ELEMENTARY MATHEMATICS

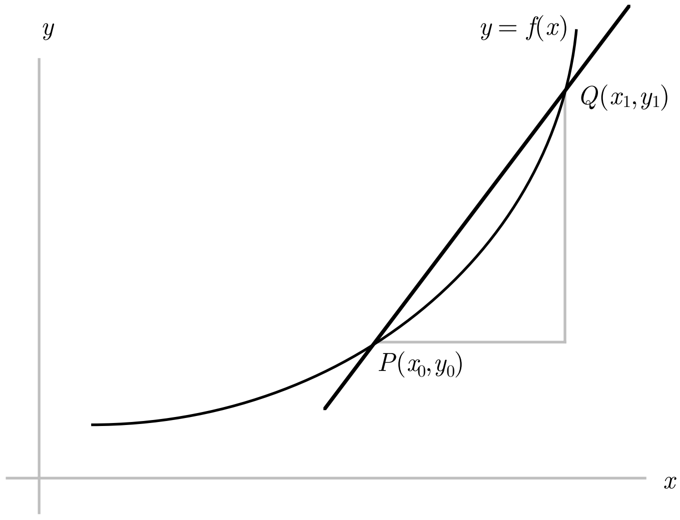
W W L CHEN and X T DUONG

Chapter 11

INTRODUCTION TO DIFFERENTIATION

11.1. Tangent to a Curve

Consider the graph of a function $y = f(x)$. Suppose that $P(x_0, y_0)$ is a point on the curve $y = f(x)$. Consider now another point $Q(x_1, y_1)$ on the curve close to the point $P(x_0, y_0)$. We draw the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$, and obtain the picture below.



Clearly the slope of this line is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

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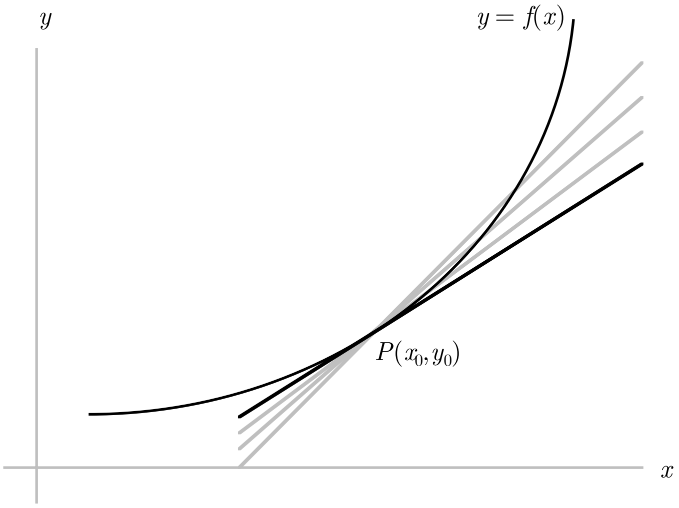
† This chapter was written at Macquarie University in 1999.

Now let us keep the point $P(x_0, y_0)$ fixed, and move the point $Q(x_1, y_1)$ along the curve towards the point P. Eventually the line PQ becomes the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$, as shown in the picture below.

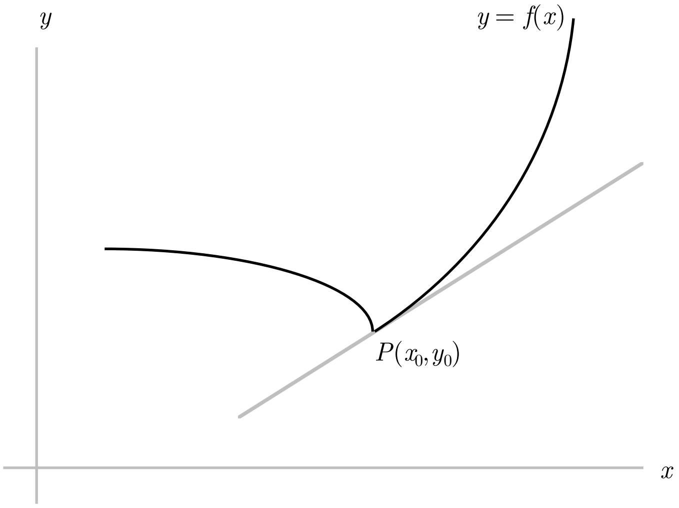


We are interested in the slope of this tangent line. Its value is called the derivative of the function $y = f(x)$ at the point $x = x_0$, and denoted by

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} \qquad \text{or} \qquad f'(x_0).$$

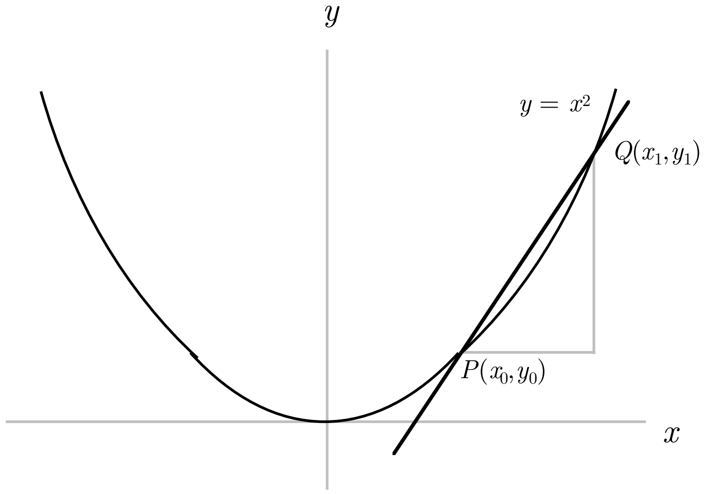
In this case, we say that the function $y = f(x)$ is differentiable at the point $x = x_0$.

Remark. Sometimes, when we move the point $Q(x_1, y_1)$ along the curve $y = f(x)$ towards the point $P(x_0, y_0)$, the line PQ does not become the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$. In this case, we say that the function $y = f(x)$ is not differentiable at the point $x = x_0$. An example of such a situation is given in the picture below.



Note that the curve $y = f(x)$ makes an abrupt turn at the point $P(x_0, y_0)$.

EXAMPLE 11.1.1. Consider the graph of the function $y = f(x) = x^2$.



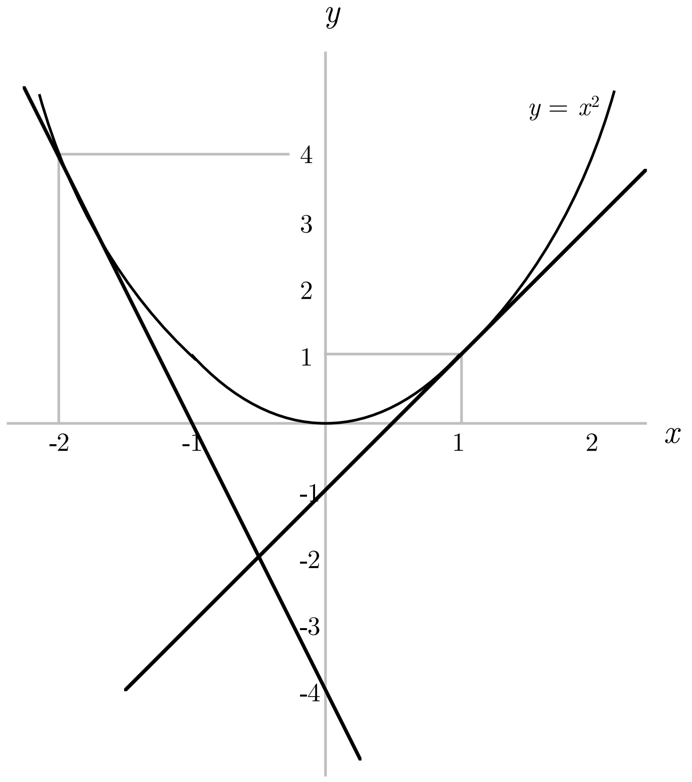
Here the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = x_1 + x_0.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will eventually be equal to $x_0 + x_0 = 2x_0$. Hence for the function $y = f(x) = x^2$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 2x_0.$$

In particular, the tangent to the curve at the point $(1, 1)$ has slope 2 and so has equation $y = 2x - 1$, whereas the tangent to the curve at the point $(-2, 4)$ has slope -4 and so has equation $y = -4x - 4$.



EXAMPLE 11.1.2. Consider the graph of the function $y = f(x) = x^3$. Here the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^3 - x_0^3}{x_1 - x_0} = x_1^2 + x_1x_0 + x_0^2.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will eventually be equal to $x_0^2 + x_0x_0 + x_0^2 = 3x_0^2$. Hence for the function $y = f(x) = x^3$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 3x_0^2.$$

In particular, the tangent to the curve at the point $(0, 0)$ has slope 0 and so has equation $y = 0$, whereas the tangent to the curve at the point $(2, 8)$ has slope 12 and so has equation $y = 12x - 16$.

EXAMPLE 11.1.3. Consider the graph of the function $y = f(x) = x$. Here the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1 - x_0}{x_1 - x_0} = 1.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will remain equal to 1. Hence for the function $y = f(x) = x$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 1.$$

EXAMPLE 11.1.4. Consider the graph of the function $y = f(x) = x^{1/2}$, defined for all real numbers $x \geq 0$. Suppose that $x_0 > 0$ and $x_1 > 0$. Then the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^{1/2} - x_0^{1/2}}{x_1 - x_0} = \frac{1}{x_1^{1/2} + x_0^{1/2}}.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will eventually be equal to

$$\frac{1}{x_0^{1/2} + x_0^{1/2}} = \frac{1}{2x_0^{1/2}} = \frac{1}{2}x_0^{-1/2}.$$

Hence for the function $y = f(x) = x^{1/2}$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = \frac{1}{2}x_0^{-1/2}.$$

The above four examples are special cases of the following result.

DERIVATIVES OF POWERS. *Suppose that n is a fixed non-zero real number. Then for the function $y = f(x) = x^n$, we have*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = nx^{n-1}$$

for every real number x for which x^{n-1} is defined.

Here and henceforth, we shall slightly abuse our notation and refer to $f'(x)$ as the derivative of the function $y = f(x)$, and write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x).$$

EXAMPLE 11.1.5. For the function $y = f(x) = x^{1/4}$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \frac{1}{4}x^{-3/4}$$

for every positive real number x.

The rule concerning derivatives of powers does not apply in the case $n = 0$.

DERIVATIVES OF CONSTANTS. *Suppose that $f(x) = c$, where c is a fixed real number. Then $f'(x) = 0$ for every real number x.*

11.2. Arithmetic of Derivatives

Very often, we need to find the derivatives of complicated functions which are constant multiples, sums, products and/or quotients of much simpler functions. To achieve this, we can make use of our knowledge concerning the derivatives of these simpler functions. We have four extremely useful results.

CONSTANT MULTIPLE RULE. *Suppose that $m(x) = cf(x)$, where c is a fixed real number. Then*

$$m'(x) = cf'(x)$$

for every real number x for which $f'(x)$ exists.

SUM RULE. *Suppose that $s(x) = f(x) + g(x)$ and $d(x) = f(x) - g(x)$. Then*

$$s'(x) = f'(x) + g'(x) \qquad \text{and} \qquad d'(x) = f'(x) - g'(x)$$

for every real number x for which $f'(x)$ and $g'(x)$ exist.

