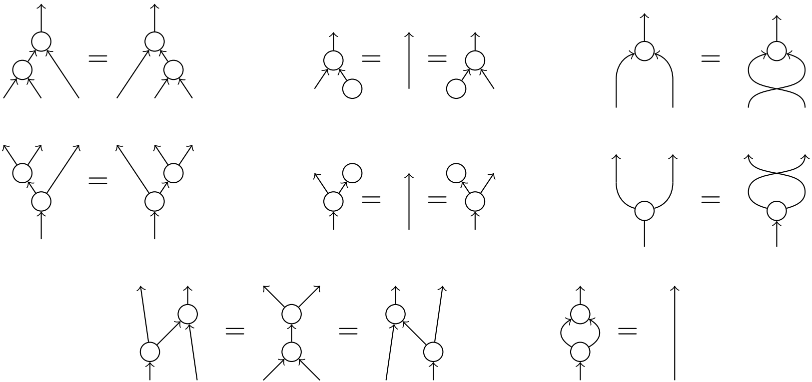
Simplification by Rotation for Frobenius/Hopf algebras

Aleks Kissinger

September 9, 2017

The goal

Simplification for special commutative Frobenius algebras:



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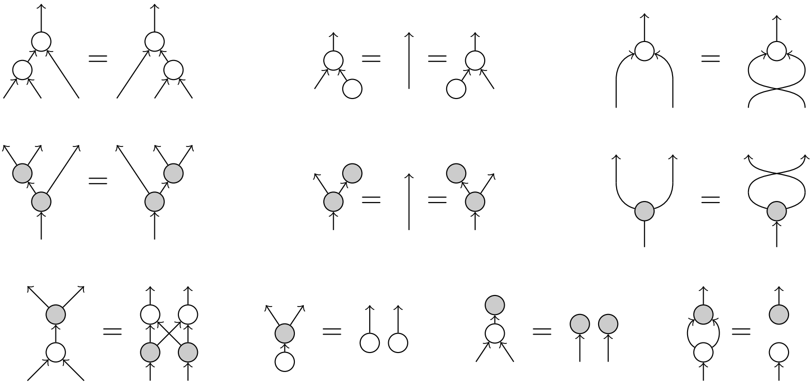
Simplification for commutative Hopf algebras:



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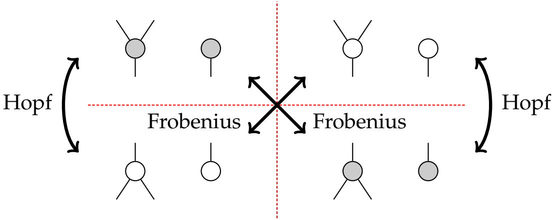
Simplification for the system $\mathbb{IB}$:



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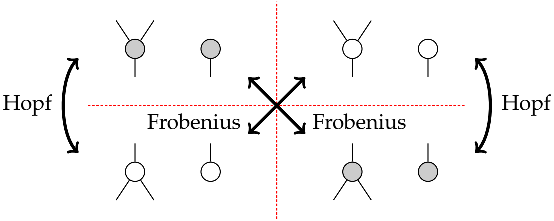

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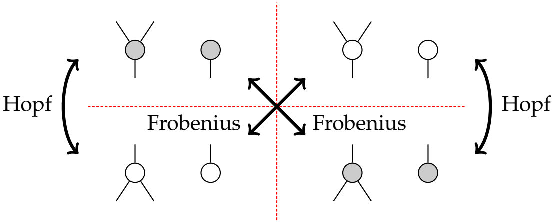

(a.k.a. the *phase-free fragment of the ZX-calculus*)

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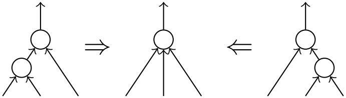

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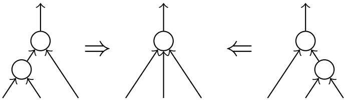
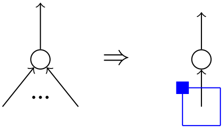
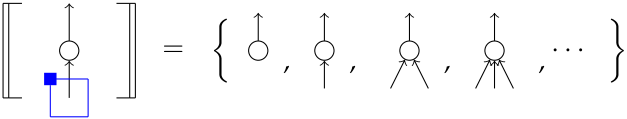


- $\implies$ need infinitely many rules, or *rule schemas*

!-boxes: simple diagram schemas





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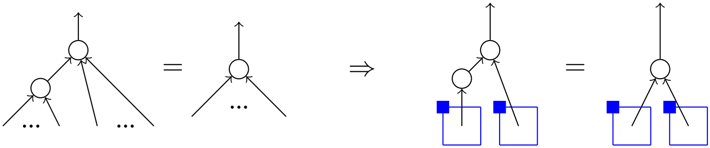

