Matrix Calculations: Linear Equations

Aleks Kissinger

Institute for Computing and Information Sciences
Radboud University Nijmegen

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Outline

First, some admin...

Lectures

- Weekly: Mondays 15:45-17:30
- Presence not compulsory...
  - But if you are going to come, actually be here! (This means laptops shut, phones away.)
- The course material consists of:
  - these slides, available via the web
  - *Linear Algebra* lecture notes by Bernd Souvignier ('LNBS')
- Course URL:

  `www.cs.ru.nl/A.Kissinger/teaching/matrixrekenen2017/`

  (Link exists in blackboard, under 'course information').
- Generally, things appear on course website (and not on blackboard!). Check there before you ask a question.

First, some admin...

Assignments

- You can work together, but exercises must be handed in individually
- Handing in is not compulsory (except for 3rd-chancers), **but**:
  - It's a tough exam. If you don't do the excercises, you are unlikely to pass.
  - Exercises give up to 1 point (out of 10) bonus on exam.
  - This could be the difference between a 5 and a 6 (...or a 9 and a 10 ☺)

First, some admin...

Werkcollege's

- Werkcollege on Friday, 10:45.
  - Presence not compulsory (except for 3rd-chancers)
  - Answers (for old assignments) & Questions (for new ones)
- Schedule:
  - New assignments on the web on Tuesday
  - Next exercise meeting (Friday) you can ask questions
  - Hand-in: **Monday before noon**, handwritten or typed, on paper in the delivery boxes, ground floor Mercator 1 (or via other means in agreement with your assistant).
- There is a separate Exercises web-page (see URL on course webpage).

First, some admin...

Werkcollege's

- There will be **no lecture** on February 27 and **no werkcollege** on March 3, on account of Carnival
- There **will** be a werkcollege this Friday
- 4 Groups:
  - **Group A:** John van de Wetering. HG00.065
  - **Group B:** Aucke Bos. HG00.308
  - **Group C:** Milan van Stiphout. HG00.310
  - **Group D:** Bart Gruppen. HG00.633
- Each assistant has a delivery box on the ground floor of the **Mercator 1 building**

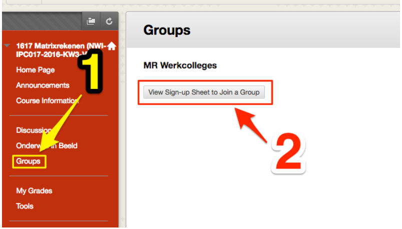
First, some admin...

There are **4** exercise classes

- You should choose a group based on your level of mathematical skill:
  - **Group A** - good at math (e.g. ≥ 7 in VWO Wiskunde B)
  - **Group B** - pretty good (e.g. ≥ 6 in VWO Wiskunde B)
  - **Group C** - okay at math
  - **Group D** - not so good/need some extra help
- please do this seriously: **it is in your own interest to be in the appropriate group**

First, some admin...

- Register for a class on Blackboard. Click ‘Groups’ in the sidebar, then the ‘View Sign-up Sheet’ button:



- Registration **must** be done by tomorrow (Tuesday) at 12:00. (Do it today, if possible.)
- I may need to shift some people to other groups. This will be finalised by **Thursday**, so double-check your group assignment then.

First, some admin...

Examination

- Final mark is computed from:
  - Average of markings of assignments: A
  - Written exam (April 4): E
  - Final mark: $F = E + \frac{A}{10}$.
- Second chance for written exam shortly thereafter.
  - you keep the outcome (average) of the assignments.
- If you fail again, you must start all over next year (including re-doing new exercises, and additional requirements)

First, some admin...

If you fail more than twice ...

- Additional requirements will be imposed
- You will have to talk to the study advisor
  - if you have not done so yet, make an appointment
  - **compulsory:** presence at all lectures, werkcollege's, handing in of all exercises
  - **sign in today** during the break (and in future lectures)
  - you exercise mark must be ≥ 5 to take the exam.

Next, some advice...

How to pass this course

- Learn by doing, not just staring at the slides (or video, or lecturer)
- Pro tip: exam questions will look a lot like the exercises
- Give this course the time it needs!
- 3ec means $3 \times 28 = 84$ hours in total
  - Let's say 20 hours for exam
  - 64 hours for 8 weeks means: 8 hours per week!
  - 4 hours in lecture and werkcollege leaves...
  - ...another 4 hours for studying & doing exercises
- Coming up-to-speed is your own responsibility
  - if you feel like you are missing some background knowledge: use Wim Gielen's notes...or wikipedia

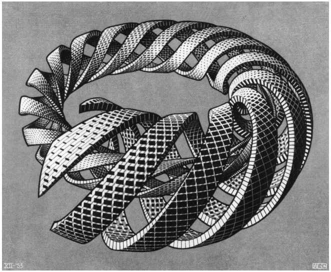
Finally, on to the good stuff...

