BINARY RELATIONS AS A FOUNDATION OF MATHEMATICS

JAN KUPER

University of Twente, Department of Computer Science, Enschede, The Netherlands
e-mail address: j.kuper@ewi.utwente.nl

Abstract. We describe a theory for binary relations in the Zermelo-Fraenkel style. We choose for ZFCU, a variant of ZFC Set theory in which the Axiom of Foundation is replaced by an axiom allowing for non-wellfounded sets. The theory of binary relations is shown to be equi-consistent ZFCU by constructing a model for the theory of binary relations in ZFU and vice versa. Thus, binary relations are a foundation for mathematics in the same sense as sets are.

Foreword

Early 1988 I had my first appointment with Henk to discuss a reformulation of the Zermelo-Fraenkel axioms to the concept of partial function. With every axiom I presented, Henk said "I buy that one". It was the start of a valuable period in my life, I still feel the pleasure to work under his supervision on my PhD thesis. Already at that time we discussed a further generalization of the Zermelo-Fraenkel axioms towards the concept of relation, but it's only now, on the occasion of Henk's 60th birthday, that I worked it out. I hope Henk will enjoy them and buy them all.

1. Introduction

Axiomatic Set Theory is a foundation for mathematics in the sense that all known mathematical theories can be modeled in Set Theory. The standard way to do so is by defining the basic objects of such a theory as a specific type of set, and to define all operations and predicates on these objects as set theoretical operations and predicates, respectively. This should be done in such a way that all set theoretical formulations of the axioms of the given theory can be proved within Set Theory. The most well known axiomatization of Set Theory is the Zermelo-Fraenkel axiom system (see e.g. [6]), usually including the Axiom of Choice. The resulting theory is denoted as ZFC. One axiom that deserves special attention is the Axiom of Foundation, by which infinitely descending $\in$-chains do not exist. Thus, a set a with $a \in a$, or sets u, v with $u \in v$ and $v \in u$, do not exist within ZFC. However, in this paper we will take a more general axiomatization of Set Theory, in which such sets *non-wellfounded sets* do exist. That is, we accept the Axiom of Universality (see [4]) instead of the Axiom of Foundation. We will denote the resulting theory as ZFCU. An essentially equivalent formulation of the Axiom of Universality is Aczel's well known Anti Foundation Axiom (AFA, see [1]). For the context of this paper, a pleasant consequence of the Axiom of Universality is that ZFCU contains a model for the Untyped Lambda Calculus such that functions are just sets of ordered pairs, and self-application and related constructions are nevertheless possible (see [14]). Other applications of non-wellfounded sets can be found in [9, 8]).

Despite the powerful character of the notion of set as a foundation of all of mathematics, it also is considered to be a rather problematic concept. Several attempts were done to search for a more tractable foundation for mathematics, without running into the intractable problems of Set Theory. It was long hoped that the concept of *function* would give such a foundational concept (see e.g. [7]). As approaches to formulate a theory for functions we mention the work of Von Neumann (see [12, 13]) and Church (see [5]). Both approaches to axiomatize the concept of function did not survive for long. Von Neumann's axiomatization of the concept of function was reformulated for sets by Bernays ([3]), and after the discovery of inconsistencies in his postulates, Church restricted his original theory for functions to the Lambda Calculus capturing computability ([2]). In [10, 11] an axiomatization for the concept of function was given along the lines of the axiom system ZFCU, and it was shown that the intractable features of the concept of set are present in the concept of function in exactly the same way.

In this paper we will further generalize the Zermelo-Fraenkel approach to the concept of *relation*. We describe in detail the Zermelo-Fraenkel style axioms for *binary* relations and we shortly indicate how to generalize that for n-ary relations. The result is an elegant theory for binary relations that has the same power as Zermelo-Fraenkel Set Theory, including the Axiom of Choice and the Axiom of Universality.

