

PHINE NUMBERS

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ABSTRACT. The binary number system is based on the classical fact that the unit interval $[0, 1]$ is the union of two intervals of size $\frac{1}{2}$. But constructing a binary representation of $0.0101 \dots + 0.00101 \dots$ is a problem: one needs overlapping subintervals. If one takes as size the reciprocal φ of the golden ratio Φ (the positive numbers given by $\varphi = \frac{1}{\Phi}$ and $\Phi = 1 + \varphi$), then the size of the overlap is φ^3 . This leads to the *phinary* number system in which sums can be calculated from left to right using only a *finite* number of states. Each natural number has a *finite* representation in base Φ .

In this article we introduce the *phine* number system. It is based on a variant with three subintervals of size φ^2 . In base Φ^2 , one needs an extra digit: either 2 or -1 . In both variants, each natural number has a *symmetrical* representation. Addition can be defined on such sequences of signed bits in a symmetrical way. The signed bit of this sum at some position only depends on the original signed bits at positions up to distance 4.

INTRODUCTION

There are several ways to represent numbers as a sequence of digits and to perform calculations on the representations. For example, $MCMXLVII + LX = MMVII$ (Roman) and $1947 + 60 = 2007$ (decimal). A *digit* like X can be identified with the *number* 10 represented by it (the number of my fingers), but a sequence of digits like $MMVII$, i.e. the finite sequence 1000, 1000, 5, 1, 1 is a *function* defined on a set $\{0, 1, 2, 3, 4\}$ of *positions*. The number represented by a sequence of digits in some number system can be calculated: $I + I + V + L - X + M - C + M = 7 + X \cdot (4 + X \cdot (9 + X \cdot 1))$.

Let us call a number a *binary decimal* if it is the sum of a (unique) finite *set* of powers of 10. Each natural number is the sum of a *sequence* of powers of 10, e.g. 2007 is the sum of 1000, 1000, 1, 1, 1, 1, 1 and 1. Such a sequence can be shortened if some power of 10 has at least 10 occurrences. So each natural number is the sum of 9 binary decimals and each sum of 10 binary decimals can be reduced to a sum of 9 decimals.

A sum of powers of 10 can be represented by a sequence of numbers $d_1, \dots, d_n$ where for each $p < n$, d_{n-p} is the number of occurrences of 10^p . In base 10, two such sequences represent the same natural number if and only if they can be rewritten to the same sequence by repeatedly applying the following steps:

- (1) add a number 0 to the left if the first number is positive
- (2) replace consecutive numbers $m, n + 10$ by $m + 1, n$

This rewrite system is confluent and terminating and the normal form of the sequence 2007 (of length 1) is the sequence 0, 2, 0, 0, 7 (of length 5). Except for initial digits 0, the normal forms are the standard decimal representations: sequences of standard digits (natural numbers $d < 10$). If one restricts the rewrite steps to sequences of natural numbers $d \leq 10$, then one cannot only use these to eliminate occurrences of the extra digit X , but also (in the opposite direction) to eliminate occurrences of the digit 0 (which also does not occur in Roman numerals): 02007, 2007, 1X07 and 19X7 represent the same number.

Addition can be performed on decimal representations by normalising the pointwise sum: $1947 + 60 = 1947 + 0060 = 19(4 + 6)7 = 19(0 + 10)7 = 1(9 + 1)07 = 1(0 + 10)07 = (1 + 1)007 = 2007$.

In a number system with base b , a sequence of digits $\dots d \dots$ represents a sum $\dots + d \cdot b \cdot b \dots b + \dots$ where the number of factors b depends on the digits at positions to the right of d . So the decimal number system has base 10 (the number of factors 10 is the same as the number of digits to the right of d) and the Roman number system has base -1 (the number of factors -1 is odd if and only if there is a larger digit to the right of d). In order to determine the number of factors -1 in the usual way, one should add a digit for 0: *MOCOMXLOVOIOI*.

The binary number system is a simplification of the decimal number system: in base 2, one only needs two digits: either the usual bits 0 and 1, or the digits 1 and 2: 60 can be represented in base 2 by 111100, but also by 22212. An advantage of this last version is that each natural number is represented by a *unique* finite sequence of digits 1 and 2. Concatenation of such sequences represents an operation on natural numbers which is, in some sense, very basic: one can give a first-order definition of addition and multiplication of natural numbers in terms of 1, 2 and this operation.

1. FROM NATURAL NUMBERS TO COMPLEX NUMBERS

One can add, multiply or compare numbers.

Complex numbers can be identified with the points in the Euclidean plane, given the points corresponding to 0 and 1 and an orientation. Adding numbers corresponds to translating, multiplying corresponds to scaling and rotating. If one draws a picture, an approximation of the exact location of a point may be sufficient.

Natural numbers can be seen as cardinalities of finite sets. The basic operations correspond to counting disjoint unions and products of finite sets. The exact result of such operations can be calculated using a suitable representation for natural numbers, like the usual decimal or binary representation. But if the calculations are only meant to compare the sizes of finite sets (the number of votes after a referendum or the number of free bytes before copying a file), then an approximation of the result is often sufficient.