Q: What is ~~matrix calculation~~ all about?
linear algebra

A: It depends on who you ask...

What is linear algebra all about?

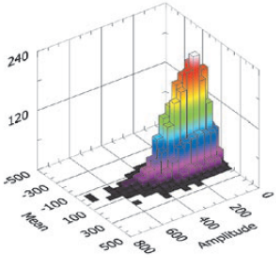
To a mathematician: linear algebra is the mathematics of **geometry** and **transformation**...



It asks: How can we represent a problem in 2D, 3D, 4D (or infinite-dimensional!) space, and **transform** it into a solution?

What is linear algebra all about?

To an engineer: linear algebra is about **numerics**...

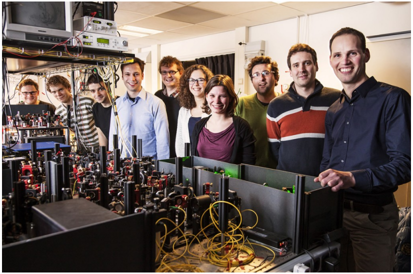


$\Rightarrow$

It asks: Can we encode a complicated question (e.g. 'Will my bridge fall down?') as a big matrix and compute the answer?

What is linear algebra all about?

To an quantum physicist (or quantum computer scientist!): linear algebra is just the way **nature behaves**...



It asks: How can we explain things that can be in **many states** at the same time, or **entangled** to distant things?

A simple example...

Let's start with something everybody knows how to do:

- Suppose I went to the pub last night, but I can't remember how many, umm...'sodas' I had.
- I remember taking out 20 EUR from the cash machine.
- Sodas cost 3 EUR.
- I discover a half-eaten kapsalon in my kitchen. That's 5 EUR.
- I have no money left. (Typical...)

By now, most people have (hopefully) figured out I had...5 sodas.That's because you can solve simple **linear equations**:

$$3x + 5 = 20 \qquad \Longrightarrow \qquad x = 5$$

An (only slightly less) simple example

I have two numbers in mind, but I don't tell you which ones

- if I add them up, the result is 12
- if I subtract, the result is 4

Which two numbers do I have in mind?

Now we have a **system** of linear equations, in two variables:

$$\begin{cases} x + y = 12 \\ x - y = 4 \end{cases} \qquad \text{with solution} \qquad x = 8,\ y = 4.$$

An (only slightly less) simple example

Let's try to find a solution, in general, for:

$$\begin{aligned} x + y &= a \\ x - y &= b \end{aligned}$$

i.e. find the values of x and y in terms of a and b.

- adding the two equations yields:

$$a + b = (x + y) + (x - y) = 2x, \qquad \text{so} \qquad \boxed{x = \frac{a+b}{2}}$$

- subtracting the two equations yields:

$$a - b = (x + y) - (x - y) = 2y, \qquad \text{so} \qquad \boxed{y = \frac{a-b}{2}}$$

Example (from the previous slide)

$a = 12, b = 4$, so $x = \frac{12+4}{2} = \frac{16}{2} = 8$ and $y = \frac{12-4}{2} = \frac{8}{2} = 4$. Yes!

A more difficult example

I have two numbers in mind, but I don't tell you which ones!

- if I add them up, the result is 12
- if I *multiply*, the result is 35

Which two number do I have in mind?

It is easy to check that $x = 5, y = 7$ is a solution.

The system of equations however, is non-linear:

$$\begin{aligned} x + y &= 12 \\ x \cdot y &= 35 \end{aligned}$$

This is already too difficult for this course. (If you don't believe me, try $x^5 + x = -1$ ...on second thought, maybe wait till later.)

We only do linear equations.

Basic definitions

Definition (linear equation and solution)

A linear equation in n variables $x_1, \cdots, x_n$ is an expression of the form:

$$a_1 x_1 + \cdots + a_n x_n = b,$$

where $a_1, \ldots, a_n, b$ are given numbers (possibly zero).

A solution for such an equation is given by n numbers $s_1, \ldots, s_n$ such that $a_1 s_1 + \cdots + a_n s_n = b$.

Example

The linear equation $3x_1 + 4x_2 = 11$ has many solutions, eg. $x_1 = 1, x_2 = 2$, or $x_1 = -3, x_2 = 5$.

