Matrix Calculations: Vector Spaces

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Outline

From last time

Homogeneous systems have 0's in the RHS of all equations.

Given a homogeneous system in n variables:

- A basic solution is a non-zero solution of the system.
- If there are n pivots in its Echelon form, $\mathbf{0} = (0, \ldots, 0)$ is the unique solution, so no basic solutions.
- If there are $p < n$ pivots in its Echelon form, it has $n - p$ linearly independent basic solutions.
- Two methods for finding them: plugging in free variables or deleting non-pivot columns, one-by-one

Non-homogeneous case: subtracting solutions

Theorem

For two solutions $\boldsymbol{s}$ *and* $\boldsymbol{p}$ *of a* **non-homogeneous** *system of equations, the difference* $\boldsymbol{s} - \boldsymbol{p}$ *is a solution of the associated* **homogeneous** *system.*

Proof: Let $a_1x_1 + \cdots + a_nx_n = b$ be an equation in the non-homogeneous system. Then:

$$\begin{aligned}
&a_1(s_1 - p_1) + \cdots + a_n(s_n - p_n) \\
&= \big(a_1s_1 - a_1p_1\big) + \cdots + \big(a_ns_n - a_np_n\big) \\
&= \big(a_1s_1 + \cdots + a_ns_n\big) - \big(a_1p_1 + \cdots + a_np_n\big) \\
&= b - b \qquad \text{since the } \boldsymbol{s} \text{ and } \boldsymbol{p} \text{ are solutions} \\
&= 0.
\end{aligned}$$

☺

General solution for non-homogeneous systems

Theorem

Assume a non-homogeneous system has a solution given by the vector $\boldsymbol{p}$, which we call a particular solution.
Then any other solution $\boldsymbol{s}$ of the non-homogeneous system can be written as

$$\boldsymbol{s} = \boldsymbol{p} + \boldsymbol{h}$$

where $\boldsymbol{h}$ is a solution of the associated homogeneous system.

Proof: Let $\boldsymbol{s}$ be a solution of the non-homogeneous system. Then $\boldsymbol{h} = \boldsymbol{s} - \boldsymbol{p}$ is a solution of the associated homogeneous system. Hence we can write $\boldsymbol{s}$ as $\boldsymbol{p} + \boldsymbol{h}$, for $\boldsymbol{h}$ some solution of the associated homogeneous system. ☺

Example: solutions of a non-homogeneous system

- Consider the non-homogeneous system $\begin{cases} x + y + 2z = 9 \\ y - 3z = 4 \end{cases}$
- with solutions: $(0, 7, 1)$ and $(5, 4, 0)$
- We can write $(0, 7, 1)$ as: $(5, 4, 0) + (-5, 3, 1)$
- where:
  - $\boldsymbol{p} = (5, 4, 0)$ is a particular solution (of the original system)
  - $(-5, 3, 1)$ is a solution of the associated homogeneous system: $\begin{cases} x + y + 2z = 0 \\ y - 3z = 0 \end{cases}$
- Similarly, $(10, 1, -1)$ is a solution of the non-homogeneous system and

$$(10, 1, -1) \;=\; (5, 4, 0) + (5, -3, -1)$$

- where:
  - $(5, -3, -1)$ is a solution of the associated homogeneous system.

General solution for non-homogeneous systems, concretely

Theorem

The general solution of a non-homogeneous system of equations in n variables is given by a parametrization as follows:

$$(s_1, \ldots, s_n) = (p_1, \ldots, p_n) + c_1(v_{11}, \ldots, v_{1n}) + \cdots c_k(v_{k1}, \ldots, v_{kn})$$

for $c_1, \ldots, c_k \in \mathbb{R}$,
where

- $(p_1, \ldots, p_n)$ *is a particular solution*
- $(v_{11}, \ldots, v_{1n}), \ldots, (v_{k1}, \ldots, v_{kn})$ *are basic solutions of the associated homogeneous system.*
- *So* $c_1(v_{11}, \ldots, v_{1n}) + \cdots + c_k(v_{k1}, \ldots, v_{kn})$ *is a general solution for the associated homogeneous system.*

Finding a particular solution

- **Recall:** we found basic solutions by setting all but one of the free variables to zero and solving the homogeneous system
- To find a particular solution, set all the free variables to zero and solving the non-homogeneous system
- In other words, remove all the non-pivot columns:

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right) \quad \mapsto \quad \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

- Solve. Then, add zeros back in for the free variables:

$$(10, -11, 4) \quad \mapsto \quad (10, 0, -11, 0, 4)$$

Elaborated example, part I

- Consider the non-homogeneous system of equations given by the augmented matrix in echelon form:

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

- It has 5 variables, 3 pivots, and thus $5 - 3 = 2$ basic solutions
- To find a particular solution, remove the non-pivot columns, and (uniquely!) solve the resulting system:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

- This has $(10, -11, 4)$ as solution; the orginal 5-variable system then has particular solution $(10, 0, -11, 0, 4)$.

