Weak interpolation in equational logic

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Let us fix an arbitrary signature consisting of functional symbols and constants including the constants $\bot$ and $\top$. We consider varieties of algebras such that $\mathbf{A} \models \bot = \top$ iff $\mathbf{A}$ is a one-element algebra. We call such algebras *bounded*; one can take bounded lattices, boolean algebras with operators and Heyting algebras as examples. We define a weak variant of interpolation property and find its algebraic equivalent.

For any algebra $\mathbf{A}$, a valuation in $\mathbf{A}$ is any homomorphism from the algebra of terms into $\mathbf{A}$. If $t = t'$ is an equation and v is a valuation, we write $\mathbf{A} \models (t = t')[v]$ whenever $v(t) = v(t')$. If Γ is a set of equations and γ is an equation, we define for any variety V:

$\Gamma \models_V \gamma \iff (\forall \mathbf{A} \in V)(\forall v)(\mathbf{A} \models \Gamma[v] \Rightarrow \mathbf{A} \models \gamma[v])$.

For any set $\mathbf{x}$ of variables, denote by $\Gamma(\mathbf{x})$ any set of equations whose all variables are in $\mathbf{x}$. We define the *weak interpolation property*:

WIP. If $\Gamma(\mathbf{x}, \mathbf{y}), \Delta(\mathbf{x}, \mathbf{z}) \models_V \bot = \top$, then there exists $\Gamma'(\mathbf{x})$ such that $\Gamma(\mathbf{x}, \mathbf{y}) \models_V \gamma$ for all $\gamma \in \Gamma'(\mathbf{x})$ and $\Gamma'(\mathbf{x}), \Delta(\mathbf{x}, \mathbf{z}) \models_V \bot = \top$.

We prove that WIP is equivalent to some weak variant of amalgamation. A class V has the *amalgamation property* if it satisfies the condition

AP. For any $\mathbf{B}, \mathbf{C} \in V$ with a common subalgebra $\mathbf{A}$, there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$.

A bounded algebra is *non-degenerate* if it contains at least two elements. We define

PRAP. For each $\mathbf{B}, \mathbf{C} \in V$ with a common subalgebra $\mathbf{A}$ there are $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ that coincide on $\mathbf{A}$, and $\mathbf{D}$ is non-degenerate whenever $\mathbf{A}$ is non-degenerate.

For a class K of algebras, $FG(K)$ denotes the class of finitely generated algebras of K. An algebra is said to be *simple* if it has exactly two congruences.

Theorem 1 *Let V be a variety of bounded algebras. If V has WIP then it has PRAP. If V has CEP and $FG(V)$ has PRAP, then V has WIP.*

Theorem 2 *Let a variety V of bounded algebras have CEP. Then the following are equivalent:*

1. *V has WIP,*
2. *the class $Sim(V)$ of simple algebras of V has AP,*
3. *the class $FG(Sim(V))$ of finitely generated and simple algebras of V has AP.*