More basic definitions

Definition

A $(m \times n)$ system of linear equations consists of m equations with n variables, written as:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

A solution for such a system consists of n numbers $s_1, \ldots, s_n$ forming a solution for **each** of the equations.

Example solution

Example

Consider the system of equations

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= 9 \\
2x_1 + 4x_2 - 3x_3 &= 1 \\
3x_1 + x_2 + x_3 &= 8.
\end{aligned}
$$

- How to find solutions, if any?
- Finding solutions requires some work.
- But checking solutions is easy, and you should always do so, just to be sure.
- Solution: $x_1 = 1, x_2 = 2, x_3 = 3.$ ✓

Easy and hard

- General systems of equations are hard to solve. But what kinds of systems are easy?
- How about this one?

$$\begin{aligned} x_1 &= 7 \\ x_2 &= -2 \\ x_3 &= 2 \end{aligned}$$

- ...this one's not too shabby either:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$$

Transformation

So, why don't we take something hard, and **transform** it into something easy?

$$\begin{cases} \quad\quad\;\; 2x_2 + x_3 = -2 \\ 3x_1 + 5x_2 - 5x_3 = 1 \\ 2x_1 + 4x_2 - 2x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ \quad\;\; x_2 + 2x_3 = 2 \\ \quad\quad\quad\quad\; x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 7 \\ x_2 = -2 \\ x_3 = 2 \end{cases}$$

Sound like something **linear algebra** might be good for?

Gaussian elimination

Gaussian elimination is the 'engine room' of all computer algebra.
It was named after this guy:

Carl Friedrich Gauss (1777-1855)
(famous for inventing: like half of mathematics)

Gaussian elimination

Gaussian elimination is the 'engine room' of all computer algebra.
...but it was probably actually invented by this guy:

Liu Hui (ca. 3rd century AD)

Variable names are inessential

The following programs are equivalent:

```
for(int i=0; i<10; i++){
    P(i);
}
```

```
for(int j=0; j<10; j++){
    P(j);
}
```

Similarly, the following systems of equations are equivalent:

$$\begin{aligned} 2x + 3y + z &= 4 \\ x + 2y + 2z &= 5 \\ 3x + y + 5z &= -1 \end{aligned} \qquad \begin{aligned} 2u + 3v + w &= 4 \\ u + 2v + 2w &= 5 \\ 3u + v + 5w &= -1 \end{aligned}$$

Matrices

The essence of the system

$$\begin{aligned} 2x + 3y + z &= 4 \\ x + 2y + 2z &= 5 \\ 3x + y + 5z &= -1 \end{aligned}$$

is not given by the variables, but by the numbers, written as:

coefficient matrix

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 5 \end{pmatrix}$$

augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 1 & 2 & 2 & 5 \\ 3 & 1 & 5 & -1 \end{array}\right)$$

Easy and hard matrices

So, the question becomes, how to we turn a *hard* matrix:

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 3 & 5 & -5 & 1 \\ 2 & 4 & -2 & 2 \end{array}\right) \leftrightarrow \left\{\begin{array}{rcl} 2x_2 + x_3 & = & -2 \\ 3x_1 + 5x_2 - 5x_3 & = & 1 \\ 2x_1 + 4x_2 - 2x_3 & = & 2 \end{array}\right.$$

...into an easy one:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \leftrightarrow \left\{\begin{array}{rcl} x_1 + 2x_2 - x_3 & = & 1 \\ x_2 + 2x_3 & = & 2 \\ x_3 & = & 2 \end{array}\right.$$

...or an *even easier* one:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \leftrightarrow \left\{\begin{array}{rcl} x_1 & = & 7 \\ x_2 & = & -2 \\ x_3 & = & 2 \end{array}\right.$$

Solving equations by row operations

- Operations on **equations** become operations on **rows**, e.g.

$$\left(\begin{array}{cc|c} 1 & 1 & -2 \\ 3 & -1 & 2 \end{array}\right) \leftrightarrow \begin{cases} x_1 + x_2 &= -2 \\ 3x_1 - x_2 &= 2 \end{cases}$$

- Multiply row 1 by 3, giving:

$$\left(\begin{array}{cc|c} 3 & 3 & -6 \\ 3 & -1 & 2 \end{array}\right) \leftrightarrow \begin{cases} 3x_1 + 3x_2 &= -6 \\ 3x_1 - x_2 &= 2 \end{cases}$$

- Subtract the first row from the second, giving:

$$\left(\begin{array}{cc|c} 3 & 3 & -6 \\ 0 & -4 & 8 \end{array}\right) \leftrightarrow \begin{cases} 3x_1 + 3x_2 &= -6 \\ -4x_2 &= 8 \end{cases}$$