In itself the generalization of the Zermelo-Fraenkel axiom system towards the concept of relation is neither very difficult, nor very surprising — the surprising thing is that (to the best of the author's knowledge) it is never done before.

2. Basic abstraction

The main step in generalizing the Zermelo-Fraenkel style axiom system towards the concept of binary relation is to make the adequate mathematical abstraction. That is to say, once the universe of objects and the elementary (meta) relation for these objects are chosen, the formulation of the axioms is fairly straightforward. We choose the objects in the universe to be *binary relations*, and on the meta level we assume the existence of a *ternary* relation $\mathfrak{R}$. The atomic formula

$$\mathfrak{R}\,(x, y, z)$$

is interpreted as: *x holds between y and z*. Thus, in this formula the variables x, y, z all denote binary relations. In this formula, the object x indeed *plays the role* of a binary relation, whereas y and z are taken as objects only. However, this way of putting it is just to indicate the connection to our informal understanding of what a relation is. Formally, x, y, z are just first order objects, called "binary relations".

Note that this abstraction essentially is the same as in Zermelo-Fraenkel Set Theory where all objects in the universe are called "sets" and the meta relation is $\in$, for "membership". In a formula like $x \in y$ both x and y denote sets even though only y indeed plays the role of a set, whereas x plays the role of an element. Of course, the same abstract point of view holds for any first order theory.

It is straightforward to generalize the above abstract perspective to n-ary relations. To do so, assume that all objects in the universe are n-ary relations, and that there is an $(n+1)$-ary relation $\mathfrak{R}^n$ on the meta level. Then, as before, the atomic formula

$$\mathfrak{R}^n\,(x, y_1, \ldots, y_n)$$

is interpreted as x *holds between* $y_1, \ldots, y_n$. We will restrict ourselves to binary relations, and leave the generalization towards n-ary relations to the reader.

In the remaining part of this paper we first describe the possibility of the Russell paradox for a naive interpretation of binary relations (Section 3). Next, we describe several axioms for binary relations (Section 4) and prove the equi-consistency of these axioms with ZFC Set Theory (Section 5).

Notation. We add some syntactic sugar to make formulas more readable. In the remaining part of this paper we will write

$$x\,(y, z)$$

for the atomic formula $\mathfrak{R}\,(x, y, z)$.

3. Russell paradox

Assume a naive comprehension principle that allows for the definition of relations by means of a conceptual description. Then, define the binary relation r for all x, y as follows (note that x, y are binary relations as well):

$$r(x, y) \;\leftrightarrow\; \neg x(y, y).$$

That is, the relation r holds for x and y iff the relation x is not reflexive in y. Substitution of r for both x and y immediately gives a contradiction:

$$r(r, r) \;\leftrightarrow\; \neg r(r, r).$$

Thus the relation r may not exist in the universe. Consequently, the naive comprehension principle is too strong and we need more restrictive axioms. We take Zermelo-Fraenkel Set Theory as a guidance to formulate a series of axioms.

4. Axioms

We describe seven axioms, covering the Zermelo-Fraenkel axiom system including the Axiom of Choice and the Axiom of Universality. First of all there is the Axiom of Extensionality saying that a relation is completely determined by the objects it relates to each other. In a way the Axiom of Extensionality puts restrictions to the existence of relations: according to it, no two different relations relating the same objects can exist.

All six other axioms prescribe the existence of specific relations, i.e., all other axioms are in a way “productive”. Only one of these productive axioms states the existence of a relation unconditionally, the five others prescribe the existence of some relation on the given existence of some possibly other relation. These five axioms can be divided into two groups: three axioms use a specific internal property, two don’t.

We start with the Axiom of Extensionality. The effect of this axiom is described above already.

Axiom 1 (Extensionality). For all relations r, r' we have

$$\forall x, y \,.\, (r(x,y) \leftrightarrow r'(x,y)) \;\rightarrow\; r = r' \qquad \square$$

Before continuing with the other axioms, we give some definitions.