EXAMPLE 11.2.1. Consider the function $h(x) = 5x^2 + 3x^5$. We can write

$$h(x) = f(x) + g(x),$$

where $f(x) = 5x^2$ and $g(x) = 3x^5$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x).$$

Next, the function $f(x) = 5x^2$ is a constant (5) multiple of the function x^2, and so it follows from the constant multiple rule and the rule on the derivatives of powers that

$$f'(x) = 5(x^2)' = 5(2x) = 10x.$$

Similarly, the function $g(x) = 3x^5$ is a constant (3) multiple of the function x^5, and so it follows from the constant multiple rule and the rule on the derivatives of powers that

$$g'(x) = 3(x^5)' = 3(5x^4) = 15x^4.$$

Hence $h'(x) = 10x + 15x^4$.

Example 11.2.2. Consider the function $h(x)=(3x)^4-(2x)^6$. We can write

$$h(x)=f(x)-g(x),$$

where $f(x)=81x^4$ and $g(x)=64x^6$. It follows from the sum rule that

$$h'(x)=f'(x)-g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x)=324x^3$ and $g'(x)=384x^5$. Hence $h'(x)=324x^3-384x^5$.

The sum rule can be extended to the sum or difference of more than two functions in the natural way. We illustrate the technique in the following three examples.

Example 11.2.3. Consider the function $h(x)=4x^3-15x^2+4x-1$. We can write

$$h(x)=f(x)-g(x)+k(x)-t(x),$$

where $f(x)=4x^3$, $g(x)=15x^2$, $k(x)=4x$ and $t(x)=1$. It follows from the sum rule that

$$h'(x)=f'(x)-g'(x)+k'(x)-t'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x)=12x^2$, $g'(x)=30x$ and $k'(x)=4$. Applying the rule on the derivatives of constants, we obtain $t'(x)=0$. Hence $h'(x)=12x^2-30x+4$.

Example 11.2.4. Consider the function $h(x)=8x^3-2(x+2)^2+3$. Then $h(x)=8x^3-2x^2-8x-5$, and so we can write

$$h(x)=f(x)-g(x)-k(x)-t(x),$$

where $f(x)=8x^3$, $g(x)=2x^2$, $k(x)=8x$ and $t(x)=5$. It follows from the sum rule that

$$h'(x)=f'(x)-g'(x)-k'(x)-t'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x)=24x^2$, $g'(x)=4x$ and $k'(x)=8$. Applying the rule on the derivatives of constants, we obtain $t'(x)=0$. Hence $h'(x)=24x^2-4x-8$.

Example 11.2.5. Consider the function $h(x)=(x^2+2x)^2$. Then $h(x)=x^4+4x^3+4x^2$, and so we can write

$$h(x)=f(x)+g(x)+k(x),$$

where $f(x)=x^4$, $g(x)=4x^3$ and $k(x)=4x^2$. It follows from the sum rule that

$$h'(x)=f'(x)+g'(x)+k'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x)=4x^3$, $g'(x)=12x^2$ and $k'(x)=8x$. Hence $h'(x)=4x^3+12x^2+8x$.

Example 11.2.6. Consider the function

$$h(x)=\frac{3}{x}+2x.$$

We can write

$$h(x)=f(x)+g(x),$$

where $f(x)=3x^{-1}$ and $g(x)=2x$. It follows from the sum rule that

$$h'(x)=f'(x)+g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x)=-3x^{-2}$ and $g'(x)=2$. Hence $h'(x)=2-3x^{-2}$.

Example 11.2.7. Consider the function

$$h(x)=6x^2\sqrt{x}-\frac{4}{\sqrt{x}}+3x^{1/3}.$$

We can write

$$h(x)=f(x)-g(x)+k(x),$$

where $f(x)=6x^{5/2}$, $g(x)=4x^{-1/2}$ and $k(x)=3x^{1/3}$. It follows from the sum rule that

$$h'(x)=f'(x)-g'(x)+k'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x)=15x^{3/2}$, $g'(x)=-2x^{-3/2}$ and $k'(x)=x^{-2/3}$. Hence $h'(x)=15x^{3/2}+2x^{-3/2}+x^{-2/3}$.

Example 11.2.8. Consider the function $h(x)=\sqrt{3x}+\sqrt[3]{2x}$. We can write

$$h(x)=f(x)+g(x),$$

where $f(x)=\sqrt{3}x^{1/2}$ and $g(x)=\sqrt[3]{2}x^{1/3}$. It follows from the sum rule that

$$h'(x)=f'(x)+g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain

$$f'(x)=\frac{\sqrt{3}}{2}x^{-1/2}\qquad\text{and}\qquad g'(x)=\frac{\sqrt[3]{2}}{3}x^{-2/3}.$$

Hence

$$h'(x)=\frac{\sqrt{3}}{2}x^{-1/2}+\frac{\sqrt[3]{2}}{3}x^{-2/3}=\sqrt{\frac{3}{4x}}+\sqrt[3]{\frac{2}{27x^2}}.$$

PRODUCT RULE. *Suppose that $p(x)=f(x)g(x)$. Then*

$$p'(x)=f'(x)g(x)+f(x)g'(x)$$

for every real number x for which $f'(x)$ and $g'(x)$ exist.

Example 11.2.9. Consider the function $h(x)=(x^3-x^5)(x^2+x^4)$. We can write

$$h(x)=f(x)g(x),$$

where $f(x)=x^3-x^5$ and $g(x)=x^2+x^4$. It follows from the product rule that

$$h'(x)=f'(x)g(x)+f(x)g'(x).$$

Applying the sum rule and the rule on the derivatives of powers, we obtain $f'(x)=3x^2-5x^4$ and $g'(x)=2x+4x^3$. Hence

$$h'(x)=(3x^2-5x^4)(x^2+x^4)+(x^3-x^5)(2x+4x^3)=5x^4-9x^8.$$

Alternatively, we observe that $h(x)=(x^3-x^5)(x^2+x^4)=x^5-x^9$. Applying the sum rule and the rule on the derivatives of powers, we obtain $h'(x)=5x^4-9x^8$ as before.

Example 11.2.10. Let us return to Example 11.2.5 and consider again the function $h(x) = (x^2+2x)^2$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = g(x) = x^2 + 2x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule, the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = g'(x) = 2x + 2$. Hence

$$h'(x) = (2x + 2)(x^2 + 2x) + (x^2 + 2x)(2x + 2) = 2(2x + 2)(x^2 + 2x) = 4x^3 + 12x^2 + 8x$$

as before. We shall return to example again in Section 12.1.

Example 11.2.11. Consider the function $h(x) = (x^2 + x)(x^3 - 6x^2 + 2x)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x^2 + x$ and $g(x) = x^3 - 6x^2 + 2x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule, the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 2x + 1$ and $g'(x) = 3x^2 - 12x + 2$. Hence

$$h'(x) = (2x + 1)(x^3 - 6x^2 + 2x) + (x^2 + x)(3x^2 - 12x + 2).$$

Example 11.2.12. Consider the function

$$h(x) = (x + \sqrt{x})\left(x - \frac{1}{\sqrt{x}}\right).$$

We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x + x^{1/2}$ and $g(x) = x - x^{-1/2}$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule and the rule on the derivatives of powers, we obtain

$$f'(x) = 1 + \frac{1}{2}x^{-1/2} \qquad \text{and} \qquad g'(x) = 1 + \frac{1}{2}x^{-3/2}.$$

Hence

$$\begin{aligned} h'(x) &= \left(1 + \frac{1}{2}x^{-1/2}\right)(x - x^{-1/2}) + (x + x^{1/2})\left(1 + \frac{1}{2}x^{-3/2}\right) \\ &= \left(x - x^{-1/2} + \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1}\right) + \left(x + \frac{1}{2}x^{-1/2} + x^{1/2} + \frac{1}{2}x^{-1}\right) \\ &= 2x + \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}. \end{aligned}$$

Alternatively, we observe that $h(x) = (x + x^{1/2})(x - x^{-1/2}) = x^2 - x^{1/2} + x^{3/2} - 1$. Applying the sum rule and the rules on the derivatives of powers and constants, we obtain

$$h'(x) = 2x - \frac{1}{2}x^{-1/2} + \frac{3}{2}x^{1/2}$$

as before.

The product rule can be extended to the product of more than two functions. The extension is at first sight somewhat less obvious than in the case of the sum rule. However, with a bit of care, it is in fact rather straightforward.

EXAMPLE 11.2.13. Consider the function $h(x) = (x^2 + 4x)(2x + 1)(6 - 2x^2)$. We can write

$$h(x) = f(x)r(x),$$

where $f(x) = x^2 + 4x$ and $r(x) = (2x + 1)(6 - 2x^2)$. It follows from the product rule that

$$h'(x) = f'(x)r(x) + f(x)r'(x).$$

We can now write

$$r(x) = g(x)k(x),$$

where $g(x) = 2x + 1$ and $k(x) = 6 - 2x^2$. It follows from the product rule that

$$r'(x) = g'(x)k(x) + g(x)k'(x).$$

Hence $h(x) = f(x)g(x)k(x)$, and

$$h'(x) = f'(x)g(x)k(x) + f(x)g'(x)k(x) + f(x)g(x)k'(x).$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 2x + 4$, $g'(x) = 2$ and $k'(x) = -4x$. Hence

$$h'(x) = (2x + 4)(2x + 1)(6 - 2x^2) + 2(x^2 + 4x)(6 - 2x^2) - 4x(x^2 + 4x)(2x + 1).$$

REMARK. The interested reader is challenged to show that if $p(x) = f(x)g(x)k(x)t(x)$, then

$$p'(x) = f'(x)g(x)k(x)t(x) + f(x)g'(x)k(x)t(x) + f(x)g(x)k'(x)t(x) + f(x)g(x)k(x)t'(x).$$

QUOTIENT RULE. *Suppose that $q(x) = f(x)/g(x)$. Then*

$$q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

for every real number x for which $f'(x)$ and $g'(x)$ exist, and for which $g(x) \neq 0$.

EXAMPLE 11.2.14. Consider the function

$$h(x) = \frac{x^2 - 1}{x^3 + 2x}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = x^2 - 1$ and $g(x) = x^3 + 2x$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 2x$ and $g'(x) = 3x^2 + 2$. Hence

$$h'(x) = \frac{2x(x^3 + 2x) - (x^2 - 1)(3x^2 + 2)}{(x^3 + 2x)^2}.$$

EXAMPLE 11.2.15. Consider the function

$$h(x)=\frac{4x^2+1}{3x}.$$

We can write

$$h(x)=\frac{f(x)}{g(x)},$$

where $f(x)=4x^2+1$ and $g(x)=3x$. It follows from the quotient rule that

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x)=8x$ and $g'(x)=3$. Hence

$$h'(x)=\frac{24x^2-3(4x^2+1)}{9x^2}.$$

EXAMPLE 11.2.16. Consider the function

$$h(x)=\frac{3x^2+4x^7}{5x^{-2}+3}.$$

We can write

$$h(x)=\frac{f(x)}{g(x)},$$

where $f(x)=3x^2+4x^7$ and $g(x)=5x^{-2}+3$. It follows from the quotient rule that

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x)=6x+28x^6$ and $g'(x)=-10x^{-3}$. Hence

$$\begin{aligned}h'(x)&=\frac{(5x^{-2}+3)(6x+28x^6)+10x^{-3}(3x^2+4x^7)}{(5x^{-2}+3)^2}=\frac{30x^{-1}+140x^4+18x+84x^6+30x^{-1}+40x^4}{25x^{-4}+30x^{-2}+9}\\&=\frac{60x^{-1}+18x+180x^4+84x^6}{25x^{-4}+30x^{-2}+9}\times\frac{x^4}{x^4}=\frac{60x^3+18x^5+180x^8+84x^{10}}{25+30x^2+9x^4}.\end{aligned}$$

Alternatively, we observe that

$$h(x)=\frac{3x^2+4x^7}{5x^{-2}+3}\times\frac{x^2}{x^2}=\frac{3x^4+4x^9}{5+3x^2}.$$

We can write

$$h(x)=\frac{k(x)}{t(x)},$$

where $k(x)=3x^4+4x^9$ and $t(x)=5+3x^2$. It follows from the quotient rule that

$$h'(x)=\frac{t(x)k'(x)-k(x)t'(x)}{t^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $k'(x)=12x^3+36x^8$ and $g'(x)=6x$. Hence

$$\begin{aligned}h'(x)&=\frac{(5+3x^2)(12x^3+36x^8)-6x(3x^4+4x^9)}{(5+3x^2)^2}=\frac{60x^3+36x^5+180x^8+108x^{10}-18x^5-24x^{10}}{25+30x^2+9x^4}\\&=\frac{60x^3+18x^5+180x^8+84x^{10}}{25+30x^2+9x^4}\end{aligned}$$

as before.