- So $x_2 = \frac{8}{-4} = -2$. The first equation becomes: $3x_1 - 6 = -6$, so $x_1 = 0$. Always check your answer. ✓

Relevant operations & notation

	on equations	**on matrices**	**LNBS**
exchange of rows	$E_i \leftrightarrow E_j$	$R_i \leftrightarrow R_j$	$W_{i,j}$
multiplication with $c \neq 0$	$E_i := cE_i$	$R_i := cR_i$	$V_i(c)$
addition with $c \neq 0$	$E_i := E_i + cE_j$	$R_i := R_i + cR_j$	$O_{i,j}(c)$

These operations on equations/matrices:

- help to find solutions
- but do not change solutions (introduce/delete them)

The goal: rowstairs!

Definition

A matrix is in Echelon form (*rijtrapvorm*) if each row starts with strictly more zeros than the previous one.

$$\textit{e.g.}\quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

A matrix in reduced Echelon form if it is in Echelon form, and each row contains at most one '1' to the left of the line.

$$\textit{e.g.}\quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Transformations example, part I

equations	**matrix**
$$\begin{array}{rcl} 2x_2 + x_3 &=& -2 \\ 3x_1 + 5x_2 - 5x_3 &=& 1 \\ 2x_1 + 4x_2 - 2x_3 &=& 2 \end{array}$$	$$\left(\begin{array}{ccc\|c} 0 & 2 & 1 & -2 \\ 3 & 5 & -5 & 1 \\ 2 & 4 & -2 & 2 \end{array}\right)$$
$E_1 \leftrightarrow E_3$	$R_1 \leftrightarrow R_3$
$$\begin{array}{rcl} 2x_1 + 4x_2 - 2x_3 &=& 2 \\ 3x_1 + 5x_2 - 5x_3 &=& 1 \\ 2x_2 + x_3 &=& -2 \end{array}$$	$$\left(\begin{array}{ccc\|c} 2 & 4 & -2 & 2 \\ 3 & 5 & -5 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right)$$
$E_1 := \frac{1}{2}E_1$	$R_1 := \frac{1}{2}R_1$
$$\begin{array}{rcl} x_1 + 2x_2 - 1x_3 &=& 1 \\ 3x_1 + 5x_2 - 5x_3 &=& 1 \\ 2x_2 + x_3 &=& -2 \end{array}$$	$$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 3 & 5 & -5 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right)$$

Transformations example, part II

equations	**matrix**

$$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ 3x_1 + 5x_2 - 5x_3 &= 1 \\ 2x_2 + x_3 &= -2 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 3 & 5 & -5 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right)$$

$$E_2 := E_2 - 3E_1 \qquad R_2 := R_2 - 3R_1$$

$$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ -x_2 - 2x_3 &= -2 \\ 2x_2 + x_3 &= -2 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 2 & 1 & -2 \end{array}\right)$$

$$E_2 := -E_2 \qquad R_2 := -R_2$$

$$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ 2x_2 + x_3 &= -2 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & -2 \end{array}\right)$$

Transformations example, part III

equations

$$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ 2x_2 + x_3 &= -2 \end{aligned}$$

$$E_3 := E_3 - 2E_2$$

$$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ -3x_3 &= -6 \end{aligned}$$

$$E_3 := -\tfrac{1}{3}E_3$$

$$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$$

matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & -2 \end{array}\right)$$

$$R_3 := R_3 - 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

$\boxed{\text{Echelon } (\textit{rijtrap}) \text{ form}}$

$$R_3 := -\tfrac{1}{3}R_3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Transformations example, part IV

equations

$$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$$

$E_1 := E_1 - 2E_2$

$$\begin{aligned} x_1 - 5x_3 &= -3 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$$

$E_2 := E_2 - 2E_3$

$$\begin{aligned} x_1 - 5x_3 &= -3 \\ x_2 &= -2 \\ x_3 &= 2 \end{aligned}$$

matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Echelon form

$R_1 := R_1 - 2R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$R_2 := R_2 - 2R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Transformations example, part V

equations

$$\begin{aligned} x_1 - 5x_3 &= -3 \\ x_2 &= -2 \\ x_3 &= 2 \end{aligned}$$

$$E_1 := E_1 + 5E_3$$

$$\begin{aligned} x_1 &= 7 \\ x_2 &= -2 \\ x_3 &= 2 \end{aligned}$$

matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$R_1 := R_1 + 5R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

reduced echelon form

Gauss elimination

- Solutions can be found by mechanically applying simple rules
    - in Dutch this is called *vegen*
    - first produce echelon form (rijtrapvorm), then either (a) finish by substitution, or (b) obtain single-variable equations, reduced echelon form (gereduceerde rijtrapvorm)
    - it is one of the most important algorithms in virtually any computer algebra system
- Applying these operations is actually easier on matrices, than on the equations themselves
- You should be able to do Gauss elimination in your sleep! It is a basic technique used throughout the course.