Natural numbers and complex numbers are related via integers, rationals and reals:

$$\{0\} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

All these sets are closed under addition and multiplication: continuous operations with respect to the usual topology on $\mathbb{C}$.

Three of these sets are vectorspaces over $\mathbb{R}$ (of dimension 0, 1 and 2). The first uncountable set in the chain is $\mathbb{R}$, the topological closure of the preceding ring. Its members are the limits of Cauchy sequences in that ring. So these are of the form $\sum_{n=0}^{\infty} q_n$ where $q_0, q_1, \dots$ is a summable sequence in that ring.

Some sets in the chain are the closure under addition and multiplication of the preceding set after adding one of the numbers 1, -1 , i , or all of the numbers $\frac{1}{p}$ (p prime). The reciprocals of the added numbers are also in the set.

Multiplication distributes over addition, so one can first close the set under multiplication and then close the resulting set under addition.

For each natural number n , the set $\{n^p : p \in \mathbb{N}\}$ is closed under multiplication. Its closure under addition is either $\mathbb{N}$ (for $n = 0$) or the set of *positive* integers.

The closure of $\{\frac{3+4i}{5}\}$ under multiplication is not a (multiplicative) group, but the topological closure of that set is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, an uncountable multiplicative group. The closure of the unit circle under addition is $\mathbb{C}$.

Rational numbers like 0, 1, -1 and $\frac{1}{2}$ can be characterised in terms of addition and multiplication. The Euclidean metric can be recovered from the topology: for all complex numbers u and v , $|u| < |v|$ if and only if v is invertible and $\lim_{n \rightarrow \infty} (\frac{u}{v})^n = 0$. The restriction of this relation to natural numbers (which can also be defined in terms of addition) is a wellordering which can be used to define addition and multiplication.

The open balls in $\mathbb{C}$ can be defined in terms of addition, multiplication and the topology. One can also work the other way around: translation and scaling can be defined in terms of intersecting lines and rotation in terms of circles. The convex hull of a finite set of points (the endpoints of a line segment or the vertices of a polygon) is the intersection of all open balls which contain these points. The line through two different points P and Q is the union of all line segments which contain P and Q . The real line is the line through the points 0 and 1.

For each $n > 0$, ζ_n is the result of rotating 1 around 0 over $\frac{360}{n}$ degrees. So $\zeta_1 = 1$, $\zeta_2 = -1$ and $\zeta_4 = i$. The ratio between the angles of triangle $0, 1, \zeta_{n+2}$ is $4 : n : n$. For $n > 1$, the powers ζ_n^p ($p \in \{0, \dots, n-1\}$) are the vertices of a regular n -gon with centre 0 and the partial sums $0, 1, 1 + \zeta_n, 1 + \zeta_n + \zeta_n^2, \dots$ are the vertices of a regular n -gon with edge $[0, 1]$. Here a 2-gon is just a line segment.

One can define the set of complex numbers (with addition, multiplication and the usual topology) in terms of real numbers: for any $z \in \mathbb{C} \setminus \mathbb{R}$, the function which maps each pair (x, y) of real numbers to the complex number $x + yz$ is a bijection. Multiplication on $\mathbb{C}$ can be represented on $\mathbb{R} \times \mathbb{R}$ in terms of the unique pair (x, y) of real numbers for which $z^2 = x + yz$. Usually one takes $z = i = \zeta_4$, so $z^2 = -1$, but one can also take $z = \zeta_3$ ($z^2 = -1 - z$) or $z = \zeta_5$ ($z^2 = -1 + \varphi z$). Here $\varphi = \zeta_5 + \zeta_5^{-1}$ is the positive real y such that $y^2 = 1 - y$. So one can identify complex numbers with points in an affine space, after having chosen three points (not on one line) corresponding to 0, 1 and z .

2. NUMBER SYSTEMS

We will identify a digit with the number represented by it, so $I = 1$ and $X = 10$. In the decimal number system, 10.1 represents the rational number $\frac{101}{10}$. One can use a floating point to distinguish a representation like 101.0 from the natural number 101 represented by it in the decimal number system. Note that the same representation 101.0 represents 10 in base 3. A sequence of 5 numbers a, b, c, d, e can be denoted by $a.bcde$, whereas $abc.de$ denotes the function $\{(-2, a), (-1, b), (0, c), (1, d), (1, e)\}$.

We do not distinguish between 1 as Roman numeral, bit, digit, position (directly to the right of a floating point), natural number, integer, rational number, real, complex number or point in the Euclidian plane. An expression with a floating point always denotes

a representation. So 1.6180 is a function which maps position 3 to digit 8. For simplicity, we assume that the set of positions is always $\mathbb{Z}$: we do not distinguish between the representations 1.6180, 1.618 and $\dots 0001.618000\dots$. So the usual (unique) representation $\dots 000f_m\dots f_{-2}f_{-1}f_0.000\dots$ of a natural number in base n ($n > 1$) is a function $f : \mathbb{Z} \rightarrow \{0, 1, \dots, n-1\}$ such that for some $m < 0$, for every $p \in \mathbb{Z}$, if $p < m$ or $p > 0$ then $f(p) = 0$.