Definition 4.1.

- x is *in the domain* of r, notation $Dom(r,x)$, if $\exists y \,.\, r(x,y)$,
- x is *in the range* of r, notation $Rng(r,x)$, if $\exists y \,.\, r(y,x)$,
- x is *in the field* of r, notation $Fld(r,x)$, if $Dom(r,x) \vee Rng(r,x)$. □

We also need several standard definitions for binary relations.

Definition 4.2. A relation r is

- *transitive*: $\forall x, y, z \,.\, r(x,y) \wedge r(y,z) \rightarrow r(x,z)$
- *symmetric*: $\forall x, y \,.\, r(x,y) \rightarrow r(y,x)$
- *anti-symmetric*: $\forall x, y \,.\, r(x,y) \wedge r(y,x) \rightarrow x = y$
- *reflexive*: $\forall x \,.\, Fld(r,x) \rightarrow r(x,x)$
- *irreflexive*: $\forall x, y \,.\, r(x,y) \rightarrow y \neq x$
- *serial*: $\forall x \,.\, Rng(r,x) \rightarrow \exists y \,.\, r(x,y)$
- *total*: $\forall x, y \,.\, Fld(r,x) \wedge Fld(r,y) \rightarrow r(x,y) \vee r(y,x)$

Furthermore, a relation r is

- a *partial order* if r is reflexive, transitive, and anti-symmetric.
- a *strict total order* if r is irreflexive, transitive, anti-symmetric, and total.

Finally,

- x is an *initial element* of a relation r if $Dom(r,x)$ and $\neg\exists y \,.\, r(y,x)$. □

As most axioms state the existence of certain relations, given the existence of some other relation, the existence of at least one relation should be stated in an unconditional way. Just as in Zermelo-Fraenkel Set Theory, we take an infinite relation for that. Given the definitions above, the Axiom of Infinity can be formulated as follows.

Axiom 2 (Infinity). There exists a strict total order r such that r has an initial element and r is serial. □

Clearly, the field of r is infinite, and thus an infinite number of relations will exist.

As mentioned before, the Axiom of Infinity states the existence of a specific relation in an unconditional way. Next we come to those axioms that do presuppose the existence of some other relations. There are five such axioms. These axioms can be divided in two groups, according to whether they prescribe an internal structure of a relation. The first group contains the axioms that do prescribe the internal structure of a relation based on some "rule": by description, by structure, and by arbitrariness. The axioms corresponding to these "rules" are the Axioms of Comprehension, Universality, and Choice, respectively.

The second group does not say anything about the internal definition of a relation (though one might reformulate these axioms in such a way that they do): the Downward Axiom and the Upward Axiom.

The Axiom of Comprehension states the existence of a relation whose behavior is described in a formula φ with (in general) two free variables. In order to avoid the Russell paradox, the formula φ should be restricted to those objects that are in the field of a given relation. Thus, the Axiom of Comprehension can be formulated as follows.

Axiom 3 (Comprehension). For each relation r there is a relation r' such that for all x, y we have

$$r'(x,y) \;\leftrightarrow\; \mathit{Fld}(r,x) \wedge \mathit{Fld}(r,y) \wedge \varphi(x,y)$$

where φ may not contain r' free. □

It is possible to be somewhat less restrictive than to allow only objects within the field of r. Let a formula $\psi(u,v)$ be a functional formula in u if

$$\forall v, v' \,.\, \psi(u,v) \wedge \psi(u,v') \rightarrow v = v'.$$

Instead of requiring in the Axiom of Comprehension that both x and y for which $r'(x,y)$ holds, are in the field of r, it is sufficient to require that there are u, v in the field of r such that $\psi_1(u,x)$ and $\psi_2(v,y)$ for two functional formulas ψ_1, ψ_2. We leave it to the reader to give the formal expression of this more general form of the Axiom of Comprehension.

