The conception of truth in game semantics and linear logic

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When Alfred Tarski defined truth as the property of what is true, he defined a semantic conception of truth which fails to capture this notion in all its generality. Indeed, his definition does not acknowledge the processes by which we (in)validate propositions and therefore becomes problematic whenever one considers like Wittgenstein that the understanding of the mathematical essence of a proposition requires to grasp the conditions underlying the truth of said propositions.

Already used to teach logic at the end of the 19th century by Lewis Carroll [1], games really made their entry in mathematical logic with Lorenz' work in the 1950s. Inspired by Wittgenstein's concept of language games, Jaako Hintikka defined in 1994 the Game-Theoretic Semantics (GTS), a semantic conceptualization of truth organized around the notion of game, in which one player tries to prove the validity of a proposition while their opponent tries to refute the same proposition. In other words, one of the players shall build a proof while the other shall build a counter-example. In this context, truth corresponds to the existence of a winning strategy, i.e. a strategy in which the player wins in all game configurations.

In this essay, we argue that the introduction of games in the philosophy of logic finds its echo in logic for computer science, identifying game strategies with programs. We focus on game semantics, linear logic and ludics, in which truth arise geometrically through the interaction of logical rules. We argue that these mathematical theories contribute to a philosophical definition of truth which does not go around the question of the foundations of the validity of logical propositions: truth arises through the interactions and compositions of strategies, rather than on a metaphysical level.

1 Tarski's conception of truth

Let us recall Tarski's conception of truth, and stress its philosophical importance.

Tarski is concerned with the definition of truth for formal languages, that is, languages whose structure can be rigorously specified within a logical framework: axioms and inference rules are explicitly given, which allows an unambiguous characterization of the class of meaningful expressions. More precisely, the focus is on the class of propositions to which the term "true" applies. From there, Tarski defines the *material adequacy condition* which defines truth as the property of what is true, declaring that 'p' is true if and only if p.

Therefore, Tarski's semantic conception of truth is defined by induction on the structure of propositions, in logical terms and without any undefined metaphysical concept. However, this definition isn't exempt from criticism.

Indeed, the liar paradox raises major issues with Tarski's definition. When a language contains at the same time its expressions, the name of its expressions, and the semantic terms true and false, contradictory propositions like "this sentence is false" hold. To avoid this issue, Tarski considers a meta language in which one constructs the definition of truth for a given language, which contains the propositions for which one intends to conceptualize truth. Hence, truth is defined outside of the language considered, and Tarski's definition creates a hierarchy of languages in which the predicate 'true' is relative to a given depth of this hierarchy. And since the definition of truth is relative to a metalanguage entirely determined by the language considered, distinct languages will not have the same definition of truth. Moreover, Tarski's conception of truth doesn't offer any way to distinguish true propositions of false ones, that is, it does not provide any criteria of truth.

Although the philosophical implications of his work are a well-studied theme in the philosophy of logic, Tarski only briefly commented on his work [7]. Taking into consideration the state-of art of the research in logic for computer science, we argue in what follows that theoretical computer science offers compelling ways to overcome the issues raised by Tarski's conception of truth.

2 Game semantics

Already introduced by Lewis Caroll's *Logic of Game* [1] at the end of the 19th century, games really make their entry in logic with the works of Lorenz and Lorenzen in the 1950s, who focused on intuistionistic logic, and later of Jaako Hintikka. A notion of asymmetric game was already there in Gentzen's consistency proof [2]: while the player must construct a proof to win, their opponent only needs a counter-example.

Game semantics introduces a referee, materialised by the game's rule, which regulates the interaction between the player, who wants to prove a proposition, and an opponent, who wants to refute the same proposition. In this context, the truth of a proposition depends on the existence of a winning strategy for one of the two players. The interaction between the two parties arise from the composition of strategies.

In theoretical computer science, games are introduced to reason about programs. The player is identified with the program and its opponent is the evaluation context. Therefore, truth is identified with well-defined typing and the meaning of a proposition is given by the interaction between the strategies.