The digits and the base need not be natural numbers.

Definition 2.1. A *digit* is a (complex) number, i.e. a point in the Euclidian plane. A *position* is an integer. A *representation* $\dots d_{-2}d_{-1}d_0.d_1d_2\dots$ is a function d which maps each position p to a digit d_p . A *number system* is a function from a set of representations to a set of (complex) numbers.

The set of all representations $d : \mathbb{Z} \rightarrow \mathbb{C}$ form a group (with pointwise addition). The *left shift* of d is $\dots d_{-2}d_{-1}d_0d_1.d_2\dots$ and the *right shift* of d is $\dots d_{-2}d_{-1}.d_0d_1d_2\dots$. The *integer part* of d is $\dots d_{-2}d_{-1}d_0.00\dots$ and the *fractional part* of d is $\dots 000.d_1d_2\dots$.

Definition 2.2. Let $z \in \mathbb{C}$, $z \neq 0$. A number system F has *base* z if its domain D is closed under left and right shift and $F(\dots a_0a_1a_2\dots) = z \cdot F(\dots a_0.a_1a_2\dots)$ for each a in D .

A number system F is *normal* if for every number a , $F(\dots 000a.000\dots) = a$.

A number system F is *additive* if its domain D is a group (with pointwise addition) and $F(\dots (a_0 + b_0).(a_1 + b_1)\dots) = F(\dots a_0.a_1\dots) + F(\dots b_0.b_1\dots)_z$ for all a and b in D .

The standard example of a normal, additive number system with base z is the following: a function $f : \mathbb{Z} \rightarrow \mathbb{C}$ represents the number $(f)_z := \sum_{p \in \mathbb{Z}} f(p)z^{-p}$, assuming that this sequence is (absolutely) summable: for every $\epsilon > 0$ there is some finite $A \subseteq \mathbb{Z}$ such that for each finite $B \subseteq \mathbb{Z} \setminus A$, $|\sum_{p \in B} f(p)z^{-p}| < \epsilon$. Note that if f is symmetrical ($f(p) = f(-p)$ for each position p) then f represents the same number in base z and in base z^{-1} .

If $|z| > 1$ then $(\dots 000.111\dots)_z = \frac{1}{z-1}$. So, if $z \in \mathbb{R}$, $z > 1$, $M \in \mathbb{R}$, $|a_1| \leq M$, $|a_2| \leq M$, $\dots$ then $|(\dots 000.a_1a_2\dots)_z| \leq \frac{M}{z-1}$.

The standard number system with base z and digits $d_0, d_1, \dots, d_n$ is the restriction of the standard example to representations of the form $f : \mathbb{Z} \rightarrow \{d_0, d_1, \dots, d_n\}$. Whether such an f represents a number only depends on the base z and the set P of positions p which are *occupied* in f , i.e. for which $f(p) \neq 0$: if $|z| = 1$, then P should be finite; if $|z| > 1$, then P should have a lower bound; if $|z| < 1$ ($\lim_{p \rightarrow \infty} z^p = 0$), then P should have an upper bound ($\lim_{p \rightarrow \infty} f(p) = 0$).

In fact, the standard example can be extended to a normal, additive number system in which each constant or repeating representation represents 0. (For $|z| > 1$ this is the only possible interpretation.) Then $(\dots 111.111\dots)_2 = 0$ and $(\dots 000.(-1)(-1)(-1)\dots)_2 = -1$, so (by pointwise addition) $(\dots 111.000\dots)_2 = -1$, just like the representation in the field of 2-adic numbers.

Compare the following number systems (in increasing order of the base):

<i>unary</i>	$1 = (0.1000\dots)_1$	$10 + 10 + 10 = 111$ in base 1 (and in no other base)
<i>phinary</i>	$1 = (0.0111\dots)_\Phi$	$1 + 1 + 10 = 101$ in base Φ and base $-\varphi$
<i>binary</i>	$1 = (0.1111\dots)_2$	$1 + 10 + 10 = 101$ in base 2
<i>phine</i>	$1 = (0.2111\dots)_{\Phi^2}$	$10 + 10 + 10 = 101$ in base Φ^2 and base φ^2
<i>ternary</i>	$1 = (0.2222\dots)_3$	$10 + 10 + 10 = 100$ in base 3

3. EXTENDING THE BINARY NUMBER SYSTEM

3.1. Integers. In order to represent all integers, one can adjust the binary representation of natural numbers in three ways:

- (1) represent negative integers by infinite sequences of bits which start with an infinite sequence of bits 1, e.g. -60 is represented by $\dots 1110110100$. In fact this is the 2-adic representation of integers: any infinite sum of powers of 2 converges in the 2-adic topology in which high powers of 2 are *small*. Addition and multiplication of infinite sequences of bits is easy: the last n bits of the result depend on the last n bits of the input by calculating modulo 2^n .
- (2) allow *signed* bits, i.e. members of $\{-1, 0, 1\}$. For every integer a there are natural numbers p and n such that $a = p - n$ (this representation is unique if one requires $pn = 0$, i.e. $p = 0$ or $n = 0$). The binary representations of p and n can be combined into a sequence of signed bits. This finite sequence is not unique, e.g. 3 can be represented by 11, 10(-1), 1(-1)1 or 1(-1)01 and so on. The first non-zero signed bit always denotes the sign of the represented integer. Note that the representation of a natural number in base 3 with signed bits is unique (except for initial digits 0), just like the usual representation (digits 0, 1 and 2) and the more efficient representation (digits 1, 2 and 3).
- (3) adjust the base: every integer is the sum of a unique finite set of powers of -2 , e.g. $60 = 64 - 8 + 4$, so it can be represented in base -2 as 1001100; negative integers are represented by sequences in which the leftmost bit 1 is followed by an *odd* number of bits.

In base -2 , each *integer* can be represented by a unique finite sequence of bits which does not start with the bit 0. In base -2 , $1 + 1 = 110$, so the output uses more digits than the input (just like $IX + IX = XVIII$).

In base -2 , two sequences of natural numbers represent the same integer if and only if they can be rewritten to the same sequence by repeatedly applying the following steps:

- (1) add a number 0 to the left if the first number is positive
- (2) replace consecutive numbers $m + 1, n + 2$ by m, n
- (3) replace consecutive numbers $m, n, p + 2$ by $m + 1, n + 1, p$

Note that the third step is also valid in base 1. For positive n , such a step is a combination (via $m, n - 1, p$) of the second step and its inverse. This rewrite system is not terminating: $1, 2 \rightarrow 0, 1, 2 \rightarrow 1, 2, 0 \rightarrow \dots$

3.2. Reals. The binary number system can be extended from natural numbers to the complex numbers via *dyadic rationals*, i.e. numbers of the form $\frac{m-n}{2^p}$ ($m, n, p \in \mathbb{N}$).

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Z}[2^{-1}] \subseteq \mathbb{Z}[2^{-1}][i] \subseteq \mathbb{C}$$

Note that the ring of dyadic numbers ($\mathbb{Z}[\frac{1}{2}]$) is dense in $\mathbb{R}$. It is finitely generated as a ring, but not as additive group.

The binary number system is based on the classical fact that each real interval $[a, b]$ is the union of two intervals of relative size $\frac{1}{2}$:

$$[a, b] = [a, \frac{a+b}{2}] \cup [\frac{a+b}{2}, b]$$

For example $[-1, 1] = [-1, 0] \cup [0, 1]$ and $[0, 1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$. Constructively this is not valid: there are arbitrarily small subintervals of $[-1, 1]$ which are neither a subinterval of the left half $[-1, 0]$, nor of the right half $[0, 1]$. This implies, even classically, that addition of non-negative reals cannot be performed in a continuous way on the set of binary representations, i.e. infinite sequences $\dots a_0.a_1\dots$ of bits 0 and 1 where $a_{-n} = 0$ for large n . In order to construct even the first bits of a binary representation of $0.0101\dots + 0.00101\dots$, all bits of the input can be relevant.

The sum of computable binary reals need not be computable: Given binary representations of x and y , try to represent $x + y$ as a binary number. Now the bit b at some position p of that representation may depend on a single bit c of the representation of y at a position far to the right:

$$\begin{array}{rcl} x & = & \dots 0010101\dots 100\dots \\ y & = & \dots 0101010\dots c00\dots \\ \hline x + y & = & \dots b\dots \\ \text{if } c = 0: & & \dots \underline{0}11111\dots 1\dots \\ \text{if } c = 1: & & \dots \underline{1}000000\dots 0\dots \end{array}$$

This shows that the usual way of adding binary numbers only works if the positions are limited to the right, e.g. for adding natural numbers, 2-adic numbers or floating point numbers with as many bits as can be represented in some computer memory. It also shows that this way of adding is not very efficient: one may have to consider each bit just to find the first bit of the sum.

This problem can be avoided by using overlapping subintervals: either more than two subintervals of relative size $\frac{1}{2}$ or two subintervals of relative size larger than $\frac{1}{2}$. Then each subinterval of $[-1, 1]$ of size less than the overlap of consecutive subintervals is part of one of these subintervals.

Three subintervals of relative size $\frac{1}{2}$ have overlaps of size $\frac{1}{4}$ relative to the original interval:

$$[-1, 1] = [-1, 0] \cup [-\frac{1}{2}, \frac{1}{2}] \cup [0, 1]$$

This leads to a representation of reals in this interval as an infinite sequence of signed bits $(-1, 0, 1)$ or, equivalently, as a difference of two reals in $[0, 1]$ which have a binary expansion. For example, $\frac{1}{5} - \frac{1}{3} = 0.00110011\dots - 0.01010101\dots = 0.0(-1)100(-1)10\dots = -\frac{2}{15}$.