Notational conventions. A finite relation may be denoted as

$$\varrho\{(a_1,b_1),\ldots,(a_n,b_n)\}.$$

Note that by the Axiom of Comprehension it is straightforward to prove the existence of this relation when all a_i and b_i are known. This also holds when $n = 0$, i.e., the *empty relation* $\varrho\{\}$ exists.

A relation, defined by a formula φ, possibly containing x, y free, may be denoted as

$$\varrho\,(x,y)\,.\,\varphi.$$

Here, ϱ is an "abstractor" that binds the variables x and y.

Note that it is possible that such a relation may not exist. For example, the Russell relation,

$$\varrho\,(x,y)\,.\,\neg x\,(y,y)$$

may not exist. To avoid undefinedness we will assume that these notational conventions are not part of the formal language.

Next we come to the Axiom of Universality. This axiom declares the existence of relations by giving their internal structure explicitly. According to this axiom, all sorts of non-wellfounded relations may exist, such as

$$r = \varrho\{(r,r)\}.$$

That is, r is a relation that only relates itself to itself. The Axiom of Universality corresponds to Boffa's Axiom of Universality in Set theory (see [4]) or to Aczel's axiom AFA (see [1]). More standard it would be to reject the existence of such non-wellfounded relations and to formulate an axiom that corresponds to the Axiom of Foundation from Set Theory. We prefer to include the Axiom of Universality since it exploits the internal structure of a relation as a positive principle for existence instead of formulating a limitation on the existence of relations as the Axiom of Foundation does. We leave it to the reader to formulate the relational equivalent of the Axiom of Foundation.

Before we can formulate the Axiom of Universality we need some more definitions.

Definition 4.3 (Function)**.** A relation r is a *function* if

$$\forall x, y, y' \,.\, r(x,y) \wedge r(x,y') \;\rightarrow\; y = y'.$$ □

When r is a function and $r(x, y)$ is the case, we will say that r is *defined for* x with value y. We will also use the standard notation for functions, i.e., $r(x) = y$.

Properties of functions like injectivity, etc., are defined as usual.

Definition 4.4 (Ordered pair)**.** The ordered pair $\langle x, y\rangle$ is defined by the relation which relates only x to y, and nothing else:

$$\langle x, y\rangle =_{\text{def}} \varrho\{(x, y)\}. \qquad \square$$

More formally, r is the ordered pair of x and y if

$$\forall x', y' \,.\, r(x', y') \leftrightarrow x' = x \wedge y' = y.$$

Clearly, $\varrho\{(x, y)\}$ answers this property.

Definition 4.5 (Relation scheme)**.** A relation s is a *relation scheme* if every element in the range of s is an ordered pair. That is to say, if for all y

$$Rng(s, y) \rightarrow \exists u, v \,.\, y = \langle u, v\rangle. \qquad \square$$

Thus, one might say that a relation scheme s gives an explicit picture of some given relations. That is to say, if for a relation scheme s it holds that

$$s(x, \langle u, v\rangle),$$

then (according to s) x holds between u and v.

To formalize this, we need to take into account that the Axiom of Extensionality holds for relations. Therefore we define that a relation scheme s is *extensional* if for every x, x' we have

$$(\forall y, z \,.\, s(x, \langle y, z\rangle) \leftrightarrow s(x', \langle y, z\rangle)) \;\rightarrow\; x = x'.$$

Definition 4.6 (Realization)**.** Let s be a relation scheme and f an injective function. Suppose that f is defined for x iff there are y, z such that $s(x, \langle y, z\rangle)$ or $s(y, \langle x, z\rangle)$ or $s(y, \langle z, x\rangle)$.

