Tentamen Matrixrekenen 2017

Tuesday, April 4, 2017, 8:30 – 11:30

- **Write the answers to EACH QUESTION on a separate sheet**
- This exam consists of **4 questions**, printed on one page, front and back. Each (sub)question indicates how many points it is worth. You can score a maximum of **100 points**.
- Please write clearly, and put on **each page (including this one!)**: your **name**, your **student number** and your exercise class group (A, B, C, or D).
- The exam is closed book. You are allowed to use a 4-function arithmetic calculator, but you are NOT allowed to use a graphing calculator, a computer, or a mobile phone. You may answer in Dutch or in English.
- You do not need to reduce expressions involving square roots. Answers like $x = 4/\sqrt{12}$ are acceptable.
- It is advised to explain your approach and to check your answers yourself.
- **Hand in ALL EXAM MATERIALS (including this sheet) when you are finished.**

This is a practice exam. For most questions, ONLY THE ANSWER IS GIVEN, so you can check if you are solving the question correctly. On the real exam, you will need to do some work in the places you see [Your work goes here...].

1. **(24 points)**

   For each of the following systems of equations, determine if the system has: (i) no solutions, (ii) a unique solution, or (iii) more than one solution. In case (ii), give the unique solution. In case (iii), give the general solution as a parametrization.

   (a) **(6 points)**

$$\begin{aligned} y + z &= -1 \\ x + 3y &= -1 \\ z - x - 3y &= 5 \end{aligned}$$

Begin Solution: ....

[Your work goes here...]

The UNIQUE SOLUTION is:

$$\begin{aligned} x &= 14 \\ y &= -5 \\ z &= 4 \end{aligned}$$

End Solution ....

   (b) **(6 points)**

$$\begin{aligned} a + 5b &= -1 \\ 2b + 4c &= 2 \\ a + 6b + 2c &= 1 \end{aligned}$$

Begin Solution: ....

[Your work goes here...]
The system has NO SOLUTIONS.

End Solution ....

(c) **(6 points)**

$$\begin{aligned} a - b + c - d &= 0 \\ 2b + 5c + d &= 0 \\ a - b - d &= 0 \\ a + 2b + d &= 0 \end{aligned}$$

Begin Solution: ....

[Your work goes here...]
There is a UNIQUE SOLUTION, namely **0***:*

$$\begin{aligned} a &= 0 \\ b &= 0 \\ c &= 0 \\ d &= 0 \end{aligned}$$

End Solution ....

(d) **(6 points)**

$$\begin{aligned} a + 3b &= 2 \\ 2b + c &= 1 \\ 2a - 3c &= 1 \end{aligned}$$

Begin Solution: ....

[Your work goes here...]
This has MANY SOLUTIONS. It has:

$$\text{1 basic solution: } \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \quad \text{and a particular solution is: } \begin{pmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

so the general solution is:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + c \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} + 3c \\ \frac{1}{2} - c \\ 2c \end{pmatrix}$$

(Note basic solutions and particular solutions are not unique, so answers may vary.)

End Solution ....

2. **(26 points)**

(a) **(6 points)** Consider the subspace:

$$U := \{(a, a, b, c) \mid a, b, c \in \mathbb{R}\} \subseteq \mathbb{R}^4$$

show that the following vectors form a basis for U:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Begin Solution: ....

We need to show that the given vectors are linearly independent and span U. For linear independence, suppose:

$$x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3 = \mathbf{0}$$

We need to show that $x = y = z = 0$.

[Your work goes here...]

For spanning, we need to show that any vector in U can be obtained from $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ by taking linear combinations. In other words, we need to solve the following equation:

$$x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3 = \begin{pmatrix} a \\ a \\ b \\ c \end{pmatrix}$$

for x, y, z in terms of a, b, c.

[Your work goes here...]

This has a solution, given by:

$$x = a$$
$$y = b$$
$$z = c - b$$

Hence $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ span U.