Elaborated example, part II

- Consider the associated homogeneous system of equations:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \end{array}$$
$$\begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

- The two basic solutions are found by removing each of the two non-pivot columns separately, and finding solutions:

$$\begin{array}{cccc} x_1 & x_3 & x_4 & x_5 \end{array} \qquad\qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_5 \end{array}$$
$$\begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix} \text{ and } \begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

- We find: $(1, -2, 1, 0)$ and $(-1, 1, 0, 0)$. Adding zeros for missing columns gives: $(1, 0, -2, 1, 0)$ and $(-1, 1, 0, 0, 0)$.

Elaborated example, part III

Wrapping up: all solutions of the system

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

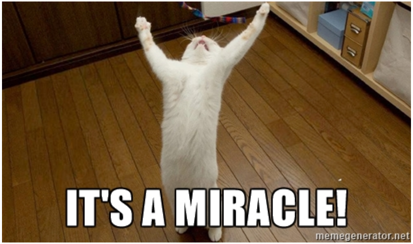
are of the form:

$$\underbrace{(10, 0, -11, 0, 4)}_{\text{particular sol.}} + \underbrace{c_1(1, 0, -2, 1, 0) + c_2(-1, 1, 0, 0, 0)}_{\text{two basic solutions}}.$$

This is the general solution of the non-homogeneous system.

What are numbers?

Suppose I don't know what numbers are...
...but I still manage to pass Mathematical Structures.



Tell me: what are numbers?

What is the *first thing* you would tell me about some numbers, e.g. the real numbers?

What are numbers?

The First Thing: numbers form a set

$$S \qquad (\longleftarrow \text{ these are some numbers!})$$

The Second Thing: numbers can be added together

$$a \in S, b \in S \qquad \Longrightarrow \qquad a + b \in S$$

Addition? Tell me more!

We have a set S, with a special operation '$+$' which satisfies:

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$

...and there's a special element $\mathbf{0} \in S$where:

3. $a + \mathbf{0} = a$

In math-speak, $(S, +, \mathbf{0})$ is called a commutative monoid, but we could also just call it a set with addition.

Examples: sets with addition

- Every kind of number you know: $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \ldots$
- The set of all polynomials:

$$(x^2 + 4x + 1) + (2x^2) := 3x^2 + 4x + 1 \qquad \mathbf{0} := 0$$

- The set of all finite sets:

$$\{1,2,3\} + \{3,4\} := \{1,2,3\} \cup \{3,4\} = \{1,2,3,4\} \qquad \mathbf{0} := \{\}$$

- Here's a small example: $\{0\}$

$$0 + 0 := 0 \qquad 0 := 0$$

- ...and (important!) the set $\mathbb{R}^n$ of all vectors of size n:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n) \qquad \mathbf{0} := (0, \ldots, 0)$$

Linear combinations

- We've been talking a lot about linear combinations:

$$a \cdot \boldsymbol{v} + b \cdot \boldsymbol{w} = \boldsymbol{u}$$

- **Q:** what is the most general kind of set, where we can take linear combinations of elements?
- **A:** a set V with addition and...scalar multiplication

$$a \in \mathbb{R}, \boldsymbol{v} \in V \qquad \Longrightarrow \qquad a \cdot \boldsymbol{v} \in V$$

Multiplication?! What does that do?

A vector space is a set with addition $(V, +, \mathbf{0})$ with an extra operation '$\cdot$', which satisfies:

1. $a \cdot (\boldsymbol{v} + \boldsymbol{w}) = a \cdot \boldsymbol{v} + a \cdot \boldsymbol{w}$
2. $(a + b) \cdot \boldsymbol{v} = a \cdot \boldsymbol{v} + b \cdot \boldsymbol{v}$
3. $a \cdot (b \cdot \boldsymbol{v}) = ab \cdot \boldsymbol{v}$
4. $1 \cdot \boldsymbol{v} = \boldsymbol{v}$
5. $0 \cdot \boldsymbol{v} = \mathbf{0}$

Example

Our **main example** is $\mathbb{R}^n$, where:

$$a \cdot (v_1, \ldots, v_n) := (av_1, \ldots, av_n)$$

Vector spaces: all together

Definition

A vector space $(V, +, \cdot, \mathbf{0})$ is a set V with a special element $\mathbf{0} \in V$ and operations '$+$' and '$\cdot$' satisfying:

1. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
2. $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$
3. $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$
4. $a \cdot (\boldsymbol{v} + \boldsymbol{w}) = a \cdot \boldsymbol{v} + a \cdot \boldsymbol{w}$
5. $(a + b) \cdot \boldsymbol{v} = a \cdot \boldsymbol{v} + b \cdot \boldsymbol{v}$
6. $a \cdot (b \cdot \boldsymbol{v}) = ab \cdot \boldsymbol{v}$
7. $1 \cdot \boldsymbol{v} = \boldsymbol{v}$
8. $0 \cdot \boldsymbol{v} = \mathbf{0}$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $a, b \in \mathbb{R}$.