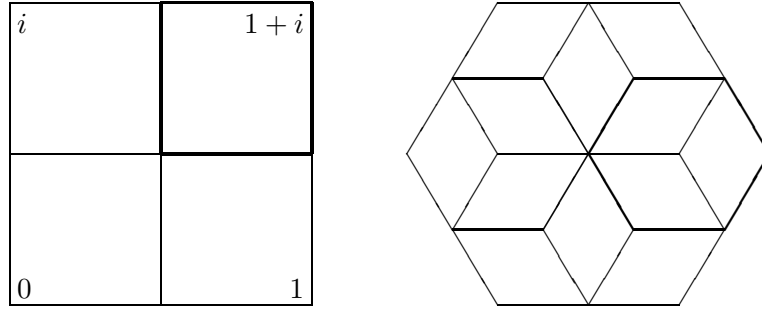
Reducing the sum of three binary numbers to the sum of two binary numbers can be performed in parallel (without needing a “carry”): just use the fact that for all bits a, b and c there are (unique) bits d and e such that $a + b + c = d + d + e$. So this also holds (pointwise) for *sequences* of bits.

3.3. Complex numbers. The binary number system can be extended to the complex numbers, using squares instead of intervals: classically, each complex number inside the unit square is the limit of a Cauchy sequence $z_0, z_1, \dots$ in $\mathbb{Z}[i, \frac{1}{2}]$ which starts at the bottom side ($z_0 = 0, z_1 = 1$) such that, for each n , $\frac{z_{n+2} - z_{n+1}}{z_{n+1} - z_n}$ is either i (“turn left”) or $\frac{1}{2}i$ (“turn left and slow down 50%”).

Using base 2 is just the simplest possibility: extending the usual number system with base n to the complex numbers corresponds to splitting up a square in n^2 similar subsquares (without common subsquares, i.e. any overlap is just a point or line segment). Each subsquare corresponds to a complex digit of the form $a + bi$ where a and b are usual digits.

One can also cover the Euclidian plane with triangles (whose vertices form the ring $\mathbb{Z}[\zeta_3]$) and split up each triangle in n^2 subtriangles, but since some triangles are upside down, this does not correspond to a number system with base n and a finite set of digits.

The other way to cover the plane with regular polygons is to use hexagons (and Cauchy sequences in $\mathbb{Z}[\zeta_6, \frac{1}{2}]$). But then the subhexagons overlap:



4. THE PHINARY NUMBER SYSTEM

The *golden ratio* Φ and its reciprocal φ are the positive numbers given by $\varphi = \frac{1}{\Phi}$ and $\Phi = 1 + \varphi$. The decreasing sequence $1, \varphi, \varphi^2, \dots$ can be characterised in terms of addition: it is the unique converging sequence $z_0, z_1, z_2, \dots$ of complex numbers such that $z_0 = 1$ and $z_n = z_{n+1} + z_{n+2}$ for each n .

Each real interval is the union of two subintervals of relative size φ . The overlap has relative size φ^3 :

$$\begin{aligned} [-1, 1] &= [-1, \varphi^3] \cup [-\varphi^3, 1] \\ [0, 1] &= [0, \varphi] \cup [\varphi^2, 1] \end{aligned}$$

This leads to the *phinary* number system (see [Knott]) in which sums can be calculated from left to right using only a *finite* number of states (see [DiGianantonio]). Any combination of consecutive digits 011 can be eliminated by replacing it by 100. Each natural number has a *finite* representation in base Φ and even a unique finite phinary representation in which no consecutive bits 011 occur: $0.11 = 1$, $2 = 10.01$ and $3 = 100.01$. But even calculating $n + 1$ from n in the phinary number system is not completely trivial. In base Φ , two representations with at a finite number of positions a positive integer and at each other position the number 0 represent the same real if and only if they can be rewritten to the same sequence of natural numbers by repeatedly applying the following steps:

- (1) replace consecutive numbers $m, n + 1, p + 1$ by $m + 1, n, p$
- (2) replace consecutive numbers $m, n + 2, p, q$ by $m + 1, n, p, q + 1$

Note that the second step is also valid in base 1. One can show that it is terminating (but the number of steps may be large): there is a bound for the position of the first non-zero digit (in terms of the real number represented by the sequence) and the sequence increases lexicographically at each step.

4.1. Complex numbers. The *phinary number system* is usually defined in terms of the ring $\mathbb{Z}[\Phi]$, where Φ is the *golden ratio* $\frac{1}{2} + \frac{1}{2}\sqrt{5}$. This ring is the real part of what I will call the ring of *phine numbers*:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Z}[\Phi] \subseteq \mathbb{Z}[\zeta_5] \subseteq \mathbb{C}$$

Let $\varphi = \zeta_5 + \zeta_5^{-1}$. Then $\varphi = \Phi^{-1}$ and $\Phi = 1 + \varphi$.