Examples

1. $$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1 \end{aligned}$$
   has a single solution, namely $x_1 = 2, x_2 = 1$

2. $$\begin{aligned} x_1 + -2x_2 - 3x_3 &= -11 \\ -x_1 + 3x_2 + 5x_3 &= 15 \end{aligned}$$
   has many solutions
   (they can be described as: $x_1 = -x_3 - 3, x_2 = 4 - 2x_3$, giving a solution for each value of x_3)

3. $$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ 6x_1 - 4x_2 &= 6 \end{aligned}$$
   has no solutions: the transformation $E_2 := E_2 - 2E_1$ yields $0 = 4$.

Solutions, geometrically

Consider systems of only two variables x, y. A linear equation $ax + by = c$ then describes a line in the plane.

For 2 such equations/lines, there are three possibilities:

1. the lines intersect in a unique point, which is the solution to both equations
2. the lines are parallel, in which case there are no joint solutions
3. the lines coincide, giving many joint solutions.

(In)consistent systems

Definition

A system of equations is consistent (*oplosbaar*) if it has one or more solutions. Otherwise, when there are no solutions, the system is called inconsistent

Thus, for a system of equations:

nr. of solutions	**terminology**
0	inconsistent
≥ 1 (one or many)	consistent

Pivots and Echelon form

Definition

A pivot (Dutch: *spil* or *draaipunt*) is the first non-zero element of a row in a matrix.

Echelon form therefore means each pivot must occur (strictly) to the right of the pivot on the previous row.

Pivots and echelon form, examples

Example ($\bullet$ = pivot)

$$\begin{pmatrix} \bullet & * & * \\ 0 & \bullet & * \\ 0 & 0 & \bullet \end{pmatrix} \quad \begin{pmatrix} \bullet & * & * & * \\ 0 & 0 & \bullet & * \end{pmatrix} \quad \begin{pmatrix} 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * \\ 0 & 0 & 0 & 0 & \bullet \end{pmatrix} \quad \begin{pmatrix} \bullet & \star \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Non-examples:

$$\begin{pmatrix} \bullet & * & * \\ 0 & \bullet & * \\ \bullet & 0 & * \end{pmatrix}, \quad \begin{pmatrix} 0 & \bullet & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \quad \begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * \\ 0 & 0 & 0 & \bullet & * \end{pmatrix}$$

Inconsistency and echelon forms

Theorem

A system of equations is inconsistent (non-solvable) if and only if in the echelon form of its augmented matrix there is a row with:

- *only zeros before the bar* $|$
- *a non-zero after the bar* $|$,

as in: $0\ 0\ \cdots\ 0 \mid c$, *where* $c \neq 0$.

Example

$$\begin{array}{rcl} 3x_1 - 2x_2 & = & 1 \\ 6x_1 - 4x_2 & = & 6 \end{array} \quad \text{gives} \quad \left(\begin{array}{cc|c} 3 & -2 & 1 \\ 6 & -4 & 6 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} 3 & -2 & 1 \\ 0 & 0 & 4 \end{array}\right)$$

(using the transformation $R_2 := R_2 - 2R_1$)

Unique solutions

Theorem

A system of equations in n variables has a unique solution if and only if in its echelon form there are n pivots.

Example ($\square$ denotes a pivot)

$$\begin{array}{rcl} x_1 + x_2 & = & 3 \\ x_1 - x_2 & = & 1 \end{array} \quad \text{gives} \quad \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} \boxed{1} & 1 & 3 \\ 0 & \boxed{1} & 1 \end{array}\right)$$

(using transformations $R_2 := R_2 - R_1$ and $R_2 := -\frac{1}{2}R_2$)

Unique solutions: earlier example

equations

$$\begin{aligned} 2x_2 + x_3 &= -2 \\ 3x_1 + 5x_2 - 5x_3 &= 1 \\ 2x_1 + 4x_2 - 2x_3 &= 2 \end{aligned}$$

matrix

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 3 & 5 & -5 & 1 \\ 2 & 4 & -2 & 2 \end{array}\right)$$

After various transformations leads to

$$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Echelon form

There are 3 variables and 3 pivots, so there is one unique solution.