Coalgebra assessed exercises sheet 1. Sam Staton. October 2014.[1]

There are two questions. Both start easy and get harder. Contact me if something is unclear: s.staton@cs.ru.nl.

1. Consider the arity $I = \{z, s\}$ with $\#z = 0$ and $\#s = 1$.
   (a) Explain what is meant by an algebra for I.
   (b) Write down an initial algebra A for I. Justify your answer (give a proof/counterexample).
   (c) Explain what is meant by a coalgebra for I.
   (d) Write down a final coalgebra Z for I. Justify your answer.
   (e) Illustrate your findings:
      i. Write down a non-trivial I-algebra X and describe the unique homomorphism from A to X.
      ii. Write down a non-trivial I-coalgebra Y and describe the unique homomorphism from Y to Z.
      iii. Is there a canonical function $A \to Z$? Justify your answer.
   (f) Let S be a set. We say that an *I-algebra on S* is an I-algebra A together with a function $f : S \to A$. We say that a homomorphism of I-algebras on S, $(A, f) \to (B, g)$ is a homomorphism of I-algebras, $h : A \to B$, such that $g = h \cdot f$.
      i. Show that the category of I-algebras on S has an initial object for any set S.
      ii. Use this to define a functor from the category **Set** of sets to the category of all I-algebras.
      iii. Can you tell a similar story for I-coalgebras?

2. Let $\mathbb{R}_+$ be the set of non-negative real numbers. Recall that a *binary relation* on $\mathbb{R}_+$ is a subset of $(\mathbb{R}_+ \times \mathbb{R}_+)$.
   (a) Write down precise definitions of the terms *category* and *functor*.
   (b) Write down a definition of a category $\mathrm{Rel}(\mathbb{R}_+)$ where the objects are binary relations on $\mathbb{R}_+$ and where $\mathrm{Mor}(R, S)$ has one inhabitant if $R \subseteq S$ and is empty otherwise.
   (c) Recall that if R and S are binary relations on a set then the composition $(S \cdot R)$ is the relation
   $$(S \cdot R) = \{(x, z) \mid \exists y.\, (x, y) \in R \ \&\ (y, z) \in S\}.$$
   Write down a definition of a functor $F : \mathrm{Rel}(\mathbb{R}_+) \to \mathrm{Rel}(\mathbb{R}_+)$ which acts on objects by
   $$F(R) = (R \cdot R).$$
   (d) What is an algebra for the functor F? Explain in elementary terms, and give an example and a non-example.
   (e) What is a coalgebra for the functor F? Explain in elementary terms, and give an example and a non-example.
   (f) Is there an initial algebra for F? Is there a final coalgebra? Justify your answer.
   (g) Consider the binary relation S on $\mathbb{R}_+$ given by
   $$S = \{(x, y) \mid y = x + 1, \text{ and } x \text{ and } y \text{ are both integers}\}.$$
   Define a functor $F_S : \mathrm{Rel}(\mathbb{R}_+) \to \mathrm{Rel}(\mathbb{R}_+)$ satisfying $F_S(R) = (S \cup F(R))$. Does F_S have an initial algebra? Justify your answer.
   (h) Consider the binary relation T on $\mathbb{R}_+$ given by
   $$T = \{(x, y) \mid x < y, \text{ and } x \text{ and } y \text{ are both in the set } [0, 1] \cup \mathbb{Z}\}.$$
   Define a functor $F_S : \mathrm{Rel}(\mathbb{R}_+) \to \mathrm{Rel}(\mathbb{R}_+)$ satisfying $F^T(R) = (T \cap F(R))$. Does F^T have a final coalgebra? Justify your answer.

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