End Solution ....

(b) **(6 points)** Consider the following linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ x + y \end{pmatrix} \qquad g(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 3y \\ 2x \end{pmatrix}$$

Write the linear maps f, g and their composition $f(g(\boldsymbol{v}))$ as matrices in the standard basis.

Begin Solution: ....

[Your work goes here...]

The matrix of f is:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

[Your work goes here...]

The matrix of g is:

$$\boldsymbol{B} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$$

[Your work goes here...]

The matrix of $f \circ g$ is:

$$\begin{pmatrix} 0 & 3 \\ 2 & 3 \end{pmatrix}$$

End Solution . . . .

(c) **(6 points)** Compute the determinants of the following matrices:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} 1 & 2 & -1 \\ -2 & -3 & 2 \\ -1 & -1 & 2 \end{pmatrix}$$

Which of these matrices is invertible?

Begin Solution: . . . .

[Your work goes here...]

$$\det \boldsymbol{A} = -2$$
$$\det \boldsymbol{B} = 0$$
$$\det \boldsymbol{C} = 1$$

Hence $\boldsymbol{A}$ *and* $\boldsymbol{C}$ *are invertible, because their determinants are non-zero.*

End Solution . . . .

(d) **(8 points)** For every invertible matrix in (d), compute the inverse.

Begin Solution: . . . .

[Your work goes here...]

$$\boldsymbol{A}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

[Your work goes here...]

$$\boldsymbol{C}^{-1} = \begin{pmatrix} -4 & -3 & 1 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

End Solution . . . .

3. **(28 points)**

Consider the following matrices:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}$$

(a) **(10 points)** Find the eigenvalues and associated eigenvectors for $\boldsymbol{A}$ and $\boldsymbol{B}$.

Begin Solution: ....

[Your work goes here...]
The eigenvalues of $\boldsymbol{A}$ *are* $\lambda_1 = 2$ *and* $\lambda_2 = 1$*, with associated eigenvectors:*

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

[Your work goes here...]
The eigenvalues of $\boldsymbol{B}$ *are* $\lambda_1 = 10$ *and* $\lambda_2 = 20$*, with associated eigenvectors:*

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

End Solution ....

(b) **(10 points)** Diagonalise $\boldsymbol{A}$.

Begin Solution: ....

[Your work goes here...]

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

End Solution ....

(c) **(8 points)** Use the diagonalisation of $\boldsymbol{A}$ to compute $\boldsymbol{A}^7$.

Begin Solution: ....

$$\boldsymbol{A}^7 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^7 \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \text{[Your work goes here...]} = \begin{pmatrix} -126 & 127 \\ -254 & 255 \end{pmatrix}$$

End Solution ....

4. **(22 points)**

Consider the following vectors in $\mathbb{R}^4$:

$$\boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 7 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

(a) **(3 points)** Compute the lengths of each vector.

Begin Solution: ....

$$\|\boldsymbol{v}_1\| = \text{[Your work goes here...]} = 5$$
$$\|\boldsymbol{v}_2\| = \text{[Your work goes here...]} = \sqrt{27}$$
$$\|\boldsymbol{v}_3\| = \text{[Your work goes here...]} = \sqrt{51}$$

End Solution . . . .

(b) **(4 points)** Compute the distance from $\boldsymbol{v}_2$ to $\boldsymbol{v}_3$.

Begin Solution: . . . .

[Your work goes here...]

$$d(\boldsymbol{v}_2, \boldsymbol{v}_3) = \sqrt{26}$$

End Solution . . . .

(c) **(5 points)** Compute $\cos\gamma$, where γ is the angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Begin Solution: . . . .

[Your work goes here...]

$$\cos\gamma = \frac{5}{\sqrt{27}}$$

End Solution . . . .

(d) **(10 points)** [This question has been removed from the practice exam, since the material is no longer covered.]