Example 11.2.17. Consider the function

$$h(x)=\frac{(x^2+4)(x-2)}{x^2+2}.$$

We can write

$$h(x)=\frac{f(x)}{g(x)},$$

where $f(x)=(x^2+4)(x-2)$ and $g(x)=x^2+2$. It follows from the quotient rule that

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

We can now write

$$f(x)=k(x)t(x),$$

where $k(x)=x^2+4$ and $t(x)=x-2$. It follows from the product rule that

$$f'(x)=k'(x)t(x)+k(x)t'(x).$$

Hence

$$h(x)=\frac{k(x)t(x)}{g(x)},$$

and

$$h'(x)=\frac{g(x)k'(x)t(x)+g(x)k(x)t'(x)-k(x)t(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $k'(x)=2x$, $t'(x)=1$ and $g'(x)=2x$. Hence

$$h'(x)=\frac{2x(x^2+2)(x-2)+(x^2+2)(x^2+4)-2x(x^2+4)(x-2)}{(x^2+2)^2}=\frac{x^4+2x^2+8x+8}{(x^2+2)^2}.$$

Alternatively, we observe that

$$f(x)=(x^2+4)(x-2)=x^3-2x^2+4x-8.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x)=3x^2-4x+4$. Hence

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}=\frac{(x^2+2)(3x^2-4x+4)-2x(x^3-2x^2+4x-8)}{(x^2+2)^2}=\frac{x^4+2x^2+8x+8}{(x^2+2)^2}$$

as before.

For those who want a small challenge, here is one more example.

Example 11.2.18. Consider the function

$$h(x)=\frac{(4x-3)(2x^2-3x)}{(2x+2)(x^3+6)}.$$

We can write

$$h(x)=\frac{f(x)}{g(x)},$$

where $f(x)=(4x-3)(2x^2-3x)$ and $g(x)=(2x+2)(x^3+6)$. It follows from the quotient rule that

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

We can now write

$$f(x) = k(x)t(x) \qquad \text{and} \qquad g(x) = u(x)v(x),$$

where $k(x) = 4x - 3$, $t(x) = 2x^2 - 3x$, $u(x) = 2x + 2$ and $v(x) = x^3 + 6$. It follows from the product rule that

$$f'(x) = k'(x)t(x) + k(x)t'(x) \qquad \text{and} \qquad g'(x) = u'(x)v(x) + u(x)v'(x).$$

Hence

$$h(x) = \frac{k(x)t(x)}{u(x)v(x)},$$

and

$$\begin{aligned} h'(x) &= \frac{u(x)v(x)k'(x)t(x) + u(x)v(x)k(x)t'(x) - k(x)t(x)u'(x)v(x) - k(x)t(x)u(x)v'(x)}{u^2(x)v^2(x)} \\ &= \frac{u(x)v(x)k'(x)t(x) + u(x)v(x)k(x)t'(x)}{u^2(x)v^2(x)} - \frac{k(x)t(x)u'(x)v(x) + k(x)t(x)u(x)v'(x)}{u^2(x)v^2(x)}. \end{aligned}$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $k'(x) = 4$, $t'(x) = 4x - 3$, $u'(x) = 2$ and $v'(x) = 3x^2$. Hence

$$\begin{aligned} h'(x) = {} & \frac{4(2x+2)(x^3+6)(2x^2-3x) + (2x+2)(x^3+6)(4x-3)^2}{(2x+2)^2(x^3+6)^2} \\ & - \frac{2(4x-3)(2x^2-3x)(x^3+6) + 3x^2(4x-3)(2x^2-3x)(2x+2)}{(2x+2)^2(x^3+6)^2}. \end{aligned}$$

11.3. Derivatives of the Trigonometric Functions

Consider the curve $y = f(x) = \sin x$. Suppose that $P(x, f(x))$ is a point on this curve. Consider another point $Q(x + h, f(x + h))$, where $h \neq 0$, which also lies on this curve. Clearly the slope of the line joining the two points P and Q is equal to

$$\frac{f(x+h) - f(x)}{(x+h) - x} = \frac{\sin(x+h) - \sin x}{h}.$$

Consider the curve $y = g(x) = \cos x$. Suppose that $R(x, g(x))$ is a point on this curve. Consider another point $S(x + h, g(x + h))$, where $h \neq 0$, which also lies on this curve. Clearly the slope of the line joining the two points R and S is equal to

$$\frac{g(x+h) - g(x)}{(x+h) - x} = \frac{\cos(x+h) - \cos x}{h}.$$

We now move the point Q along the curve $y = f(x) = \sin x$ towards the point P, and move the point S along the curve $y = g(x) = \cos x$ towards the point R. Recall Example 3.3.9, that when h is very close to 0, we have

$$\frac{\sin(x+h) - \sin x}{h} \approx \cos x \qquad \text{and} \qquad \frac{\cos(x+h) - \cos x}{h} \approx -\sin x.$$

We have established the first two parts of the result below.

DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS.
(a) *If* $f(x) = \sin x$, *then* $f'(x) = \cos x$.
(b) *If* $g(x) = \cos x$, *then* $g'(x) = -\sin x$.
(c) *If* $t(x) = \tan x$, *then* $t'(x) = \sec^2 x$.
(d) *If* $t(x) = \cot x$, *then* $t'(x) = -\csc^2 x$.
(e) *If* $t(x) = \sec x$, *then* $t'(x) = \tan x \sec x$.
(f) *If* $t(x) = \csc x$, *then* $t'(x) = -\cot x \csc x$.

PROOF. The proofs of parts (c)–(f) depend on the quotient rule as well as parts (a) and (b). For the sake of convenience, we use the functions $f(x)=\sin x$ and $g(x)=\cos x$ throughout this proof, as well as the function $c(x)=1$, with $c'(x)=0$.

(c) Suppose that $t(x)=\tan x$. Then $t(x)=f(x)/g(x)$. It follows from the quotient rule that

$$t'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x.$$

(d) Suppose that $t(x)=\cot x$. Then $t(x)=g(x)/f(x)$. It follows from the quotient rule that

$$t'(x)=\frac{f(x)g'(x)-g(x)f'(x)}{f^2(x)}=\frac{-\sin^2 x-\cos^2 x}{\sin^2 x}=-\frac{1}{\sin^2 x}=-\csc^2 x.$$

(e) Suppose that $t(x)=\sec x$. Then $t(x)=c(x)/g(x)$. It follows from the quotient rule that

$$t'(x)=\frac{g(x)c'(x)-c(x)g'(x)}{g^2(x)}=\frac{\sin x}{\cos^2 x}=\frac{\sin x}{\cos x}\times\frac{1}{\cos x}=\tan x\sec x.$$

(f) Suppose that $t(x)=\csc x$. Then $t(x)=c(x)/f(x)$. It follows from the quotient rule that

$$t'(x)=\frac{f(x)c'(x)-c(x)f'(x)}{f^2(x)}=-\frac{\cos x}{\sin^2 x}=-\frac{\cos x}{\sin x}\times\frac{1}{\sin x}=-\cot x\csc x.$$ ♣

We next combine our knowledge on trigonometric functions with the arithmetic of derivatives. The reader is advised to identify the rules used at each step in the following examples.

EXAMPLE 11.3.1. Consider the function $h(x)=(x^3-2)(\sin x+\cos x)$. We can write

$$h(x)=f(x)g(x),$$

where $f(x)=x^3-2$ and $g(x)=\sin x+\cos x$. It follows that

$$h'(x)=f'(x)g(x)+f(x)g'(x).$$

Observe next that $f'(x)=3x^2$ and $g'(x)=\cos x-\sin x$. Hence

$$h'(x)=3x^2(\sin x+\cos x)+(x^3-2)(\cos x-\sin x).$$

EXAMPLE 11.3.2. Consider the function

$$h(x)=\frac{\sin x}{x}.$$

We can write

$$h(x)=\frac{f(x)}{g(x)},$$

where $f(x)=\sin x$ and $g(x)=x$. It follows that

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x)=\cos x$ and $g'(x)=1$. Hence

$$h'(x)=\frac{x\cos x-\sin x}{x^2}.$$

Example 11.3.3. Consider the function $h(x)=\sin^2 x$. We can write

$$h(x)=f(x)g(x),$$

where $f(x)=g(x)=\sin x$. It follows that

$$h'(x)=f'(x)g(x)+f(x)g'(x).$$

Observe next that $f'(x)=g'(x)=\cos x$. Hence

$$h'(x)=\cos x\sin x+\sin x\cos x=2\sin x\cos x.$$

Example 11.3.4. Consider the function $y=\sin 2x$. We can write

$$h(x)=2f(x)g(x),$$

where $f(x)=\sin x$ and $g(x)=\cos x$. It follows that

$$h'(x)=2(f'(x)g(x)+f(x)g'(x)).$$

Observe next that $f'(x)=\cos x$ and $g'(x)=-\sin x$. Hence

$$h'(x)=2(\cos^2 x-\sin^2 x)=2\cos 2x.$$

We shall return to Examples 11.3.3 and 11.3.4 in Section 12.1.

Example 11.3.5. Consider the function

$$h(x)=\frac{\cos x}{x^2-x}.$$

We can write

$$h(x)=\frac{f(x)}{g(x)},$$

where $f(x)=\cos x$ and $g(x)=x^2-x$. It follows that

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x)=-\sin x$ and $g'(x)=2x-1$. Hence

$$h'(x)=\frac{(x-x^2)\sin x-(2x-1)\cos x}{(x^2-x)^2}.$$

Example 11.3.6. Consider the function

$$h(x)=\frac{\sin x+\cos x}{1-x^4}.$$

We can write

$$h(x)=\frac{f(x)}{g(x)},$$

where $f(x)=\sin x+\cos x$ and $g(x)=1-x^4$. It follows that

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x) = \cos x - \sin x$ and $g'(x) = -4x^3$. Hence

$$h'(x) = \frac{(1-x^4)(\cos x - \sin x) + 4x^3(\sin x + \cos x)}{(1-x^4)^2}.$$

EXAMPLE 11.3.7. Consider the function

$$h(x) = \left(\frac{x^2+1}{\cos x}\right)\sin x.$$

We can write

$$h(x) = f(x)g(x),$$

where

$$f(x) = \frac{x^2+1}{\cos x} \qquad \text{and} \qquad g(x) = \sin x.$$

It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

We can also write

$$f(x) = \frac{k(x)}{t(x)},$$

where $k(x) = x^2 + 1$ and $t(x) = \cos x$. It follows that

$$f'(x) = \frac{t(x)k'(x) - k(x)t'(x)}{t^2(x)},$$

and so

$$h'(x) = \frac{t(x)k'(x) - k(x)t'(x)}{t^2(x)}g(x) + \frac{k(x)}{t(x)}g'(x).$$

Observe next that $k'(x) = 2x$, $t'(x) = -\sin x$ and $g'(x) = \cos x$. Hence

$$\begin{aligned} h'(x) &= \left(\frac{2x\cos x + (x^2+1)\sin x}{\cos^2 x}\right)\sin x + \left(\frac{x^2+1}{\cos x}\right)\cos x \\ &= 2x\tan x + (x^2+1)\tan^2 x + (x^2+1) = 2x\tan x + (x^2+1)\sec^2 x. \end{aligned}$$

Alternatively, we observe that $h(x) = (x^2+1)\tan x$. We can write

$$h(x) = u(x)v(x),$$

where $u(x) = x^2 + 1$ and $v(x) = \tan x$. It follows that

$$h'(x) = u'(x)v(x) + u(x)v'(x).$$

Observe next that $u'(x) = 2x$ and $v'(x) = \sec^2 x$. Hence $h'(x) = 2x\tan x + (x^2+1)\sec^2 x$ as before.