Hintikka argues in [5] that a suitable definition of truth must be sufficiently rich enough to express mathematical reasoning and must offer a way to codify the process by which we verify or falsify propositions, instead of being an abstract correspondence between propositions and facts. He tried to put this philosophy in practice in [6].

The game-theoretic semantics elaborated by Hintikka intends to explain how abstract logical relations can be associated to the operations by which human beings verify or falsify propositions in formal languages. This view relies on the following principle: two players agree to play a game about a specific proposition. Again, the first player, V, must verify the proposition while the second player, F, must falsify it.

When the proposition is of the form $P_1 \vee P_2$, the player V decides to start a new sub-game on the proposition P_1 or the proposition P_2. When the proposition is of the form $S_1 \wedge S_2$, the player F is the one who makes this choice. When the proposition is of the form $\forall x.S$, the player V must fix the value of the variable x and then starts a new sub-game on S. When the proposition is of the form $\exists x.S$, the player F makes this choice. Moreover, the two players must switch roles when the proposition is of the form $\neg S$. Finally, for atomic propositions, the game only lasts one turn: the winner is the player V when the atomic proposition holds, and the player F otherwise. Then, a proposition is true if and only if the player V has a winning strategy, that is, a strategy which allows him to win against any opponent.

Propositions such as the liar paradox end up being neither true nor false, and are therefore banned because they do not accept any truth value. However, affirming the existence of such strategy is a second-order proposition in his logic, making the claim that we're avoiding the use of a meta-theory debatable.

3 Ludics

In Girard's work, truth isn't an *a priori* norm but a by-product of the interaction between propositions. This vision goes beyond game semantics which, while focused on interactions, presuppose the existence of a logical theory which underlies the game-theoretic interpretation.

Tarski's conception of truth requires the metaphysical assumption of the existence of truth in a given language to establish it in a metalanguage. For Girard, the use of a metalanguage is a pleonasm. Instead, he suggests an immanent conceptualization of truth, in which the signification of the rules is given by the interaction between the rules. There, proofs are opposed to counter-proofs rather than counter-models. In his ludics [3], Girard focuses on a geometric approach which intends to uncover the structure underlying the logic. Therefore, the choice of a particular logic only occurs after the formalization of the game. The structures obtained through this process constitute a skeleton of the proof as much as a trace of the interaction between the two players. In short, truth only exists as a by-product of the interaction between logical entities: the notion of winning strategy encompass the notion of truth [4].

Girard's conception of truth is relative to a viewpoint, which allows to understand the coherence of propositions, by a construction which mirrors the use of the propositions. A viewpoint is a commutative sub-algebra. For example, diagonal matrices with respect to a fixed base.

There is a paradox behind this conceptualization: not only can we have a system which isn't true or false (when there's no winning strategy in the system or its negation), but it might be that the same system is true for some viewpoints, false for others. However, it would be a mistake to believe that this paradox makes Girard's definition relativistic. For example, to accomplish the modus ponens, one must use the same viewpoint for $A \multimap B$, A and B. By inference, we obtain a logical system which is not subjective but inter-subjective. This way, the meaning of the proposition A is taken in its inter-subjective context, formalized by the construction of a common viewpoint shares by all the possible inferences starting from A.

4 A Wittgensteinian perspective

I am of the opinion that game semantics and ludics share the same form of symmetrization of logics. Instead of placing the concept of truth at the center, one adopts a symmetrising approach in which truth does not arise on the first plan, which consequently puts the focus on interactions between mathematical propositions.

In his Tractatus Logico-Philosophicus, Wittgenstein writes that "in order to be able to say "p" is true (or false) I must have determined under which conditions I call "p" true, and thereby I determine the sense of the proposition." [8, 4.063]. Therefore, a proof must be understood within the network of proofs it is part of. Girard's ludics realize this philosophical intuition by stressing the geometric properties of mathematical proofs. On a more general note, it seems that game-theoretic semantics realize the Wittgensteinian ambition of perceiving a proposition by the interaction between the conditions of its validity.

References

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