Matrix Calculations: Vector Spaces

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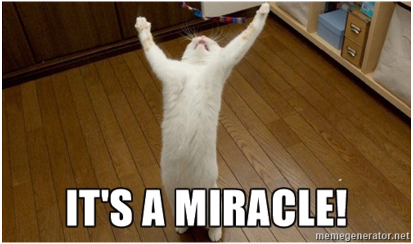
Version: spring 2017

Outline

What are numbers?

Suppose I don't know what numbers are...
...but I passed Wiskundige Structuren.



Tell me: what are numbers?

What is the *first thing* you would tell me about some numbers, e.g. the real numbers?

What are numbers?

The First Thing: numbers form a set

$$S \qquad (\longleftarrow \text{ these are some numbers!})$$

The Second Thing: numbers can be added together

$$a \in S, b \in S \qquad \Longrightarrow \qquad a + b \in S$$

Addition? Tell me more!

We have a set S, with a special operation '$+$' which satisfies:

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$

...and there's a special element $\mathbf{0} \in S$where:

3. $a + \mathbf{0} = a$

In math-speak, $(S, +, \mathbf{0})$ is called a commutative monoid, but we could also just call it a set with addition.

Examples: sets with addition

- Every kind of number you know: $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \ldots$
- The set of all polynomials:

$$(x^2 + 4x + 1) + (2x^2) := 3x^2 + 4x + 1 \qquad \mathbf{0} := 0$$

- The set of all finite sets:

$$\{1, 2, 3\} + \{3, 4\} := \{1, 2, 3\} \cup \{3, 4\} = \{1, 2, 3, 4\} \qquad \mathbf{0} := \{\}$$

- Here's a small example: $\{0\}$

$$0 + 0 := 0 \qquad \mathbf{0} := 0$$

- ...and (important!) the set $\mathbb{R}^n$ of all vectors of size n:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n) \qquad \mathbf{0} := (0, \ldots, 0)$$

Linear combinations

- Last time, we talked a lot about linear combinations:

$$a \cdot \boldsymbol{v} + b \cdot \boldsymbol{w} = \boldsymbol{u}$$

- **Q:** what is the most general kind of set, where we can take linear combinations of elements?
- **A:** a set V with addition and...scalar multiplication

$$a \in \mathbb{R}, \boldsymbol{v} \in V \qquad \Longrightarrow \qquad a \cdot \boldsymbol{v} \in V$$

Multiplication?! What does that do?

A vector space is a set with addition $(V, +, \mathbf{0})$ with an extra operation '$\cdot$', which satisfies:

1. $a \cdot (\boldsymbol{v} + \boldsymbol{w}) = a \cdot \boldsymbol{v} + a \cdot \boldsymbol{w}$
2. $(a + b) \cdot \boldsymbol{v} = a \cdot \boldsymbol{v} + b \cdot \boldsymbol{v}$
3. $a \cdot (b \cdot \boldsymbol{v}) = ab \cdot \boldsymbol{v}$
4. $1 \cdot \boldsymbol{v} = \boldsymbol{v}$
5. $0 \cdot \boldsymbol{v} = \mathbf{0}$

Example

Our **main example** is $\mathbb{R}^n$, where:

$$a \cdot (v_1, \ldots, v_n) := (av_1, \ldots, av_n)$$

Vector spaces: all together

Definition

A vector space $(V, +, \cdot, \mathbf{0})$ is a set V with a special element $\mathbf{0} \in V$ and operations '+' and '·' satisfying:

1. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
2. $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$
3. $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$
4. $a \cdot (\boldsymbol{v} + \boldsymbol{w}) = a \cdot \boldsymbol{v} + a \cdot \boldsymbol{w}$
5. $(a + b) \cdot \boldsymbol{v} = a \cdot \boldsymbol{v} + b \cdot \boldsymbol{v}$
6. $a \cdot (b \cdot \boldsymbol{v}) = ab \cdot \boldsymbol{v}$
7. $1 \cdot \boldsymbol{v} = \boldsymbol{v}$
8. $0 \cdot \boldsymbol{v} = \mathbf{0}$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $a, b \in \mathbb{R}$.

Vector spaces: Main Example

Our **main example**:

$$\mathbb{R}^n = \{(v_1, \ldots, v_n) \mid v_1, \ldots, v_n \in \mathbb{R}\}$$

$$= \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \ldots, v_n \in \mathbb{R} \right\}$$

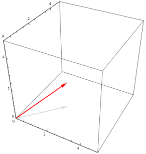
The operations:

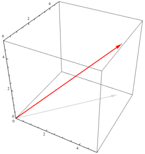
$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} \qquad a \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} av_1 \\ \vdots \\ av_n \end{pmatrix}$$

have a clear geometric interpretation.