EXAMPLE 11.3.8. Consider the function $h(x) = \sin^2 x + \cos^2 x$. We can write

$$h(x) = f(x)g(x) + k(x)t(x),$$

where $f(x) = g(x) = \sin x$ and $k(x) = t(x) = \cos x$. It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x) + u'(x)v(x) + u(x)v'(x).$$

Observe next that $f'(x) = g'(x) = \cos x$ and $k'(x) = t'(x) = -\sin x$. Hence

$$h'(x) = \cos x \sin x + \sin x \cos x - \sin x \cos x - \cos x \sin x = 0.$$

A far simpler way to obtain the same result is to merely observe that $h(x) = 1$.

PROBLEMS FOR CHAPTER 11

1. For each of the following functions $f(x)$, write down the derivative $f'(x)$ as a function of x, and find the slope of the tangent at the point $P(1, f(1))$:
   a) $f(x) = x^4$
   b) $f(x) = 5x^2$
   c) $f(x) = \frac{1}{6}x^{-3}$
   d) $f(x) = \pi x^{1.5}$

2. Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants and sums:
   a) $h(x) = 6x^3$
   b) $h(x) = 5x^{-7}$
   c) $h(x) = 12x - 3x^2$
   d) $h(x) = x^3 + 4x$
   e) $h(x) = 6x^2 - 40x$
   f) $h(x) = x^7 + 6x^5 - 8x^2 + 3x$
   g) $h(x) = -\dfrac{3}{x}$
   h) $h(x) = \dfrac{7}{x^6}$
   i) $h(x) = \dfrac{6}{x^2}$
   j) $h(x) = x^3 + 3x - \dfrac{5}{x^3}$
   k) $h(x) = x^2 - 10x + 100 + \dfrac{4}{x}$
   l) $h(x) = x^{100} + 50x + 1 - 2x^{-3} + 7x^{-6}$
   m) $h(x) = \pi x^3 - \dfrac{\pi^2}{x^6}$
   n) $h(x) = x^2(x^3 + 3x)$
   o) $h(x) = (x^2 + 3)(2x - 5)$
   p) $h(x) = -5\sqrt{x}$
   q) $h(x) = \dfrac{\sqrt{x}}{x^3}$
   r) $h(x) = \sqrt{3x}$
   s) $h(x) = \sqrt{4x} + \sqrt{\dfrac{4}{x}}$
   t) $h(x) = x^5 + 6x^{-3/2}$

3. Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants, sums and products as appropriate:
   a) $h(x) = (x^2 + 3)(2x - 5)$
   b) $h(x) = (x^2 - x + 2)(x^2 - 2)$
   c) $h(x) = (x^2 + 5)(x^3 - 4x^2)$
   d) $h(x) = (x^4 - 3x^3 + 2x)(3x^2 + 4x)$
   e) $h(x) = (x^9 + 2x^3)x^{-4}$
   f) $h(x) = (x^4 - 2x^3 + 7x + 8)^2$
   g) $h(x) = x^{2/3}(x + 2)$
   h) $h(x) = (x + 3)(x - 5)(x^2 - 4)$
   i) $h(x) = x^{1/2}(x^3 + x - 2)(3x + 1)$
   j) $h(x) = x(x - 1)(x - 2)$

4. Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants, sums, products and quotients as appropriate:
   a) $h(x) = \dfrac{1}{x^4 + x^3 + 1}$
   b) $h(x) = 1 + \dfrac{3}{x} - \dfrac{2}{x^2}$
   c) $h(x) = \dfrac{x - 2}{x + 1}$
   d) $h(x) = \dfrac{1 + x^2}{1 - x^2}$
   e) $h(x) = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$
   f) $h(x) = \dfrac{x}{x + x^{-1}}$
   g) $h(x) = \dfrac{2x + 3}{3x + 2}$
   h) $h(x) = \dfrac{2x + 1}{x - 1}$

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ELEMENTARY MATHEMATICS

W W L CHEN and X T DUONG

Chapter 12

FURTHER TECHNIQUES OF DIFFERENTIATION

12.1. The Chain Rule

We begin by re-examining a few examples discussed in the previous chapter.

Example 12.1.1. Recall Examples 11.2.5 and 11.2.10, that for the function

$$y = h(x) = (x^2 + 2x)^2,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) = 4x^3 + 12x^2 + 8x.$$

On the other hand, we can build a chain and describe the function $y = h(x)$ by writing

$$y = g(u) = u^2 \qquad \text{and} \qquad u = f(x) = x^2 + 2x.$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 2u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 2,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u(2x + 2) = 2(x^2 + 2x)(2x + 2) = 4x^3 + 12x^2 + 8x.$$

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† This chapter was written at Macquarie University in 1999.

Example 12.1.2. Recall Example 11.3.3, that for the function

$$y = h(x) = \sin^2 x,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) = 2\sin x\cos x.$$

On the other hand, we can build a chain and describe the function $y = h(x)$ by writing

$$y = g(u) = u^2 \qquad \text{and} \qquad u = f(x) = \sin x.$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 2u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u\cos x = 2\sin x\cos x.$$

Example 12.1.3. Recall Example 11.3.4, that for the function

$$y = h(x) = \sin 2x,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) = 2\cos 2x.$$

On the other hand, we can build a chain and describe the function $y = h(x)$ by writing

$$y = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = 2x.$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos u = 2\cos 2x.$$

In these three examples, we consider functions of the form $y = h(x)$ which can be described in a chain by $y = g(u)$ and $u = f(x)$, where u is some intermediate variable. Suppose that $x_0, x_1 \in \mathbb{R}$. Write $u_0 = f(x_0)$ and $u_1 = f(x_1)$, and write $y_0 = g(u_0)$ and $y_1 = g(u_1)$. Then clearly $h(x_0) = g(f(x_0))$ and $h(x_1) = g(f(x_1))$. Heuristically, we have

$$\frac{h(x_1) - h(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{u_1 - u_0} \times \frac{u_1 - u_0}{x_1 - x_0} = \frac{g(u_1) - g(u_0)}{u_1 - u_0} \times \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

If x_1 is close to x_0, then we expect that u_1 is close to u_0, and so the product

$$\frac{g(u_1) - g(u_0)}{u_1 - u_0} \times \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

is close to $g'(u_0)f'(x_0)$, while the product

$$\frac{h(x_1) - h(x_0)}{x_1 - x_0}$$

is close to $h'(x_0)$. It is therefore not unreasonable to expect the following result, although a formal proof is somewhat more complicated.

CHAIN RULE. *Suppose that* $y = g(u)$ *and* $u = f(x)$*. Then*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x},$$

provided that the two derivatives on the right hand side exist.

We can interpret the rule in the following way. As we vary x, the value $u = f(x)$ changes at the rate of $\mathrm{d}u/\mathrm{d}x$. This change in the value of $u = f(x)$ in turn causes a change in the value of $y = g(u)$ at the rate of $\mathrm{d}y/\mathrm{d}u$.

EXAMPLE 12.1.4. Consider the function $y = h(x) = (x^2 - 6x + 5)^3$. We can set up a chain by writing

$$y = g(u) = u^3 \qquad \text{and} \qquad u = f(x) = x^2 - 6x + 5.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 3u^2 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 6,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 3u^2(2x - 6) = 6(x^2 - 6x + 5)^2(x - 3).$$

EXAMPLE 12.1.5. Consider the function $y = h(x) = \sin^4 x$. We can set up a chain by writing

$$y = g(u) = u^4 \qquad \text{and} \qquad u = f(x) = \sin x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 4u^3 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 4u^3 \cos x = 4 \sin^3 x \cos x.$$

EXAMPLE 12.1.6. Consider the function $y = h(x) = \sec(x^4)$. We can set up a chain by writing

$$y = g(u) = \sec u \qquad \text{and} \qquad u = f(x) = x^4.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \tan u \sec u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3 \tan u \sec u = 4x^3 \tan(x^4) \sec(x^4).$$

EXAMPLE 12.1.7. Consider the function $y = h(x) = \tan(x^2 - 3x + 4)$. We can set up a chain by writing

$$y = g(u) = \tan u \qquad \text{and} \qquad u = f(x) = x^2 - 3x + 4.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \sec^2 u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 3,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=(2x-3)\sec^2 u=(2x-3)\sec^2(x^2-3x+4).$$

EXAMPLE 12.1.8. Consider the function $y=h(x)=(x^2+5x-1)^{2/3}$. We can set up a chain by writing

$$y=g(u)=u^{2/3}\qquad\text{and}\qquad u=f(x)=x^2+5x-1.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{2}{3}u^{-1/3}\qquad\text{and}\qquad\frac{\mathrm{d}u}{\mathrm{d}x}=2x+5,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2}{3}u^{-1/3}(2x+5)=\frac{2}{3}(x^2+5x-1)^{-1/3}(2x+5).$$

EXAMPLE 12.1.9. Consider the function

$$y=h(x)=\frac{1}{\cos^3 x}.$$

We can set up a chain by writing

$$y=g(u)=\frac{1}{u^3}=u^{-3}\qquad\text{and}\qquad u=f(x)=\cos x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u}=-3u^{-4}\qquad\text{and}\qquad\frac{\mathrm{d}u}{\mathrm{d}x}=-\sin x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=3u^{-4}\sin x=\frac{3\sin x}{u^4}=\frac{3\sin x}{\cos^4 x}.$$

EXAMPLE 12.1.10. Consider the function

$$y=h(x)=\frac{1}{(2x^3-5x+1)^4}.$$

We can set up a chain by writing

$$y=g(u)=\frac{1}{u^4}=u^{-4}\qquad\text{and}\qquad u=f(x)=2x^3-5x+1.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u}=-4u^{-5}\qquad\text{and}\qquad\frac{\mathrm{d}u}{\mathrm{d}x}=6x^2-5,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=4u^{-5}(5-6x^2)=\frac{4(5-6x^2)}{u^5}=\frac{4(5-6x^2)}{(2x^3-5x+1)^5}.$$

EXAMPLE 12.1.11. Consider the function $y = h(x) = \sin(\cos x)$. We can set up a chain by writing

$$y = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = \cos x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -\cos u \sin x = -\cos(\cos x)\sin x.$$

EXAMPLE 12.1.12. Consider the function

$$y = h(x) = \frac{1}{2(x+1)} + \frac{1}{4(x+1)^2}.$$

We can set up a chain by writing

$$y = g(u) = \frac{1}{2u} + \frac{1}{4u^2} = \frac{1}{2}u^{-1} + \frac{1}{4}u^{-2} \qquad \text{and} \qquad u = f(x) = x+1.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = -\frac{1}{2}u^{-2} - \frac{1}{2}u^{-3} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{2}u^{-2} - \frac{1}{2}u^{-3} = -\frac{1}{2(x+1)^2} - \frac{1}{2(x+1)^3}.$$

EXAMPLE 12.1.13. Consider the function

$$y = h(x) = \left(\frac{x-1}{x+1}\right)^3.$$

We can set up a chain by writing

$$y = g(u) = u^3 \qquad \text{and} \qquad u = f(x) = \frac{x-1}{x+1}.$$

Clearly we have (using the quotient rule for the latter)

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 3u^2 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{(x+1)^2},$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 3u^2 \times \frac{2}{(x+1)^2} = 3\left(\frac{x-1}{x+1}\right)^2 \times \frac{2}{(x+1)^2} = \frac{6(x-1)^2}{(x+1)^4}.$$

The chain rule can be extended to chains of more than two functions. We illustrate the ideas by considering the next four examples.

