Quasi-reducibility and functions without fixed-points

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Shoenfild [1] defined the quasi-reducibility of sets: $A \leq_Q B$ if there is a computable function f such that $x \in A \Leftrightarrow W_{f(x)} \subseteq B$. Arslanov (see, for example, [2]) proved the following criterion for the T-reducibility: an c.e. set X is T-complete iff there is a Turing computable in X function f without fixed-points, i.e. $(\forall x)(W_x \neq W_{f(x)})$. We prove a similar criterion for the quasi-reducibility and use this criterion to obtain the quasi-completeness of Chaitin Ω numbers.

We say that a function f is quasi-reducible to a set A ($f \leq_Q A$), if there are computable functions a, b and g such that

$$f(x) = \begin{cases} a(x), & \text{if } W_{g(x)} \subseteq A \\ b(x), & \text{if } W_{g(x)} \not\subseteq A. \end{cases}$$

Denote the set $B_{f,A} = \{x | W_{g(x)} \subseteq A\}$.

Theorem 1. For each set $A \subseteq \omega$ following conditions are equivalent:
1) All c.e. sets are Q-reducible to the set A,
2) There is a function $f \leq_Q A$ without fixed-points, i.e. $(\forall x)(W_x \neq W_{f(x)})$, and $B_{f,A}$ is a c.e. set.

Fix an universal Chaitin computer U, denote by Σ^* the set of all finite binary strings and for arbitrary $x \in \Sigma^*$ denote by $H(x)$ the program-size (Chaitin) complexity induced by Chaitin's computer U. The halting probability of U is denoted by Ω.

We say that a set C is quasi-reducible to the real Ω ($B \leq_Q \Omega$) if $C \leq_Q A$, where $A = \{q \in (0,1) \bigcap Q | \ q < \Omega\}$.

Theorem 2. The set $\mathcal{H} = \{(x,n) | \ x \in \Sigma^*, n \in \omega, H(x) \leq n\}$ is Q-complete.

Theorem 3. The set $\mathcal{H}$ is Q-reducible to Ω.

[1] Shoenfield, J.R. Quasicreative sets //Proceedings of the American Mathematical Society. 1957. vol 8. p.964-967.

[2] Soare, R.I. Recursively Enumersable Sets and Degrees, Springer-Verlag, Berlin, 1987.