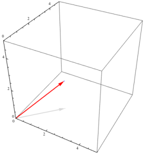
Vector spaces: geometric interpretation

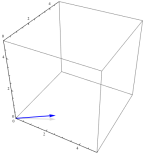
$a \cdot \boldsymbol{v}$ makes a vector shorter or longer:

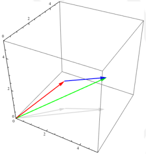
$\boldsymbol{v} :=$ 

$2 \cdot \boldsymbol{v} =$ 

$\boldsymbol{v} + \boldsymbol{w}$ stacks vectors together:

$\boldsymbol{v} :=$ 

$\boldsymbol{w} :=$ 

$\boldsymbol{v} + \boldsymbol{w} :=$ 

Example: subspaces

Certain subsets $V \subseteq \mathbb{R}^n$ are also vector spaces, e.g.

$$V = \{(v_1, v_2, 0) \mid v_1, v_2 \in \mathbb{R}\} \subseteq \mathbb{R}^3$$

$$W = \{(x, 2x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

as long as they have $\mathbf{0}$, and they are closed under '+' and '·':

$$\boldsymbol{v}, \boldsymbol{w} \in V \implies \boldsymbol{v} + \boldsymbol{w} \in V$$

$$\boldsymbol{v} \in V, a \in \mathbb{R} \implies a \cdot \boldsymbol{v} \in V$$

These are called *subspaces* of $\mathbb{R}^n$.

Vector space example

We've seen this example before!

Example

The set of solutions of a homogeneous system of equations is a vector space.

Let S be the set of solutions of a homogeneous system of equations, with n variables. Then $S \subseteq \mathbb{R}^n$, and as we learned last week:

$$\boldsymbol{s}, \boldsymbol{t} \in S \implies \boldsymbol{s} + \boldsymbol{t} \in S$$

$$\boldsymbol{s} \in S, a \in \mathbb{R} \implies a \cdot \boldsymbol{s} \in S$$

Vector spaces: 'weirder' examples

$\mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ are the only things we'll use in this course...but there are other examples:

- $\{0\}$ is still an example
- Polynomials are still an example: $5 \cdot (2x^2 + 1) = 10x^2 + 5$
- ...but finite sets are not!

$$5 \cdot \{\text{sandwich}, \text{Tuesday}\} = ???$$

- Functions $\mathcal{F}(X) := \{f : X \to \mathbb{R}\}$ are an example. If f, g are functions, then '$f + g$' and $a \cdot f$ are also functions, defined by:

$$(f + g)(x) := f(x) + g(x) \qquad\qquad (a \cdot f)(x) = af(x)$$

Exercise: show that, if $X = \{1, 2, \ldots, n\}$, then $\mathcal{F}(X)$ is basically the same as $\mathbb{R}^n$.

Our first theorem about vector spaces

We've got a Definition, let's prove a Theorem!

Theorem

Vector spaces have additive inverses. That is, for all $\boldsymbol{v} \in V$, there exists a vector $-\boldsymbol{v}$ such that $-\boldsymbol{v} + \boldsymbol{v} = \mathbf{0}$.

Proof. Let $-\boldsymbol{v} := (-1) \cdot \boldsymbol{v}$. Then, we use rules (1)-(6):

$$\begin{aligned} -\boldsymbol{v} + \boldsymbol{v} &= (-1) \cdot \boldsymbol{v} + 1 \cdot \boldsymbol{v} \\ &= (-1 + 1) \cdot \boldsymbol{v} \\ &= 0 \cdot \boldsymbol{v} \\ &= \mathbf{0} \end{aligned}$$

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Maps between vector spaces

We can send vectors $\boldsymbol{v} \in V$ in one vector space to other vectors $\boldsymbol{w} \in W$ in another (or possibly the same) vector space?

V, W are vector spaces, so they are sets with extra stuff (namely: $+$, $\cdot$, $\mathbf{0}$).

A common theme in mathematics: study functions $f : V \to W$ which preserve the extra stuff.