The set $\mathbb{Z}$ of integers is the least set of complex numbers which contains ζ_2 and is closed under addition and multiplication. For every integer z there are unique natural numbers a and b such that $z = a\zeta_2 + b$ and $ab = 0$ (i.e. at least one of the numbers is 0). The ring $\mathbb{Z}[i]$ can be defined in a similar way (with 4 instead of 2). The following definition is similar, but with 5 instead of 2. The main advantage is that it defines a finitely generated (additive) group which is dense in $\mathbb{C}$.

Definition 4.1. The set $\mathbb{P}$ of *phine numbers* is the least set of complex numbers which contains ζ_5 and is closed under addition and multiplication.

Note that the phine numbers form the ring $\mathbb{Z}[\zeta_5]$. This follows from the fact that $0 = 1 + -1$, $1 = \zeta_5^5$ and $-1 = \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4$. For every phine number z there are unique integers a, b, c, d such that $z = a\zeta_5 + b\zeta_5^2 + c\zeta_5^3 + d\zeta_5^4$. There are unique natural numbers a, b, c, d, e such that $z = a\zeta_5 + b\zeta_5^2 + c\zeta_5^3 + d\zeta_5^4 + e$ and $abcde = 0$ (i.e. at least one of the numbers is 0).

Since $\zeta_{10} = -\zeta_5^3$ and $\zeta_5 = \zeta_{10}^2$, we also have that $\mathbb{P} = \mathbb{Z}[\zeta_{10}]$.

Let f be the ring automorphism of $\mathbb{P}$ which maps ζ_5 to ζ_5^2 . Then each ring automorphism is an iterate of f . One can show that for each phine number z : $f(z) = z$ if and only if $z \in \mathbb{Z}$; $f(f(z)) = z$ if and only if $z \in \mathbb{R}$; z is invertible in $\mathbb{P}$ if and only if $zf(z)f(f(z))f(f(f(z)))$ is invertible in $\mathbb{Z}$, i.e. is either 1 or -1 . In fact, the invertible members of this ring are the members of the form $\varphi^m \zeta_{10}^n$ (where m and n are integers). So the units (invertible members) of the ring of phine numbers form decagons with center 0 and relative sizes powers of the golden ratio.

For each phine number z , $f(f(z)) = \bar{z}$, the complex conjugate of z . So ζ_5 and ζ_5^4 cannot be distinguished in terms of addition, multiplication and the topology on $\mathbb{C}$. But ζ_5 and ζ_5^2 can: $\varphi = \zeta_5 + \zeta_5^4$ and $f(\varphi) = \zeta_5^2 + \zeta_5^3 = -1 - \varphi = -\Phi$ are the solutions of $z + z^2 = 1$, but only $z = \varphi$ satisfies $\lim_{p \rightarrow \infty} z^p = 0$.

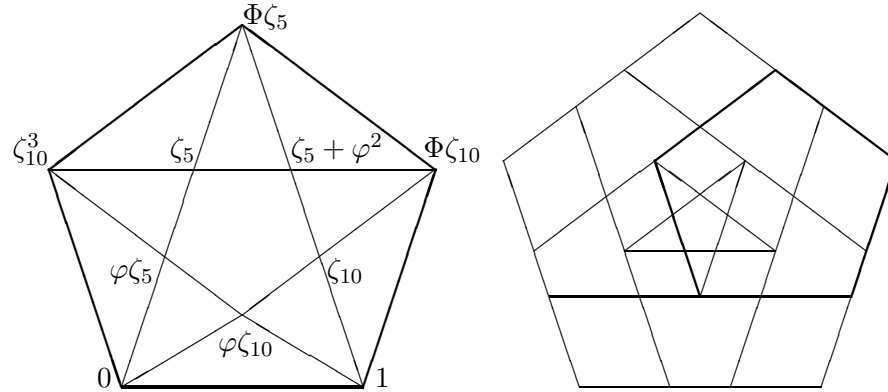
Since $\varphi = \zeta_5 + \zeta_5^{-1}$, the intersection of the ring with $\mathbb{R}$ is $\mathbb{Z}[\varphi]$. Since this is dense in $\mathbb{R}$, $\mathbb{Z}[\zeta_5]$ itself is dense in $\mathbb{C}$.

One can also construct phine numbers in a more geometrical way, starting with the numbers 0 and 1 and closing under rotations over 36 degrees around already constructed points.

Each phine number is the sum of a sequence of powers of ζ_5 . Modulo 5, the length of this sequence is fixed. This splits the ring of phine numbers into 5 equivalence classes which are invariant under rotation over 72 degrees around any phine number. One can construct all phine numbers, starting with the natural numbers $n < 5$ (or with the vertices of a regular pentagon in which 0 and 1 form a side) and closing under rotations over 72 degrees around already constructed points.