Then f is a *realization of* s if the following conditions hold:

– let $f(x) = r$, $f(y) = u$, $f(z) = v$ and suppose $s(x, \langle y, z\rangle)$. Then also $r(u, v)$ holds,
– let $f(x) = r$ and suppose $r(u, v)$. Then there are y, z with $f(y) = u$ and $f(z) = v$ such that $s(x, \langle y, z\rangle)$. $\square$

In other words, f is a realization of s if there is a collection of relations such that the structure that is made explicit by s is *isomorphic* to this collection of relations.

The Axiom of Universality now can be formulated as follows.

Axiom 4 (Universality). For every extensional relation scheme there is a realization. $\square$

As an example of relations that exist according to the Axiom of Universality is the already mentioned relation

$$r = \varrho\{(r, r)\},$$

i.e., r relates only itself to itself and nothing else. To show that r exists, define

$$s = \varrho\{(a, \langle a, a\rangle)\}.$$

Then s is an extensional relation scheme which exists according to the Axiom of Comprehension. According to the Axiom of Universality there exists a realization f for s. Let $f(a) = r$, then $r(r, r)$, which concludes the proof.

Likewise, there are relations r, r', with $r \neq r'$, such that

$$r = \varrho\{(r', r')\},\ r' = \varrho\{(r, r)\}.$$

Note that the Axiom of Universality is not only concerned with non-wellfounded relations. For example, let $t = \varrho\{(a, \langle b, b\rangle)\}$, let f be a function which is only defined for a and b, and let e be the empty relation $\varrho\{\}$. Then f is a realization of t if $f(a) = \varrho\{(e, e)\}$, and $f(b) = e$. Here, both $f(a)$ and $f(b)$ are wellfounded relations. In fact, this f is the only realization of t.

Next we move to the Axiom of Choice, the third axiom that states the existence of a relation based on its internal structure — even though the Axiom of Choice only uses *arbitrariness* as characterization of this internal structure.

One well known equivalent of the Axiom of Choice that uses relations is that within every non-empty relation r there exists a function f with the same domain.

Axiom 5 (Choice). For every relation r there exists a function f such that

$$\forall x\,.\,\mathit{Dom}(f, x) \leftrightarrow \mathit{Dom}(r, x),$$
$$\forall x, y\,.\,f(x) = y \rightarrow r(x, y). \qquad \square$$

Next we turn to the last two axioms: the Downward Axiom and the Upward Axiom. These axioms correspond to the Sumset Axiom and the Powerset Axiom from ZFC Set Theory, respectively. They declare the existence of some relations without saying anything about the internal structure of these relations.

Axiom 6 (Downward). For each relation r there is an r' such that

$$\forall x\,.\,\mathit{Fld}(r', x)\ \leftrightarrow\ \exists y\,.\,\mathit{Fld}(r, y) \wedge \mathit{Fld}(y, x) \qquad \square$$

Thus, the Downward Axiom states the existence of a relation r that relates those objects that are "indirectly" related by r.

Axiom 7 (Upward). For each relation r there is an r' such that

$$\forall s\,.\,\mathit{Fld}(r', s)\ \leftrightarrow\ \forall x\,.\,\mathit{Fld}(s, x) \leftrightarrow \mathit{Fld}(r, x). \qquad \square$$

Thus, the Upward Axiom relates those relations that have the same field as r.

Note that both the Upward and the Downward Axiom do not say anything about the internal structure of the relation r'. However, it is straightforward to extend these axioms such that they do by adding an additional requirement such as

$$r'(x, y) \leftrightarrow x = y.$$

Such an additional requirement may be chosen arbitrarily and therefore we prefer to leave it out. Note that the additional requirement $x = y$ is always possible, whereas in the Axiom of Choice such additional requirements are principally impossible.

5. Equi-consistency with ZFC Set Theory

To prove that the above described theory for binary relations two things have to be done:

- define an interpretation of relations and of the relation $\mathfrak{R}$ in ZFCU (Zermelo-Fraenkel Set Theory with the Axiom of Choice and with the Axiom of Universality) and prove that all the above axioms are true in this interpretation,
- define an interpretation of sets and the $\in$-relation in terms of the universe of binary relations such that all axioms of ZFCU are true in this interpretation.

