Fixed points in non-standard models of subsystems of second order arithmetic

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The proof-theoretic ordinal $|\mathsf{T}|$ of a subsystem T of second order arithmetic is the least ordinal α for which there is no primitive recursive well-ordering $\prec$ of ordertype α such that T proves the well-foundedness of $\prec$. Thus, given a primitive recursive well-ordering $\lhd$ of ordertype $|\mathsf{T}|$, the extension $\mathsf{T}^\dagger$ of T by the axiom $\neg\mathsf{Wo}(\lhd)$ is still consistent.

We work within a non-standard model $\langle\mathfrak{N},\mathcal{S}\rangle$ ($\mathcal{S}\subseteq\mathsf{P}(\mathfrak{N})$) of $\mathsf{T}^\dagger$. First, we show how to construct a fixed point of a monotone operator F^A on the powerset of $\mathfrak{N}$ induced by a U-positive, arithmetical formula $A(U,u)$: Let $\mathcal{O}$ be the class of all $\alpha\in\mathsf{Field}(\lhd)$ for which the αth level I^A_α of the inductive definition A is a set in $\mathcal{S}$. For a suitable theory T, we have that $\mathcal{Z}:=\bigcup_{\alpha\in\mathcal{O}}I^A_\alpha$ is a Σ^1_1 definable fixed point of the operator F^A which is properly contained in the Π^1_1 definable class $\bigcap\{X : F^A(X)\subseteq X\}$. Then, we use this fixed point construction to transform suitable models of $\Sigma^1_1\text{-}\mathsf{AC}^\dagger$ into models of various fixed point theories. Moreover, we apply this method to obtain a non-standard model of Kripke-Platek set theory where not all sets have a transitive closure and the collection of all n-element subsets of a set x may not be a set itself.

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