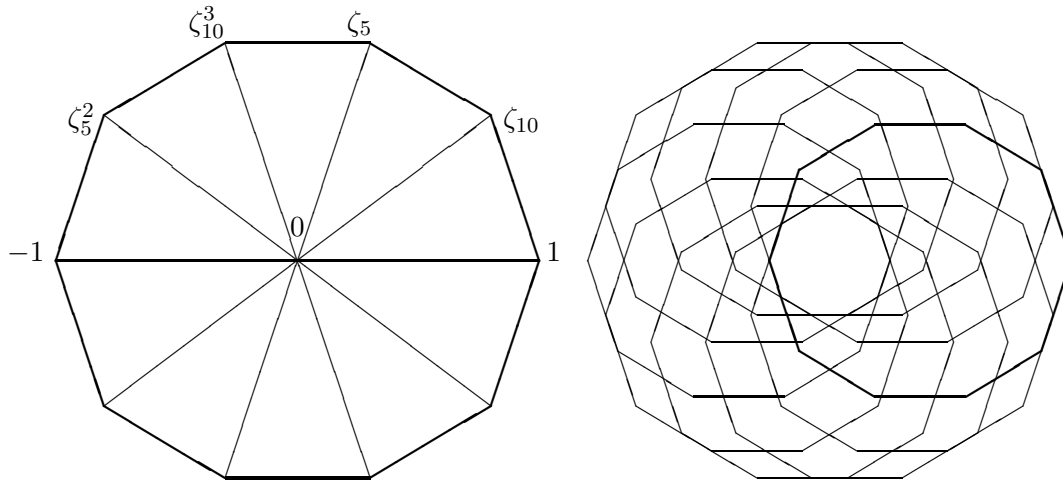
The phinary number system can be extended to the complex numbers, using *pentagons* instead of intervals: a regular pentagon with edges of length 1 (and diagonals of length Φ) is covered by 5 overlapping pentagons with edges of size φ . The intersection of these pentagons is a pentagon with edges of size φ^4 . The phinary number system can be extended to the complex numbers, using pentagons instead of intervals. If each of the edges of a pentagon

is parallel to some diagonal, then the size of any edge relative to the parallel diagonal is the golden ratio φ . This holds in particular for any regular pentagon.



Now each such pentagon is covered by 5 overlapping pentagons of relative size φ . The intersection of these 5 pentagons is a pentagon of relative size φ^4 .

Note that a triangle is *not* covered by 3 subtriangles of relative size φ , whereas pointsymmetric convex polygons like parallelograms or regular decagons are covered by similar subpolygons of relative size φ (one for each vertex). The overlap is a similar subpolygon of relative size φ^3 and every smaller similar subpolygon of the original polygon is part of one of the subpolygons of relative size φ .



5. THE PHINE NUMBER SYSTEM

We now introduce the *phine* number system. It is based on a variant with three subintervals of size φ^2 . Three subintervals of relative size the *square* φ^2 of the golden ratio (given by $\varphi^2 + \varphi^{-2} = 3$) have overlaps of relative size $\frac{1}{2}\varphi^4$:

$$[-1, 1] = [-1, -\varphi^3] \cup [-\varphi^2, \varphi^2] \cup [\varphi^3, 1]$$

This leads to a new *phine* number system in which each real is represented by at least one infinite sequence of signed bits (-1 , 0 or 1). It is clear that $0.000\dots$ is the only representation of the number 0 : every representation of a positive (negative) number has a 1 (-1) as leftmost non-zero signed bit; a number x has a representation of the form $1.a_1a_2\dots$ if

and only if $x \in [\varphi^2, \Phi]$ so for each non-zero number there are at most two possible positions for the first non-zero signed bit. A non-zero number has a unique representation if and only if it has a representation with an infinite number of non-zero signed bits whose signs alternate, e.g. $\frac{1}{1+\varphi^4} = 1.0(-1)010(-1)0\dots$. The number φ has exactly two representations: a finite one $(1.(-1))$ and an infinite one without negative bits $(0.111\dots)$. The number $\frac{1}{2}$ has countably many repeating representations (like $0.110(-1)(-1)0110(-1)(-1)\dots$ and $1.(-1)(-1)1(-1)(-1)\dots$) and uncountably many nonrepeating representations. In fact, all possible representations of $\frac{1}{2}$ can be generated as follows, starting at the position before the “decimal” point in state (i):

(i): either write $1(-1)(-1)$ and go to (i) or write 011 and go to (ii)

(ii): either write $(-1)11$ and go to (ii) or write $0(-1)(-1)$ and go to (ii)

Addition can be defined on these sequences in a symmetrical way. The signed bit of this sum at some position only depends on the original signed bits at positions up to distance 4.

In base Φ^2 , one needs an extra digit: either 2 or -1 . In both variants, each natural number has a *symmetrical* representation: $5 = 12.1 = 1(-1)1.(-1)1$, $7 = 21.2 = 100.01 = 1(-1)(-1)(-1).(-1)(-1)1$ and $60 = 10120.2101$. Addition can be defined on such sequences of signed bits in a symmetrical way. The signed bit of this sum at some position only depends on the original signed bits at positions up to distance 4.

