Big)\\
&= F\pi_1\circ\bigsqcup_{n\in\mathbb{N}}F\operatorname{Tr}^{(n)}(f)\\
&= \pi_1\circ\phi_B\circ\bigsqcup_{n\in\mathbb{N}}F\operatorname{Tr}^{(n)}(f)\\
&= \pi_1\circ\bigsqcup_{n\in\mathbb{N}}\Big(\phi_B\circ F\operatorname{Tr}^{(n)}(f)\Big)\ ,
\end{aligned}
$$

while

$$\operatorname{Tr}(\phi_B\circ Ff\circ\phi_A^{-1})=\pi_1\circ\bigsqcup_{n\in\mathbb{N}}\operatorname{Tr}^{(n)}(\phi_B\circ Ff\circ\phi_A^{-1})\ .$$

Then, to show

$$\bigsqcup_{n\in\mathbb{N}}\Big(\phi_B\circ F\operatorname{Tr}^{(n)}(f)\Big)=\bigsqcup_{n\in\mathbb{N}}\operatorname{Tr}^{(n)}(\phi_B\circ Ff\circ\phi_A^{-1})\ ,$$

it suffices to show

$$\operatorname{Tr}^{(n)}(\phi_B\circ Ff\circ\phi_A^{-1})\le\phi_B\circ F\operatorname{Tr}^{(n)}(f)\le\operatorname{Tr}^{(n+1)}(\phi_B\circ Ff\circ\phi_A^{-1})$$

for all $n\in\mathbb{N}$. We will prove it by induction on n.

(i) Base case ($n=0$):

$$\operatorname{Tr}^{(0)}(\phi_B\circ Ff\circ\phi_A^{-1})=\bot\le\phi_B\circ F\operatorname{Tr}^{(0)}(f)$$

shows the first inequality, and

$$
\begin{aligned}
&\operatorname{Tr}^{(1)}(\phi_B\circ Ff\circ\phi_A^{-1})\\
&=\phi_B\circ Ff\circ\phi_A^{-1}\circ\langle\mathrm{id}_{FA},\pi_2\circ\operatorname{Tr}^{(0)}(\phi_B\circ Ff\circ\phi_A^{-1})\rangle\\
&\ge\phi_B\circ\bot\\
&=\phi_B\circ F\bot && \text{by strictness of } F\\
&=\phi_B\circ F\operatorname{Tr}^{(0)}(f)
\end{aligned}
$$

shows the second inequality.

(ii) Induction step:

$$
\begin{aligned}
&\phi_B\circ F\operatorname{Tr}^{(n+1)}(f)\\
&=\phi_B\circ F\big(f\circ\langle\mathrm{id}_A,\pi_2\circ\operatorname{Tr}^{(n)}(f)\rangle\\
&=\phi_B\circ Ff\circ\phi_A^{-1}\circ\big\langle F\mathrm{id}_A,F\big(\pi_2\circ\operatorname{Tr}^{(n)}(f)\big)\big\rangle\\
&=\phi_B\circ Ff\circ\phi_A^{-1}\circ\big\langle\mathrm{id}_{FA},\pi_2\circ\phi_B\circ\operatorname{Tr}^{(n)}(f)\big\rangle\\
&\ge\phi_B\circ Ff\circ\phi_A^{-1}\circ\big\langle\mathrm{id}_{FA},\pi_2\circ\operatorname{Tr}^{(n)}(\phi_B\circ Ff\circ\phi_A^{-1})\big\rangle && \text{by I.H.}\\
&=\operatorname{Tr}^{(n+1)}(\phi_B\circ Ff\circ\phi_A^{-1})
\end{aligned}
$$

shows the first inequality, and similarly

$$\begin{aligned}
&\phi_B \circ F\,\mathrm{Tr}^{(n+1)}(f) \\
&= \phi_B \circ Ff \circ \phi_A^{-1} \circ \langle \mathrm{id}_{FA}, \pi_2 \circ \phi_B \circ \mathrm{Tr}^{(n)}(f)\rangle \\
&\le \phi_B \circ Ff \circ \phi_A^{-1} \circ \langle \mathrm{id}_{FA}, \pi_2 \circ \mathrm{Tr}^{(n+1)}(\phi_B \circ Ff \circ \phi_A^{-1})\rangle \qquad \text{by I.H.} \\
&= \mathrm{Tr}^{(n+2)}(\phi_B \circ Ff \circ \phi_A^{-1})
\end{aligned}$$

shows the second inequality.

Therefore we have

$$F\,\mathrm{Tr}(f) = \mathrm{Tr}(\phi_B \circ Ff \circ \phi_A^{-1}) \ . \qquad \blacksquare$$