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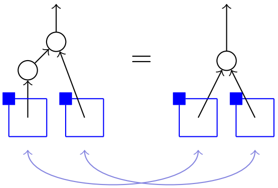

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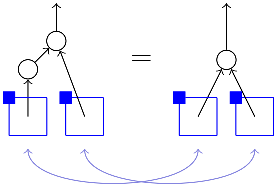
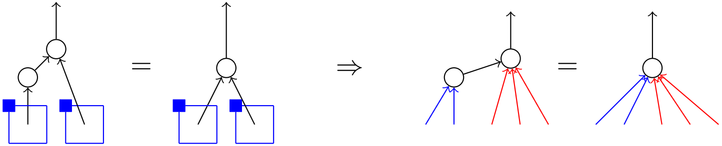
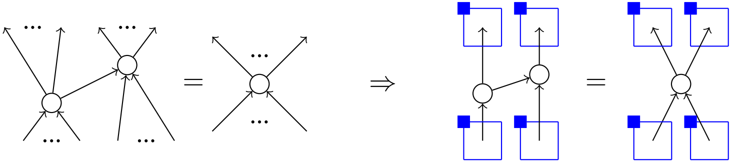
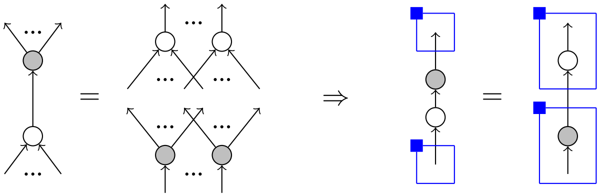




Unbiased Frobenius algebras



Unbiased bialgebras



To quanto!

Interacting bialgebras are linear relations

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- $\mathrm{LinRel}_{\mathbb{Z}_2}$ has:
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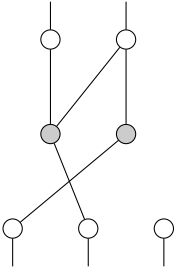
  - **tensor** is $\oplus$, **composition** is relation-style
- Pseudo-normal forms can be interpreted as:
  - **white spiders** := place-holders
  - **grey spiders** := vectors spanning the subspace

Lets see how this works...

- Subspaces can be represented as:



$$\leftrightarrow \quad \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ \hline 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \hline 0 \\ 1 \end{pmatrix} \right\rangle$$

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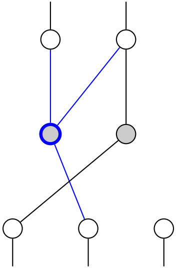

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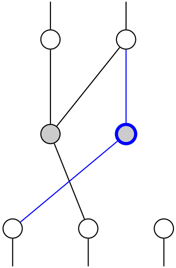

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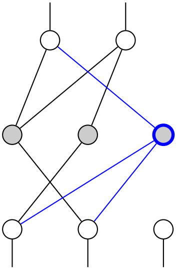
Lets see how this works...

- Not unique! We can always add or remove a vector that is the sum of two other spanning vectors and get the same space:



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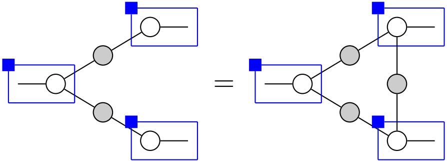
Addition is a !-box rule

- 'Addition' operation can be written as a !-box rule:



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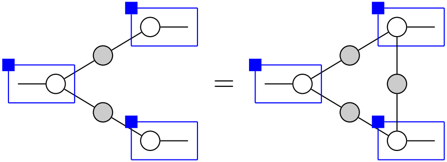
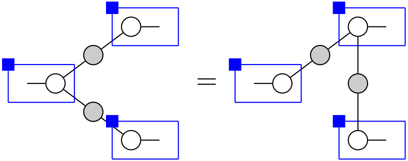


- We can also apply this forward then backward to get a 'rotation' rule:



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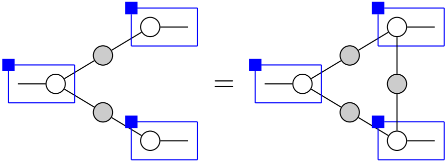

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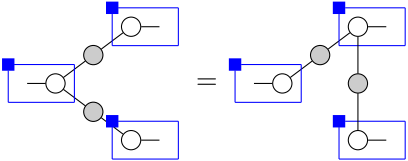

- Note this rule decreases the arity of the white dot on the left by 1.

Thanks!

- Joint work with Lucas Dixon, Alex Merry, Ross Duncan, Vladimir Zamdzhiev, David Quick, Hector Miller-Bakewell and others
- See: `quantomatic.github.io`