EXAMPLE 12.1.14. Consider the function $y = h(x) = \sin^3(x^2+2)$. We can set up a chain by writing

$$y = k(v) = v^3, \qquad v = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = x^2 + 2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = 3v^2, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 6xv^2 \cos u = 6x \sin^2 u \cos u = 6x \sin^2(x^2+2)\cos(x^2+2).$$

Example 12.1.15. Consider the function $y = h(x) = (1 + (1 + x)^{1/2})^5$. We can set up a chain by writing

$$y = k(v) = v^5, \qquad v = g(u) = 1 + u^{1/2} \qquad \text{and} \qquad u = f(x) = 1 + x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = 5v^4, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1}{2}u^{-1/2} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{5}{2}v^4u^{-1/2} = \frac{5}{2}(1 + u^{1/2})^4u^{-1/2} = \frac{5(1 + (1 + x)^{1/2})^4}{2(1 + x)^{1/2}}.$$

Example 12.1.16. Consider the function $y = h(x) = \tan((x^4 - 3x)^3)$. We can set up a chain by writing

$$y = k(v) = \tan v, \qquad v = g(u) = u^3 \qquad \text{and} \qquad u = f(x) = x^4 - 3x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \sec^2 v, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = 3u^2 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3 - 3,$$

so it follows from the chain rule that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} &= 3u^2(4x^3 - 3)\sec^2 v = 3u^2(4x^3 - 3)\sec^2(u^3) \\ &= 3(x^4 - 3x)^2(4x^3 - 3)\sec^2((x^4 - 3x)^3).\end{aligned}$$

Example 12.1.17. Consider the function $y = h(x) = \sqrt{x^2 + \sin(x^2)}$. We can set up a chain by writing

$$y = k(v) = v^{1/2}, \qquad v = g(u) = u + \sin u \qquad \text{and} \qquad u = f(x) = x^2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \frac{1}{2v^{1/2}}, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = 1 + \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x(1 + \cos u)}{v^{1/2}} = \frac{x(1 + \cos u)}{(u + \sin u)^{1/2}} = \frac{x(1 + \cos(x^2))}{\sqrt{x^2 + \sin(x^2)}}.$$

We conclude this section by studying three examples where the chain rule is used only in part of the argument. These examples are rather hard, and the reader is advised to concentrate on the ideas and not to get overly worried about the arithmetic details. For accuracy, it is absolutely crucial that we exercise great care.

Example 12.1.18. Consider the function $y = h(x) = (x^2 - 1)^{1/2}(x^2 + 4x + 3)$. We can write

$$h(x) = f(x)g(x),$$

where

$$f(x) = (x^2 - 1)^{1/2} \qquad \text{and} \qquad g(x) = x^2 + 4x + 3.$$

It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

It is easy to see that $g'(x) = 2x + 4$. To find $f'(x)$, we shall use the chain rule. Let

$$z = f(x) = (x^2 - 1)^{1/2}.$$

We can set up a chain by writing

$$z = u^{1/2} \qquad \text{and} \qquad u = x^2 - 1.$$

Clearly we have

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{1}{2u^{1/2}} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

so it follows from the chain rule that

$$f'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x}{u^{1/2}} = \frac{x}{(x^2 - 1)^{1/2}}.$$

Hence

$$h'(x) = \frac{x(x^2 + 4x + 3)}{(x^2 - 1)^{1/2}} + (x^2 - 1)^{1/2}(2x + 4).$$

EXAMPLE 12.1.19. Consider the function

$$y = h(x) = \frac{(1 - x^3)^2}{(1 + 2x + 3x^2)^2}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where

$$f(x) = (1 - x^3)^2 \qquad \text{and} \qquad g(x) = (1 + 2x + 3x^2)^2.$$

It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

To find $f'(x)$ and $g'(x)$, we shall use the chain rule. Let

$$z = f(x) = (1 - x^3)^2 \qquad \text{and} \qquad w = g(x) = (1 + 2x + 3x^2)^2.$$

We can set up a chain by writing

$$z = u^2 \qquad \text{and} \qquad u = 1 - x^3.$$

Then

$$f'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u \times (-3x^2) = -6ux^2 = -6(1 - x^3)x^2.$$

Similarly, we can set up a chain by writing

$$w = v^2 \qquad \text{and} \qquad v = 1 + 2x + 3x^2.$$

Then

$$g'(x) = \frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}x} = 2v \times (2 + 6x) = 4v(1 + 3x) = 4(1 + 2x + 3x^2)(1 + 3x).$$

Hence

$$h'(x) = \frac{-6(1+2x+3x^2)^2(1-x^3)x^2 - 4(1-x^3)^2(1+2x+3x^2)(1+3x)}{(1+2x+3x^2)^4}$$
$$= -\frac{6(1+2x+3x^2)(1-x^3)x^2 + 4(1-x^3)^2(1+3x)}{(1+2x+3x^2)^3}.$$

Alternatively, observe that we can set up a chain by writing

$$y = s^2 \qquad \text{and} \qquad s = \frac{1-x^3}{1+2x+3x^2}.$$

Clearly we have (using the quotient rule for the latter)

$$\frac{\mathrm{d}y}{\mathrm{d}s} = 2s \qquad \text{and} \qquad \frac{\mathrm{d}s}{\mathrm{d}x} = \frac{-3(1+2x+3x^2)x^2 - (1-x^3)(2+6x)}{(1+2x+3x^2)^2},$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}x} = -2s \times \frac{3(1+2x+3x^2)x^2 + (1-x^3)(2+6x)}{(1+2x+3x^2)^2}$$
$$= -\frac{2(1-x^3)}{1+2x+3x^2} \times \frac{3(1+2x+3x^2)x^2 + 2(1-x^3)(1+3x)}{(1+2x+3x^2)^2}.$$

It can be easily checked that the answer is the same as before.

Example 12.1.20. Consider the function $y = h(x) = (x^2 + (x^3 + x^5)^7)^{11}$. We can set up a chain by writing

$$y = g(u) = u^{11} \qquad \text{and} \qquad u = f(x) = x^2 + (x^3 + x^5)^7.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 11u^{10}.$$

On the other hand, we have $f(x) = k(x) + t(x)$, where $k(x) = x^2$ and $t(x) = (x^3 + x^5)^7$. It follows that $f'(x) = k'(x) + t'(x)$. Note that $k'(x) = 2x$. To find $t'(x)$, we shall use the chain rule. Let

$$z = t(x) = (x^3 + x^5)^7.$$

We can set up a chain by writing

$$z = v^7 \qquad \text{and} \qquad v = x^3 + x^5.$$

Clearly we have

$$\frac{\mathrm{d}z}{\mathrm{d}v} = 7v^6 \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 3x^2 + 5x^4,$$

so it follows from the chain rule that

$$t'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}x} = 7v^6(3x^2 + 5x^4) = 7(x^3 + x^5)^6(3x^2 + 5x^4),$$

and so

$$f'(x) = \frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 7(x^3 + x^5)^6(3x^2 + 5x^4).$$

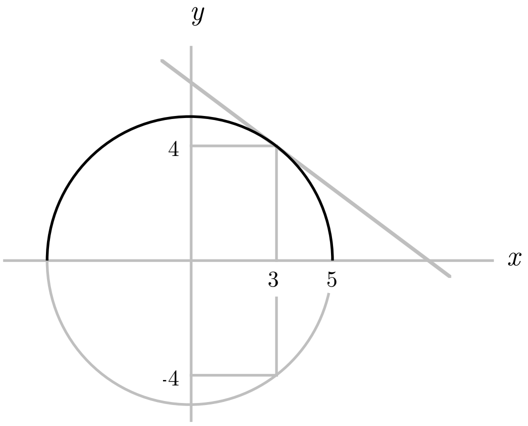
It then follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 11u^{10}(2x + 7(x^3 + x^5)^6(3x^2 + 5x^4))$$
$$= 11(x^2 + (x^3 + x^5)^7)^{10}(2x + 7(x^3 + x^5)^6(3x^2 + 5x^4)).$$

12.2. Implicit Differentiation

A function $y = f(x)$ can usually be viewed as a curve on the xy-plane, and gives a relationship between the (independent) variable x and the (dependent) variable y by describing y explicitly in terms of x. However, a relationship between two variables x and y cannot always be expressed as a function $y = f(x)$. Moreover, we may even choose to describe a function $y = f(x)$ implicitly by simply giving some relationship between the variables x and y, and not describing y explicitly in terms of x.

EXAMPLE 12.2.1. Consider the equation $x^2 + y^2 = 25$, representing a circle of radius 5 and centred at the origin $(0, 0)$. This equation expresses a relationship between the two variables x and y, but y is not given explicitly in terms of x. Indeed, it is not possible to give y explicitly in terms of x, as this equation does not represent a function $y = f(x)$. To see this, note that if $x = 3$, then both $y = 4$ and $y = -4$ will satisfy the equation, so it is meaningless to talk of $f(3)$. On the other hand, we see that the point $(3, 4)$ is on the circle, and clearly there is a tangent line to the circle at the point $(3, 4)$, as shown in the picture below.



If we restrict our attention to the upper semicircle, then we can express the variable y explicitly as a function of the variable x by writing

$$y = (25 - x^2)^{1/2}.$$

We can set up a chain by writing

$$y = u^{1/2} \qquad \text{and} \qquad u = 25 - x^2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{2}u^{-1/2} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -2x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -xu^{-1/2} = -x(25 - x^2)^{-1/2}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(3,4)} = -\frac{3}{4}.$$

Note that our argument here involves obtaining an explicit expression for the variable y in terms of the variable x from similar information given implicitly by the equation $x^2 + y^2 = 25$. Now let us see whether we can obtain a similar conclusion concerning the slope of the tangent line at the point $(3, 4)$ without first having to obtain the explicit expression $y = (25 - x^2)^{1/2}$ of the upper semicircle. Let us start from the equation $x^2 + y^2 = 25$ of the circle. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(25).$$

Using the rule on the derivatives of constants, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(25) = 0.$$

Using the sum rule and the rule on the derivatives of powers, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = 2x + \frac{\mathrm{d}}{\mathrm{d}x}(y^2).$$

We next set up a chain by writing

$$z = y^2 \qquad \text{and} \qquad y = f(x),$$

where there is no need to know precisely what $f(x)$ is. Then using the chain rule and the rule on the derivatives of powers, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(3,4)} = -\frac{3}{4}$$

as before.