Each integer has a symmetrical representation as finite sum of different numbers of the form $\pm\varphi^{2a}$ (a integer). This representation is not unique, e.g. 3 can be represented as 10.1 but also as $1(-1)(-1).(-1)1$. Note that one could leave out the floating point if one only considers symmetrical representations. The first non-zero signed bit not only denotes the sign of the represented integer, but also indicates the order: if the first 1 of a representation of m is at least two positions to the left of the first 1 in a representation of n , then $m > n$ (in fact, $\frac{n}{m} \leq \varphi$). If one only wants to represent natural numbers, then one can use digits 0, 1 and 2 (just like in ternary notation). One can calculate $n+1$ from n in this representation by first adding 1 to the central digit and then repeatedly eliminate any digit 3 by replacing it by 0, while adding 1 to the neighbouring digits. Note that this terminates: the sum of all digits decreases by 1 each time we replace three consecutive digits $a, b+3, c$ by $a+1, b, c+1$. If we start with exactly one digit 3, then eliminating it may introduce a new digit 3 to the left and/or to the right of the position of the original 3. Eliminating such a new occurrence of 3 may only introduce a single new 3: immediately after eliminating a 3 from some position, the value at that position is 0, which may raise to 1 or 2 if a 3 is eliminated from one or both of the neighbouring positions.

In general, if we start with a (two-sided infinite) sequence $\dots, d_n, d_{n+1}, \dots$ of natural numbers which are non-zero at finitely many positions, then the result of eliminating all numbers > 2 is independent of the order of elimination steps: Let D be the set of all positions n such that $d_n > 2$. Then any maximal sequence of elimination steps will contain, for each $n \in D$, at least one step which decreases the number at position n (which may be $> d_n$ by then). The steps can be reordered in such a way that the steps at these positions n take place before any other steps. This can be repeated until all numbers > 2 are eliminated.

Now suppose that we start with a finite sequence of digits 0, 1, 2 and 3. Then, by induction to the number of elimination steps, at each position the digit is decreased by 3 at most once and increased by 1 at most twice. So while eliminating digits 3 or higher, no digit higher than 5 will be introduced.

This does *not* imply that each natural number has a unique representation in base Φ^2 with digits 0, 1 and 2: the least counterexample is 7, which has representations 21.2 and 100.01. Consecutive digits 0, 2, 2, 0 and 1, 0, 0, 1 are also equivalent. But by allowing negative digits, one can reduce e.g. 21.2 via $1(-1)2.2$ and $10(-1).3$ to 100.01.

A sequence of digits $-2, -1, 0, 1$ and 2 can be reduced to a sequence without 2 or -2 . Each digit in the output sequence only depends on the input digits from 4 positions to the left to 4 positions to the right.

Definition 5.1. The *carry sequence* of a representation b in base Φ^2 is the sequence c defined by $c(n) = b(n+1) - 3b(n) + b(n-1)$ for each position n .

Note that each carry sequence represents the number 0. One can show that two representations represent the same real number if and only if their pointwise difference is a carry sequence.

Theorem 5.2. *Let a and b be sequences of signed bits. Then there are sequences c and d of signed bits such that for each integer n , $d(n) = a(n) + b(n) - 3c(n) + c(n-1) + c(n+1)$. In fact, we can choose $c(n)$ as a (symmetrical) function of the values $a(m) + b(m)$ for $|m - n| \leq 3$, so the sign $d(n)$ only depends on the signs $a(m)$ and $b(m)$ for $|m - n| \leq 4$.*

One can show that this result is sharp and there is no such function in which the role of the bits 1 and -1 are symmetrical.

Let us call a position n at which $c(n) = 1$ an *overflow* position (reducing the digit by 3 and adding 1 to the digits to the left and the right) and a position n at which $c(n) = -1$ an *underflow* position. The algorithm consists of 3 steps:

- (1) let $a + b$ underflow at each position n at which $a(n) + b(n) < 0$;
- (2) let the result of the first step overflow at each position with digit 2 and also at consecutive positions with positive digits
- (3) let the result of the second step overflow at positions with digit 2 and underflow at positions with digit -2

5.1. Complex numbers. The phine number system can be extended to the complex numbers, using parallelograms (4 vertices and 4 edges) or decagons (10 vertices and 10 edges). Such a polygon is covered by 9 parallelograms or 21 decagons: shrink the original polygon by ratio φ^2 relative to each vertex, the middle of each edge and the centre of the polygon.

REFERENCES

- [Knott] Ron Knott. Using Powers of Phi to Represent Integers (Base Phi) <http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/phigits.html>.
 [DiGianantonio] Pietro Di Gianantonio. A Golden Ratio Notation for the Real Numbers. *CWI technical report*. <http://www.dimi.uniud.it/~pietro>.

