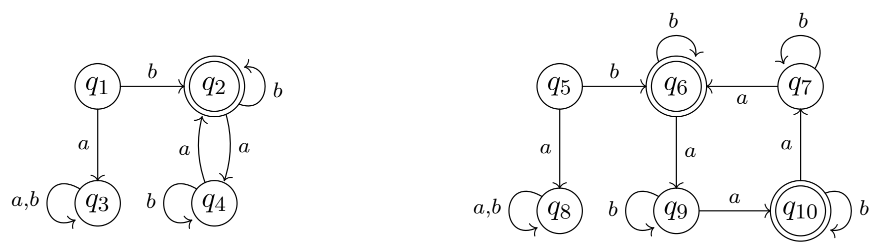
Homework 2

Coalgebra 2014-2015

Deadline is January 14th. Please email me alexandra@cs.ru.nl if you have questions.

Exercise 1. Consider the following two deterministic automata. Show that states q_1 and q_5 accept the same language.



Exercise 2. Consider the functor $M(X) = (B \times X)^A$. Coalgebras for M are the so-called Mealy machines, which are very similar to normal automata. Formally, a Mealy machine is a pair (S, f) where S is a set of states and $f: S \to (B \times S)^A$. Typically, transitions are represented by

$$x \xrightarrow{a|b} y \iff f(x)(a) = \langle b, y \rangle.$$

Define when a relation $R \subseteq S \times S$ is a bisimulation for Mealy machines.

Exercise 3. The set T_A of node-labelled, infinite binary trees, where each node is assigned a label in A, can be formally defined by:

$$T_A = \{t \mid t: \{L, R\}^* \to A\}$$

The set T_A carries a coalgebra structure for the functor $F(X) = X \times A \times X$ consisting of the following function:

$$\begin{array}{rcccl} \langle l, i, r \rangle & : & T_A & \to & T_A \times A \times T_A \\ & & t & \mapsto & \langle \lambda w.t(Lw), t(\varepsilon), \lambda w.t(Rw) \rangle \end{array}$$

where l and r return the left and right subtrees, respectively, and i gives the label of the root node of the tree. Here, ε denotes the empty word and, for $b \in \{L, R\}$, bw denotes the word resulting from prefixing the word w with the letter b.

Show that T_A is a final F-coalgebra.

Exercise 4. Consider the monad $T(X) = 1 + \mathcal{P}(X)$. with unit η and multiplication μ defined as

$$\eta(x) = \kappa_2(\{x\}) \qquad\qquad \mu(U) = \begin{cases} \top & \text{if } U = \top \\ \top & \text{if } U = \kappa_2(S) \text{ and } \top \in S \\ \bigcup_{U' \in S} U' & \text{if } U = \kappa_2(S) \text{ and } \top \notin S \end{cases}$$

where $\top$ stands for $\kappa_1(*)$.

(i) Show that $1 + \mathcal{PP}A$ carries a T-algebra structure: define $h \colon T(1 + \mathcal{PP}A) \to 1 + \mathcal{PP}A$ and show the algebra laws.

(ii) For the functor $F = (1 + \mathcal{PP}A) \times X$ and for every $FT - coalgebra\ f \colon X \to FTX$ it is always possible to build a T-algebra map $f^\sharp \colon TX \to FTX$. Explain why and define $f^\sharp$. (Hint: use (i))

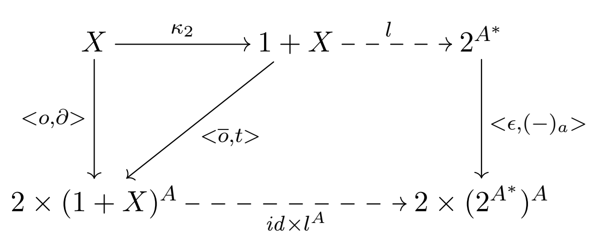
Exercise 5. A partial automaton (PA) over the input alphabet A is a pair $(X, \langle o, \partial \rangle)$ consisting of a set of states X and a pair of functions $\langle o, \partial \rangle : X \to 2 \times (1 + X)^A$. Here, $o \colon X \to 2$ determines whether the state is final or not. The function $\partial \colon X \to (1 + X)^A$ is a transition function that sends any state $x \in X$ to a function $\partial(x) \colon A \to 1 + X$, which for any input letter $a \in A$ is either undefined (no a-labelled transition takes place) or specifies the next state that is reached.

Given a PA $\mathcal{A}$, we can construct a total (deterministic) automaton $\mathbf{tot}(\mathcal{A})$ by adding an extra *sink* state to the state space: every undefined a-transition from a state x is then replaced by an a-labelled transition from x to the sink state.

PA's are coalgebras for the functor $H(X) = 2 \times (1 + X)^A$. Note that $H = G \circ T$ where $G(X) = 2 \times X^A$ and $T(X) = 1 + X$ (which is a monad). Now answer the following questions:

(i) Show that, for any set X, $H(X)$ is an algebra for the monad T (that is, define a map $h \colon TH(X) \to H(X)$ and show the algebra laws).

(ii) Define $\langle \bar{o}, t \rangle$ which makes the below diagram commute.



(iii) Provide the explicit definition of l. Which language does the newly added state recognize?