The second part of the example above is a case of using implicit differentiation, where we find the derivative of a function $y = f(x)$ without knowing any explicit expression for the variable y in terms of the variable x. We shall describe this technique further by discussing a few more examples. In some of these examples, it may be very difficult, if not impossible, to find any explicit expression for the variable y in terms of the variable x.

EXAMPLE 12.2.2. Suppose that $y^2 - x^2 = 4$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2 - x^2) = \frac{\mathrm{d}}{\mathrm{d}x}(4) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2 - x^2) = \frac{\mathrm{d}}{\mathrm{d}x}(y^2) - \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = \frac{\mathrm{d}}{\mathrm{d}x}(y^2) - 2x = 0.$$

We next set up a chain by writing

$$z = y^2 \qquad \text{and} \qquad y = f(x),$$

where there is no need to know precisely what $f(x)$ is. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}.$$

Example 12.2.3. Suppose that $y^3+\sin x=3$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3+\sin x)=\frac{\mathrm{d}}{\mathrm{d}x}(3)=0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3+\sin x)=\frac{\mathrm{d}}{\mathrm{d}x}(y^3)+\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\frac{\mathrm{d}}{\mathrm{d}x}(y^3)+\cos x=0.$$

We next set up a chain by writing

$$z=y^3 \qquad \text{and} \qquad y=f(x),$$

where there is no need to know precisely what $f(x)$ is. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3)=\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\mathrm{d}z}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}x}=3y^2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x}+\cos x=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\cos x}{3y^2}.$$

Example 12.2.4. Suppose that $y^5+3y^2-2x^2+4=0$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^5+3y^2-2x^2+4)=\frac{\mathrm{d}}{\mathrm{d}x}(0)=0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^5+3y^2-2x^2+4)=\frac{\mathrm{d}}{\mathrm{d}x}(y^5)+3\frac{\mathrm{d}}{\mathrm{d}x}(y^2)-2\frac{\mathrm{d}}{\mathrm{d}x}(x^2)+\frac{\mathrm{d}}{\mathrm{d}x}(4)=\frac{\mathrm{d}}{\mathrm{d}x}(y^5)+3\frac{\mathrm{d}}{\mathrm{d}x}(y^2)-4x=0.$$

Using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^5)=\frac{\mathrm{d}}{\mathrm{d}y}(y^5)\times\frac{\mathrm{d}y}{\mathrm{d}x}=5y^4\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(y^2)=\frac{\mathrm{d}}{\mathrm{d}y}(y^2)\times\frac{\mathrm{d}y}{\mathrm{d}x}=2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$(5y^4+6y)\frac{\mathrm{d}y}{\mathrm{d}x}-4x=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4x}{5y^4+6y}.$$

Example 12.2.5. Suppose that $xy=6$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy)=\frac{\mathrm{d}}{\mathrm{d}x}(6)=0.$$

It follows from the product rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy)=\frac{\mathrm{d}}{\mathrm{d}x}(x)\times y+x\times\frac{\mathrm{d}}{\mathrm{d}x}(y)=y+x\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y}{x}.$$

EXAMPLE 12.2.6. Suppose that $x^3+2x^2y^3+3y^4=6$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x^2y^3+3y^4)=\frac{\mathrm{d}}{\mathrm{d}x}(6)=0.$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x^2y^3+3y^4)&=\frac{\mathrm{d}}{\mathrm{d}x}(x^3)+2\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\times y^3+x^2\times\frac{\mathrm{d}}{\mathrm{d}x}(y^3)\right)+3\frac{\mathrm{d}}{\mathrm{d}x}(y^4)\\&=3x^2+4xy^3+2x^2\frac{\mathrm{d}}{\mathrm{d}x}(y^3)+3\frac{\mathrm{d}}{\mathrm{d}x}(y^4)=0.\end{aligned}$$

Using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3)=\frac{\mathrm{d}}{\mathrm{d}y}(y^3)\times\frac{\mathrm{d}y}{\mathrm{d}x}=3y^2\frac{\mathrm{d}y}{\mathrm{d}x}\qquad\text{and}\qquad\frac{\mathrm{d}}{\mathrm{d}x}(y^4)=\frac{\mathrm{d}}{\mathrm{d}y}(y^4)\times\frac{\mathrm{d}y}{\mathrm{d}x}=4y^3\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$3x^2+4xy^3+(6x^2y^2+12y^3)\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3x^2+4xy^3}{6x^2y^2+12y^3}.$$

Note next that the point $(1,1)$ satisfies the equation. It follows that

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)}=-\frac{7}{18}.$$

Check that the equation of the tangent line at this point is given by $7x+18y=25$.

EXAMPLE 12.2.7. Suppose that $(x^2+y^3)^2=9$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}((x^2+y^3)^2)=\frac{\mathrm{d}}{\mathrm{d}x}(9)=0.$$

Let $w=(x^2+y^3)^2$. We can set up a chain by writing

$$w=z^2\qquad\text{and}\qquad z=x^2+y^3,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}((x^2+y^3)^2)=\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{\mathrm{d}w}{\mathrm{d}z}\times\frac{\mathrm{d}z}{\mathrm{d}x}=2z\frac{\mathrm{d}z}{\mathrm{d}x}=2(x^2+y^3)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^3).$$

Hence

$$(x^2+y^3)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^3)=0.$$

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^3)=\frac{\mathrm{d}}{\mathrm{d}x}(x^2)+\frac{\mathrm{d}}{\mathrm{d}x}(y^3)=2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x},$$

where we have used the chain rule at the last step. Summarizing, we obtain

$$(x^2+y^3)\left(2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0.$$

It is clear that $x^2+y^3\neq0$ for any point (x,y) satisfying the equation. It follows that

$$2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{3y^2}.$$

Note next that the point $(2,-1)$ satisfies the equation. It follows that

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(2,-1)} = -\frac{4}{3}.$$

Check that the equation of the tangent line at this point is given by $4x+3y=5$.

EXAMPLE 12.2.8. The point $(1,1)$ is one of the intersection points of the parabola $y-x^2=0$ and the ellipse $x^2+2y^2=3$. We shall show that the two tangents at $(1,1)$ are perpendicular to each other. Consider first of all the parabola $y-x^2=0$. Here we can write $y=x^2$, so that $\mathrm{d}y/\mathrm{d}x=2x$. Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)} = 2.$$

Consider next the ellipse $x^2+2y^2=3$. Using implicit differentiation, it is not difficult to show that

$$2x+4y\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{2y}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)} = -\frac{1}{2}.$$

Since the product of the two derivatives is equal to -1, it follows that the two tangents are perpendicular to each other.

12.3. Derivatives of the Exponential and Logarithmic Functions

We shall state without proof the following result.

DERIVATIVE OF THE EXPONENTIAL FUNCTION. *If* $f(x)=\mathrm{e}^x$, *then* $f'(x)=\mathrm{e}^x$.

EXAMPLE 12.3.1. Consider the function $y=h(x)=\mathrm{e}^x(\sin x+2\cos x)$. We can write

$$h(x)=f(x)g(x),$$

where $f(x)=\mathrm{e}^x$ and $g(x)=\sin x+2\cos x$. It follows from the product rule that

$$h'(x)=f'(x)g(x)+f(x)g'(x).$$

Clearly

$$f'(x)=\mathrm{e}^x \qquad \text{and} \qquad g'(x)=\cos x-2\sin x.$$

Hence

$$h'(x)=\mathrm{e}^x(\sin x+2\cos x)+\mathrm{e}^x(\cos x-2\sin x)=\mathrm{e}^x(3\cos x-\sin x).$$

EXAMPLE 12.3.2. Consider the function $y = h(x) = \mathrm{e}^x(x^2 + x + 2)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = \mathrm{e}^x$ and $g(x) = x^2 + x + 2$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Clearly

$$f'(x) = \mathrm{e}^x \qquad \text{and} \qquad g'(x) = 2x + 1.$$

Hence

$$h'(x) = \mathrm{e}^x(x^2 + x + 2) + \mathrm{e}^x(2x + 1) = \mathrm{e}^x(x^2 + 3x + 3).$$

EXAMPLE 12.3.3. Consider the function $y = h(x) = \mathrm{e}^{2x}$. We can set up a chain by writing

$$y = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = 2x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2\mathrm{e}^u = 2\mathrm{e}^{2x}.$$

Alternatively, we can set up a chain by writing

$$y = t(v) = v^2 \qquad \text{and} \qquad v = k(x) = \mathrm{e}^x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = 2v \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}x} = 2v\mathrm{e}^x = 2\mathrm{e}^x\mathrm{e}^x = 2\mathrm{e}^{2x}.$$

Yet another alternative is to observe that $h(x) = \mathrm{e}^x\mathrm{e}^x$. It follows that we can use the product rule instead of the chain rule. Try it!

EXAMPLE 12.3.4. Consider the function $y = h(x) = \mathrm{e}^{x^3}$. We can set up a chain by writing

$$y = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = x^3.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2\mathrm{e}^u = 3x^2\mathrm{e}^{x^3}.$$

Example 12.3.5. Consider the function $y = h(x) = \mathrm{e}^{\sin x + 4\cos x}$. We can set up a chain by writing

$$y = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = \sin x + 4\cos x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x - 4\sin x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^u(\cos x - 4\sin x) = \mathrm{e}^{\sin x + 4\cos x}(\cos x - 4\sin x).$$

Example 12.3.6. Consider the function

$$y = h(x) = \sin^3\left(\mathrm{e}^{4x^2}\right).$$

We can set up a chain by writing

$$y = t(w) = w^3, \qquad w = k(v) = \sin v, \qquad v = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = 4x^2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}w} = 3w^2, \qquad \frac{\mathrm{d}w}{\mathrm{d}v} = \cos v, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 8x,$$

so it follows from the chain rule that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}w} \times \frac{\mathrm{d}w}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} &= 24x\mathrm{e}^u w^2 \cos v = 24x\mathrm{e}^u \sin^2 v \cos v = 24x\mathrm{e}^u \sin^2(\mathrm{e}^u)\cos(\mathrm{e}^u) \\ &= 24x\mathrm{e}^{4x^2}\sin^2(\mathrm{e}^{4x^2})\cos(\mathrm{e}^{4x^2}).\end{aligned}$$

Example 12.3.7. Suppose that $\mathrm{e}^{2x} + y^2 = 5$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x} + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(5) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x} + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = 2\mathrm{e}^{2x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

using Example 12.3.3 and the chain rule. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\mathrm{e}^{2x}}{y}.$$

The next example is rather complicated, and the reader is advised to concentrate on the ideas and not to get overly worried about the arithmetic details. For accuracy, it is absolutely crucial that we exercise great care.

Example 12.3.8. Suppose that $\mathrm{e}^{2x}\sin 3y + x^2y^3 = 3$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}\sin 3y + x^2y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(3) = 0.$$

Using the sum and product rules, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}\sin 3y + x^2y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) \times \sin 3y + \mathrm{e}^{2x} \times \frac{\mathrm{d}}{\mathrm{d}x}(\sin 3y) + \frac{\mathrm{d}}{\mathrm{d}x}(x^2) \times y^3 + x^2 \times \frac{\mathrm{d}}{\mathrm{d}x}(y^3).$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) = 2\mathrm{e}^{2x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x.$$

Writing $z = 3y$ and using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin 3y) = \frac{\mathrm{d}}{\mathrm{d}y}(\sin 3y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}y}(\sin z) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}z}(\sin z) \times \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos z \frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos 3y \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Using the chain rule, we also obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3) = \frac{\mathrm{d}}{\mathrm{d}y}(y^3) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$2\mathrm{e}^{2x}\sin 3y + 3\mathrm{e}^{2x}\cos 3y \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^3 + 3x^2y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 2(\mathrm{e}^{2x}\sin 3y + xy^3) + 3(\mathrm{e}^{2x}\cos 3y + x^2y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2(\mathrm{e}^{2x}\sin 3y + xy^3)}{3(\mathrm{e}^{2x}\cos 3y + x^2y^2)}.$$

Next, we turn to the logarithmic function. Using implicit differentiation, we can establish the following result.