Functions

- A function f is an operation that sends elements of one set X to another set Y.
  - in that case we write $f: X \to Y$ or sometimes $X \xrightarrow{f} Y$
  - this f sends $x \in X$ to $f(x) \in Y$
  - X is called the domain and Y the codomain of the function f
- Example. $f(n) = \frac{1}{n+1}$ can be seen as function $\mathbb{N} \to \mathbb{Q}$, that is from the *natural* numbers $\mathbb{N}$ to the *rational* numbers $\mathbb{Q}$
- On each set X there is the identity function $\mathrm{id}: X \to X$ that does nothing: $\mathrm{id}(x) = x$.
- Also one can compose 2 functions $X \xrightarrow{f} Y \xrightarrow{g} Z$ to a function:

$$g \circ f : X \longrightarrow Z \quad \text{given by} \quad (g \circ f)(x) = g(f(x))$$

Linear maps

A linear map is a function that preserves the extra stuff in a vector space:

Definition

Let V, W be two vector spaces, and $f\colon V \to W$ a map between them; f is called linear if it preserves both:

- addition: for all $\mathbf{v}, \mathbf{v}' \in V$,

$$f(\underbrace{\mathbf{v} + \mathbf{v}'}_{\text{in } V}) = \underbrace{f(\mathbf{v}) + f(\mathbf{v}')}_{\text{in } W}$$

- scalar multiplication: for each $\mathbf{v} \in V$ and $a \in \mathbb{R}$,

$$f(\underbrace{a \cdot \mathbf{v}}_{\text{in } V}) = \underbrace{a \cdot f(\mathbf{v})}_{\text{in } W}$$

Linear maps preserve zero and minus

Theorem

Each linear map $f : V \to W$ preserves:

- *zero:* $f(\mathbf{0}) = \mathbf{0}$.
- *minus:* $f(-\boldsymbol{v}) = -f(\boldsymbol{v})$

Proof:

$$\begin{aligned} f(\mathbf{0}) &= f(0 \cdot \mathbf{0}) \\ &= 0 \cdot f(\mathbf{0}) \\ &= \mathbf{0} \end{aligned} \qquad \begin{aligned} f(-\boldsymbol{v}) &= f((-1) \cdot \boldsymbol{v}) \\ &= (-1) \cdot f(\boldsymbol{v}) \\ &= -f(\boldsymbol{v}) \end{aligned}$$

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Linear map examples I

$\mathbb{R}$ is a vector space. Let's consider maps $f : \mathbb{R} \to \mathbb{R}$.
Most of them are *not linear*, like, for instance:

- $f(x) = 1 + x$, since $f(0) = 1 \neq 0$
- $f(x) = x^2$, since $f(-1) = 1 = f(1) \neq -f(1)$.

So: linear maps $\mathbb{R} \to \mathbb{R}$ can only be very simple.

Theorem

Each linear map $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = c \cdot x$, for some $c \in \mathbb{R}$.

Proof:

$$f(x) = f(x \cdot 1) = x \cdot f(1) = f(1) \cdot x = c \cdot x, \quad \text{for } c = f(1).$$

Linear map examples II

Linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ start to get more interesting:

$$s\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} av_1 \\ v_2 \end{pmatrix} \qquad t\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} v_1 \\ bv_2 \end{pmatrix}$$

...these scale a vector on the X- and Y-axis.

We can show these are linear by checking the two linearity equations:

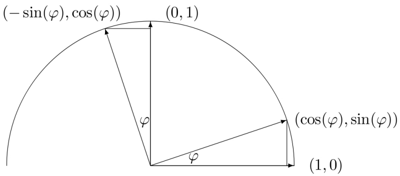
$$f(\boldsymbol{v} + \boldsymbol{w}) = f(\boldsymbol{v}) + f(\boldsymbol{w}) \qquad f(a \cdot \boldsymbol{v}) = a \cdot f(\boldsymbol{v})$$

Linear map examples III

Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} v_1\cos(\varphi) - v_2\sin(\varphi) \\ v_1\sin(\varphi) + v_2\cos(\varphi) \end{pmatrix}$$

This map describes rotation in the plane, with angle φ:



We can also check linearity equations.

Linear map examples IV

These extend naturally to 3D, i.e. linear maps $\mathbb{R}^3 \to \mathbb{R}^3$:

$$sx(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}) = \begin{pmatrix} av_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad sy(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}) = \begin{pmatrix} v_1 \\ bv_2 \\ v_3 \end{pmatrix} \qquad sz(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}) = \begin{pmatrix} v_1 \\ v_2 \\ cv_3 \end{pmatrix}$$

Q: How do we do rotation?
A: Keep one coordinate fixed (axis of rotation), and 2D rotate the other two, e.g.

$$rz(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}) = \begin{pmatrix} v_1 \cos(\varphi) - v_2 \sin(\varphi) \\ v_1 \sin(\varphi) + v_2 \cos(\varphi) \\ v_3 \end{pmatrix}$$

And it works!

These kinds of linear maps are the basis of all 3D graphics, animation, physics, etc.