In fact, these interpretations are straightforward. To prove the first obligation, take the standard definition of binary relation in set theory, i.e., a relation is a set of ordered pairs. Let a binary relation r be pure if for all $\langle x, y\rangle \in r$ we have that x, y are again pure relations. The proof that all axioms from section 4 are true in this interpretation is straightforward and left to the reader.

To prove the other way around requires a definition of sets in the universe of relations. First we define purely reflexive relations:

Definition 5.1 (Purely reflexive)**.** A relation r is *purely reflexive* if

$$\forall x, y\,.\,r(x, y) \rightarrow x = y. \qquad \Box$$

Now, a *set* is represented by a purely reflexive relation. Clearly, if a is a set, we may write $x \in a$ for $a(x, x)$. Note that $\in$ does *not* exist as an object in the universe of binary relations.

Notation. In formulas we now may allow restrictions on quantifiers by sets, as in $\forall x \in a\,.\,\varphi$, with the obvious meaning. Here it is understood that a is a set.

Theorem 5.2. *For each relation r there exist* sets *for the domain, range, and field of the relation r.*

Proof. By the Axiom of Comprehension it is immediate that the following relation exists:

$$a(x, y) \;\leftrightarrow\; Dom(r, x) \wedge x = y.$$

Clearly, a is a purely reflexive relation, thus a set, and we have

$$x \in a \;\leftrightarrow\; Dom(r, x).$$

Likewise for the range and the field of a relation r. $\Box$

Now take as interpretation for sets in the universe of Set Theory the *pure* sets, i.e., those sets whose elements again are sets. The proof that all axioms of ZFCU are true in this interpretation again is straightforward and left to the reader.

References

[1] P. Aczel. *Non-Well-Founded Sets*. CSLI, Stanford, 1988.

[2] H.P. Barendregt. *The Lambda Calculus – Its Syntax and Semantics* (second revised printing). North-Holland, Amsterdam, 1984.

[3] P. Bernays. A system of axiomatic set theory – part I–VII. *The Journal of Symbolic Logic*, 1937–1954.

[4] M. Boffa. Sur la théorie des ensembles sans axiome de fondement. *Bulletin de la Société Mathématique de Belgique*, 21:16–56, 1969.

[5] A. Church. A set of postulates for the foundation of logic (first and second paper). *Annals of Mathematics*, 33–34, 1932–1933.

[6] D. van Dalen, H.C. Doets, and H.C.M. de Swart. *Sets: Naive, Axiomatic and Applied*. Wolters-Noordhoff, Groningen, 1978.

[7] H.C. Doets. Cantor's paradise. *Nieuw Archief voor Wiskunde*, 4:290–344, 1983.

[8] J. Gerbrandy and W. Groeneveld. Reasoning about information change. *Journal of Logic, Language and Information*, 6(2):147–169, 1997.

[9] J. Kuper. An application of non-wellfounded sets to the foundations of geometry. *Zeitschr. f. math. Logik und Grundlagen d. Math.*, 37:257–264, 1991.

[10] J. Kuper. Axiomatic function theory. *Information and Computation*, 107:104–150, 1993.

[11] J. Kuper. *Partiality in Logic and Computation – Aspects of Undefinedness*. PhD thesis, University of Twente, Enschede, The Netherlands, 1994.

[12] J. von Neumann. Eine Axiomatisierung der Mengenlehre. *Zeitschrift für reine und angewandte Mathematik*, 154:219–240, 1925.

[13] J. von Neumann. Die Axiomatisierung der Mengenlehre. *Mathematisches Zeitschrift*, 27:669–752, 1928.

[14] M. von Rimscha. Mengentheoretische Modelle des λK-Kalküls. *Arch. Math. Logik Grundl.*, 21:65–73, 1980.