DERIVATIVE OF THE LOGARITHMIC FUNCTION. *If $f(x) = \log x$, then $f'(x) = 1/x$.*

Proof. Suppose that $y = \log x$. Then $\mathrm{e}^y = x$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^y) = \frac{\mathrm{d}}{\mathrm{d}x}(x) = 1.$$

Using the chain rule and the rule on the derivative of the exponential function, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^y) = \frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^y \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$\mathrm{e}^y \frac{\mathrm{d}y}{\mathrm{d}x} = 1,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{e}^y} = \frac{1}{x}.$$ ♣

Example 12.3.9. Consider the function $y = h(x) = x\log x$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x$ and $g(x) = \log x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Clearly $f'(x) = 1$ and $g'(x) = 1/x$. Hence $h'(x) = \log x + 1$.

Example 12.3.10. Consider the function

$$y = h(x) = \frac{x \log x + \sin x}{\mathrm{e}^x}.$$

We can write

$$h(x) = \frac{f(x) + k(x)}{g(x)},$$

where $f(x) = x \log x$, $k(x) = \sin x$ and $g(x) = \mathrm{e}^x$. It follows from the sum and quotient rules that

$$h'(x) = \frac{g(x)(f'(x) + k'(x)) - (f(x) + k(x))g'(x)}{g^2(x)}.$$

Clearly $k'(x) = \cos x$ and $g'(x) = \mathrm{e}^x$. Observe also from Example 12.3.9 that $f'(x) = \log x + 1$. Hence

$$h'(x) = \frac{\mathrm{e}^x(\log x + 1 + \cos x) - (x \log x + \sin x)\mathrm{e}^x}{\mathrm{e}^{2x}} = \frac{(1 - x)\log x + 1 + \cos x - \sin x}{\mathrm{e}^x}.$$

Example 12.3.11. Consider the function $y = h(x) = \log(5x^2 + 3)$. We can set up a chain by writing

$$y = g(u) = \log u \qquad \text{and} \qquad u = f(x) = 5x^2 + 3.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 10x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{10x}{u} = \frac{10x}{5x^2 + 3}.$$

Example 12.3.12. Consider the function $y = h(x) = \log(\tan x + \sec x)$. We can set up a chain by writing

$$y = g(u) = \log u \qquad \text{and} \qquad u = f(x) = \tan x + \sec x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x + \tan x \sec x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\sec^2 x + \tan x \sec x}{u} = \frac{\sec^2 x + \tan x \sec x}{\tan x + \sec x} = \sec x.$$

Example 12.3.13. Consider the function $y = h(x) = \log(\cot x + \csc x)$. We can set up a chain by writing

$$y = g(u) = \log u \qquad \text{and} \qquad u = f(x) = \cot x + \csc x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\csc^2 x - \cot x \csc x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-\csc^2 x - \cot x \csc x}{u} = -\frac{\csc^2 x + \cot x \csc x}{\cot x + \csc x} = -\csc x.$$

EXAMPLE 12.3.14. Consider the function $y = h(x) = \log(\sin(x^{1/2}))$. We can set up a chain by writing

$$y = k(v) = \log v, \qquad v = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = x^{1/2}.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \frac{1}{v}, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2x^{1/2}},$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\cos u}{2x^{1/2}v} = \frac{\cos u}{2x^{1/2}\sin u} = \frac{\cos(x^{1/2})}{2x^{1/2}\sin(x^{1/2})}.$$

EXAMPLE 12.3.15. Suppose that $x \log y + y^2 = 4$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \log y + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(4) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \log y + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \times \log y + x \times \frac{\mathrm{d}}{\mathrm{d}x}(\log y) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \log y + x\frac{\mathrm{d}}{\mathrm{d}x}(\log y) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2).$$

By the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log y) = \frac{\mathrm{d}}{\mathrm{d}y}(\log y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}}{\mathrm{d}y}(y^2) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$\log y + \left(\frac{x}{y} + 2y\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y \log y}{x + 2y^2}.$$

EXAMPLE 12.3.16. Suppose that $\log(xy^2) = 2x^2$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log(xy^2)) = \frac{\mathrm{d}}{\mathrm{d}x}(2x^2) = 4x.$$

Let $z = \log(xy^2)$. We can set up a chain by writing

$$z = \log u \qquad \text{and} \qquad u = xy^2.$$

Then it follows from the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log(xy^2)) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{xy^2} \times \frac{\mathrm{d}}{\mathrm{d}x}(xy^2).$$

Next, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \times y^2 + x \times \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = y^2 + x \times \frac{\mathrm{d}}{\mathrm{d}y}(y^2) \times \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^2y^2,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(4x^2 - 1)y^2}{2xy}.$$

12.4. Derivatives of the Inverse Trigonometric Functions

The purpose of this last section is to determine the derivatives of the inverse trigonometric functions by using implicit differentiation and our knowledge on the derivatives of the trigonometric functions.

For notational purposes, we shall write

$$y = \sin^{-1} x \qquad \text{if and only if} \qquad x = \sin y,$$

and similarly for the other trigonometric functions. These inverse trigonometric functions are well defined, provided that we restrict the values for x to suitable intervals of real numbers.

DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS.

(a) If $y = \sin^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{1-x^2}}$.

(b) If $y = \cos^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{\sqrt{1-x^2}}$.

(c) If $y = \tan^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{1+x^2}$.

(d) If $y = \cot^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{1+x^2}$.

(e) If $y = \sec^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x\sqrt{x^2-1}}$.

(f) If $y = \csc^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{x\sqrt{x^2-1}}$.

SKETCH OF PROOF. For simplicity, we shall assume that $0 < y < \pi/2$, so that y is in the first quadrant, and so all the trigonometric functions have positive values.

(a) If $y = \sin^{-1} x$, then $x = \sin y$. Differentiating with respect to x, we obtain

$$1 = \cos y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

(b) If $y = \cos^{-1} x$, then $x = \cos y$. Differentiating with respect to x, we obtain

$$1 = -\sin y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}.$$

(c) If $y = \tan^{-1} x$, then $x = \tan y$. Differentiating with respect to x, we obtain

$$1 = \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

(d) If $y = \cot^{-1} x$, then $x = \cot y$. Differentiating with respect to x, we obtain

$$1 = -\csc^2 y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}.$$

(e) If $y=\sec^{-1}x$, then $x=\sec y$. Differentiating with respect to x, we obtain

$$1=\tan y\sec y\frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\tan y\sec y}=\frac{1}{(\sec^2 y-1)^{1/2}\sec y}=\frac{1}{x\sqrt{x^2-1}}.$$

(f) If $y=\csc^{-1}x$, then $x=\csc y$. Differentiating with respect to x, we obtain

$$1=-\cot y\csc y\frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\cot y\csc y}=-\frac{1}{(\csc^2 y-1)^{1/2}\csc y}=-\frac{1}{x\sqrt{x^2-1}}.$$ ♣

There is no need to remember the derivatives of any of these inverse trigonometric functions.

PROBLEMS FOR CHAPTER 12

1. By making suitable use of the chain rule and other rules as appropriate, find the derivative of each of the following functions:
   - a) $h(x)=\sqrt{1-\cos x}$
   - b) $h(x)=\sin(3x)$
   - c) $h(x)=\cos(\sin x)$
   - d) $h(x)=x^2\cos x$
   - e) $h(x)=\sin(2x)\sin(3x)$
   - f) $h(x)=2x\sin(3x)$
   - g) $h(x)=\tan(3x)$
   - h) $h(x)=4\sec(5x)$
   - i) $h(x)=\cos(x^3)$
   - j) $h(x)=\cos^3 x$
   - k) $h(x)=(1+\cos^2 x)^6$
   - l) $h(x)=\tan(x^2)+\tan^2 x$
   - m) $h(x)=\cos(\tan x)$
   - n) $h(x)=\sin(\sin x)$

2. Find the derivative of each of the following functions:
   - a) $h(x)=\sin(\mathrm{e}^x)$
   - b) $h(x)=\mathrm{e}^{\sin x}$
   - c) $h(x)=\mathrm{e}^{-2x}\sin x$
   - d) $h(x)=\mathrm{e}^x\sin(2x)$
   - e) $h(x)=\tan(\mathrm{e}^{-3x})$
   - f) $h(x)=\tan(\mathrm{e}^x)$

3. a) For each $k=0,1,2,3,\ldots$, find a function $f_k(x)$ such that $f_k'(x)=x^k$.
   b) For each $k=-2,-3,-4,\ldots$, find a function $f_k(x)$ such that $f_k'(x)=x^k$.
   c) Find a function $f_{-1}(x)$ such that $f_{-1}'(x)=x^{-1}$.

4. Find the derivative of each of the following functions:
   - a) $h(x)=(3x^2+\pi)(\mathrm{e}^x-4)$
   - b) $h(x)=x^5+3x^2+\dfrac{2}{x^4}+1$
   - c) $h(x)=2x-\dfrac{1}{\sqrt[3]{x}}+\mathrm{e}^{2x}$
   - d) $h(x)=2\mathrm{e}^x+x\mathrm{e}^{3x}$
   - e) $h(x)=\mathrm{e}^{\tan x}$
   - f) $h(x)=2x\mathrm{e}^x-x^{-2}$
   - g) $h(x)=\log(\log(2x^3))$
   - h) $h(x)=\sqrt{x+5}$
   - i) $h(x)=\dfrac{x+2}{x^2+1}$
   - j) $h(x)=\sin(2x+3)$
   - k) $h(x)=\cos^2(2x)$
   - l) $h(x)=\log(\mathrm{e}^{-x}-1)$
   - m) $h(x)=\mathrm{e}^{\mathrm{e}^x+\mathrm{e}^{-x}}$
   - n) $h(x)=\dfrac{x^2+1}{\sqrt{x}}$
   - o) $h(x)=(x+3)^2$
   - p) $h(x)=\log(2x+3)$
   - q) $h(x)=\tan(3x+2x^2)$
   - r) $h(x)=\cos(\mathrm{e}^{2x})$

5. Use implicit differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following relations:
   - a) $x^2+xy-y^3=xy^2$
   - b) $x^2+y^2=\sqrt{7}$
   - c) $\sqrt{x}+\sqrt{y}=25$
   - d) $\sin(xy)=2x+5$
   - e) $x\log y+y^3=\log x$
   - f) $y^3-xy=-6$
   - g) $x^2-xy+y^4=x^2y$
   - h) $\sin(xy)=3x^2-2$

6. For each of the following, verify first that the given point satisfies the relation defining the curve, then find the equation of the tangent line to the curve at the point:

a) $xy^2 = 1$ at $(1, 1)$	b) $y^2 = \dfrac{x^2}{xy - 4}$ at $(4, 2)$
c) $y + \sin y + x^2 = 9$ at $(3, 0)$	d) $x^{2/3} + y^{2/3} = a^{2/3}$ at $(a, 0)$

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ELEMENTARY MATHEMATICS

W W L CHEN and X T DUONG

Chapter 13

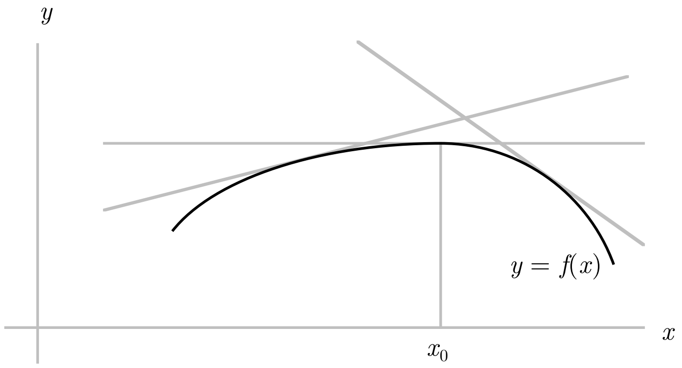
APPLICATIONS OF DIFFERENTIATION

13.1. Second Derivatives

Recall that for a function $y = f(x)$, the derivative $f'(x)$ represents the slope of the tangent. It is easy to see from a picture that if the derivative $f'(x) > 0$, then the function $f(x)$ is increasing; in other words, $f(x)$ increases in value as x increases. On the other hand, if the derivative $f'(x) < 0$, then the function $f(x)$ is decreasing; in other words, $f(x)$ decreases in value as x increases. We are interested in the case when the derivative $f'(x) = 0$. Values $x = x_0$ such that $f'(x_0) = 0$ are called stationary points.