Vector spaces: Main Example

Our **main example**:

$$\mathbb{R}^n = \{(v_1, \ldots, v_n) \mid v_1, \ldots, v_n \in \mathbb{R}\}$$
$$= \{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \ldots, v_n \in \mathbb{R}\}$$

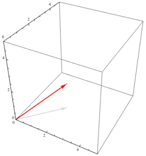
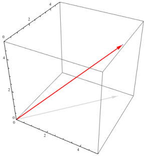
The operations:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} \qquad a \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} av_1 \\ \vdots \\ av_n \end{pmatrix}$$

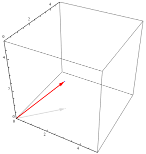
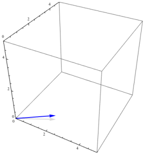
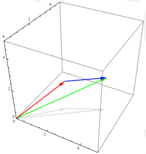
have a clear geometric interpretation.

Vector spaces: geometric interpretation

$a \cdot \boldsymbol{v}$ makes a vector shorter or longer:

$\boldsymbol{v} :=$  $2 \cdot \boldsymbol{v} =$ 

$\boldsymbol{v} + \boldsymbol{w}$ stacks vectors together:

$\boldsymbol{v} :=$  $\boldsymbol{w} :=$  $\boldsymbol{v} + \boldsymbol{w} :=$ 

Example: subspaces

Certain subsets $V \subseteq \mathbb{R}^n$ are also vector spaces, e.g.

$$V = \{(v_1, v_2, 0) \mid v_1, v_2 \in \mathbb{R}\} \subseteq \mathbb{R}^3$$

$$W = \{(x, 2x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

as long as they have $\mathbf{0}$, and they are closed under '$+$' and '$\cdot$':

$$\boldsymbol{v}, \boldsymbol{w} \in V \implies \boldsymbol{v} + \boldsymbol{w} \in V$$

$$\boldsymbol{v} \in V, a \in \mathbb{R} \implies a \cdot \boldsymbol{v} \in V$$

These are called *subspaces* of $\mathbb{R}^n$.

Vector space example

We've seen this example before!

Example

The set of solutions of a homogeneous system of equations is a vector space.

Let S be the set of solutions of a homogeneous system of equations, with n variables. Then $S \subseteq \mathbb{R}^n$, and as we learned last week:

$$\boldsymbol{s}, \boldsymbol{t} \in S \implies \boldsymbol{s} + \boldsymbol{t} \in S$$

$$\boldsymbol{s} \in S, a \in \mathbb{R} \implies a \cdot \boldsymbol{s} \in S$$

Proving something is a subspace

In summary, to *prove* something is a subspace, there are 2 things to check:

1. $$\mathbf{v}, \mathbf{w} \in V \implies \mathbf{v} + \mathbf{w} \in V$$

2. $$\mathbf{v} \in V, \lambda \in \mathbb{R} \implies \lambda \cdot \mathbf{v} \in V$$

To prove something is *not* a subspace, show that one of these things fails:

- e.g. find $\mathbf{v}, \mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} \notin V$
- ...or find $\mathbf{v}$ such that $2 \cdot \mathbf{v} \notin V$,
- ...or give some other *counter-example*.

Proving something is a subspace: example

Example: Prove $V = \{(2x, y, x+y) \mid x, y \in \mathbb{R}\}$ is a subspace.

1. Let $\boldsymbol{v} = (2a, b, a+b)$ and $\boldsymbol{w} = (2c, d, c+d)$ are two arbitrary elements of V. Then:

$$\begin{aligned} \boldsymbol{v} + \boldsymbol{w} &= (2a + 2c, b + d, a + b + c + d) \\ &= (2(a+c), b+d, (a+c) + (b+d)) \end{aligned}$$

   is in V. (In this case, $x = a + c$ and $y = b + d$.)

2. Let $\boldsymbol{v} = (2a, b, a+b)$ and λ be some arbitrary vector in V and some arbitrary real number. Then:

$$\begin{aligned} \lambda \cdot \boldsymbol{v} &= (2\lambda a, \lambda b, \lambda(a+b)) \\ &= (2\lambda a, \lambda b, \lambda a + \lambda b) \end{aligned}$$

   is in V. (In this case, $x = \lambda a$ and $y = \lambda b$.)

Vector spaces: 'weirder' examples

$\mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ are the only things we'll use in this course...but there are other examples:

- $\{0\}$ is still an example
- Polynomials are still an example: $5 \cdot (2x^2 + 1) = 10x^2 + 5$
- ...but finite sets are not!

$$5 \cdot \{\text{sandwich}, \text{Tuesday}\} = ???$$

- Functions $\mathcal{F}(X) := \{f : X \to \mathbb{R}\}$ are an example. If f, g are functions, then '$f + g$' and $a \cdot f$ are also functions, defined by:

$$(f + g)(x) := f(x) + g(x) \qquad (a \cdot f)(x) = af(x)$$

Exercise: show that, if $X = \{1, 2, \ldots, n\}$, then $\mathcal{F}(X)$ is basically the same as $\mathbb{R}^n$.