Let us introduce the second derivative $f''(x)$ of the function $f(x)$. This is defined to be the derivative of the derivative $f'(x)$. With the same reasoning as before but applied to the function $f'(x)$ instead of the function $f(x)$, we conclude that if the second derivative $f''(x) > 0$, then the derivative $f'(x)$ is increasing. Similarly, if the second derivative $f''(x) < 0$, then the derivative $f'(x)$ is decreasing.

Suppose that $f'(x_0) = 0$ and $f''(x_0) < 0$. The condition $f''(x_0) < 0$ tells us that the derivative $f'(x)$ is decreasing near the point $x = x_0$. Since $f'(x_0) = 0$, this suggests that $f'(x) > 0$ when x is a little smaller than x_0, and that $f'(x) < 0$ when x is a little greater than x_0, as indicated in the picture below.



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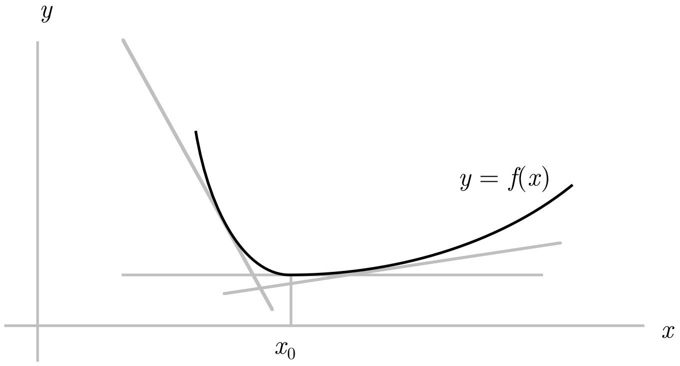
† This chapter was written at Macquarie University in 1999.

In this case, we say that the function has a local maximum at the point $x = x_0$. This means that if we restrict our attention to real values x near enough to the point $x = x_0$, then $f(x) \leq f(x_0)$ for all such real values x.

LOCAL MAXIMUM. *Suppose that $f'(x_0) = 0$ and $f''(x_0) < 0$. Then the function $f(x)$ has a local maximum at the point $x = x_0$.*

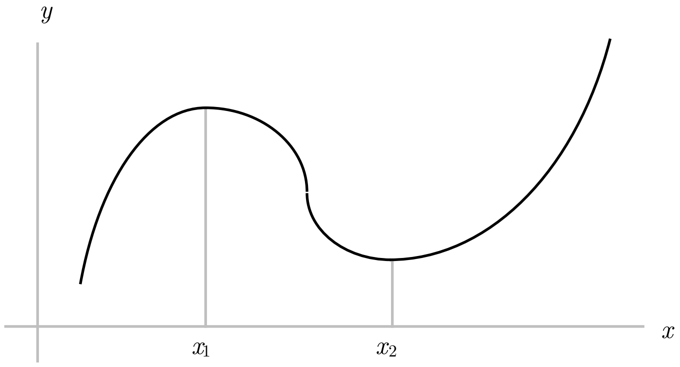
Suppose next that $f'(x_0) = 0$ and $f''(x_0) > 0$. The condition $f''(x_0) > 0$ tells us that the derivative $f'(x)$ is increasing near the point $x = x_0$. Since $f'(x_0) = 0$, this suggests that $f'(x) < 0$ when x is a little smaller than x_0, and that $f'(x) > 0$ when x is a little greater than x_0, as indicated in the picture below.



In this case, we say that the function has a local minimum at the point $x = x_0$. This means that if we restrict our attention to real values x near enough to the point $x = x_0$, then $f(x) \geq f(x_0)$ for all such real values x.

LOCAL MINIMUM. *Suppose that $f'(x_0) = 0$ and $f''(x_0) > 0$. Then the function $f(x)$ has a local minimum at the point $x = x_0$.*

Remark. These stationary points are called local maxima or local minima because such points may not maximize or minimize the functions in question. Consider the picture below, with a local maximum at $x = x_1$ and a local minumum at $x = x_2$.



We also say that a point $x = x_0$ is a point of inflection if $f''(x_0) = 0$, irrespective of whether $f'(x_0) = 0$ or not. A simple way of visualizing the graph of a function at a point of inflection is to imagine that one is steering a car along the curve. A point of inflection then corresponds to the place on the curve where the steering wheel of the car is momentarily straight while being turned from a little left to a little right, or while being turned from a little right to a little left.

EXAMPLE 13.1.1. Consider the function $f(x) = \cos x$. Since $f'(x) = -\sin x = 0$ whenever $x = k\pi$, where $k \in \mathbb{Z}$, it follows that the function $f(x) = \cos x$ has a stationary point at $x = k\pi$ for every $k \in \mathbb{Z}$. Next, note that $f''(x) = -\cos x$. If k is even, then $f''(k\pi) = -1$, so that $f(x)$ has a local maximum at $x = k\pi$. If k is odd, then $f''(k\pi) = 1$, so that $f(x)$ has a local minimum at $x = k\pi$. See the graph of this function in Chapter 3.

EXAMPLE 13.1.2. Consider the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$. Since

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x-1)(x+2),$$

it follows that the function $f(x)$ has stationary points at $x = 0$, $x = 1$ and $x = -2$. On the other hand, we have $f''(x) = 36x^2 + 24x - 24$. Since $f''(0) = -24$, $f''(1) = 36$ and $f''(-2) = 72$, it follows that $f(x)$ has a local maximum at $x = 0$ and local minima at $x = 1$ and $x = -2$.

EXAMPLE 13.1.3. Consider the function $f(x) = x^3 - 3x^2 + 2$. Since $f'(x) = 3x^2 - 6x = 3x(x-2)$, it follows that the function $f(x)$ has stationary points at $x = 0$ and $x = 2$. On the other hand, we have $f''(x) = 6x - 6$. Since $f''(0) = -6$ and $f''(2) = 6$, it follows that $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$. Observe also that there is a point of inflection at $x = 1$.

EXAMPLE 13.1.4. Consider the function $f(x) = x^4 - 2x^2 + 7$. Since

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1),$$

it follows that the function $f(x)$ has stationary points at $x = 0$, $x = 1$ and $x = -1$. On the other hand, we have $f''(x) = 12x^2 - 4$. Since $f''(0) = -4$, $f''(1) = 8$ and $f''(-1) = 8$, it follows that $f(x)$ has a local maximum at $x = 0$ and local minima at $x = 1$ and $x = -1$. Note also that $f''(x) = 0$ if $x = \pm\sqrt{1/3}$, so there are points of inflection at $x = \pm\sqrt{1/3}$.

EXAMPLE 13.1.5. Consider the function $f(x) = 3x^4 - 16x^3 + 24x^2 - 1$. Since

$$f'(x) = 12x^3 - 48x^2 + 48x = 12x(x^2 - 4x + 4) = 12x(x-2)^2,$$

it follows that the function $f(x)$ has stationary points at $x = 0$ and $x = 2$. On the other hand, we have $f''(x) = 36x^2 - 96x + 48$. Since $f''(0) = 48$ and $f''(2) = 0$, it follows that $f(x)$ has a local minimum at $x = 0$ and a point of inflection at $x = 2$. Note also that $36x^2 - 96x + 48 = 12(x-2)(3x-2)$, so there is another point of inflection at $x = 2/3$.

EXAMPLE 13.1.6. Consider the function

$$f(x) = \frac{1}{x^2+1}, \qquad \text{with} \qquad f'(x) = -\frac{2x}{(x^2+1)^2}.$$

Clearly $f(x)$ has a stationary point at $x = 0$. On the other hand, it is easy to check that

$$f''(x) = \frac{6x^2 - 2}{(x^2+1)^3}.$$

Since $f''(0) = -2$, it follows that $f(x)$ has a local maximum at $x = 0$. We also have points of inflection when $6x^2 - 2 = 0$; in other words, when $x = \pm\sqrt{1/3}$.

EXAMPLE 13.1.7. Consider the function

$$f(x) = \frac{x}{x^2+1}, \qquad \text{with} \qquad f'(x) = \frac{1-x^2}{(x^2+1)^2}.$$

Clearly $f(x)$ has stationary points at $x = 1$ and $x = -1$. On the other hand, it is easy to check that

$$f''(x) = -\frac{2x(x^2+1) + 4x(1-x^2)}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Since $f''(1) = -1/2$ and $f''(-1) = 1/2$, it follows that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = -1$. We also have points of inflection when $2x(x^2 - 3) = 0$; in other words, when $x = 0$ or $x = \pm\sqrt{3}$.

EXAMPLE 13.1.8. Consider the function $f(x) = \mathrm{e}^x + \mathrm{e}^{-x}$. Since $f'(x) = \mathrm{e}^x - \mathrm{e}^{-x}$, it follows that the function $f(x)$ has a stationary point at $x = 0$. On the other hand, we have $f''(x) = \mathrm{e}^x + \mathrm{e}^{-x}$. Since $f''(0) = 2$, it follows that $f(x)$ has a local minimum at $x = 0$.

EXAMPLE 13.1.9. Consider the function $f(x) = \sin x - \cos^2 x$, restricted to the interval $0 \le x \le 2\pi$. It is easy to see that

$$f'(x) = \cos x + 2\cos x \sin x = (1 + 2\sin x)\cos x.$$

We therefore have stationary points when $\cos x = 0$ or $\sin x = -1/2$. There are four stationary points in the interval $0 \le x \le 2\pi$, namely

$$x = \frac{\pi}{2}, \qquad x = \frac{3\pi}{2}, \qquad x = \frac{7\pi}{6}, \qquad x = \frac{11\pi}{6}.$$

Next, note that we can write $f'(x) = \cos x + \sin 2x$, so that $f''(x) = 2\cos 2x - \sin x$. It is easy to check that

$$f''\left(\frac{\pi}{2}\right) = -3, \qquad f''\left(\frac{3\pi}{2}\right) = -1, \qquad f''\left(\frac{7\pi}{6}\right) = \frac{3}{2}, \qquad f''\left(\frac{11\pi}{6}\right) = \frac{3}{2}.$$

Hence $f(x)$ has local maxima at $x = \pi/2$ and $x = 3\pi/2$, and local minima at $x = 7\pi/6$ and $x = 11\pi/6$.

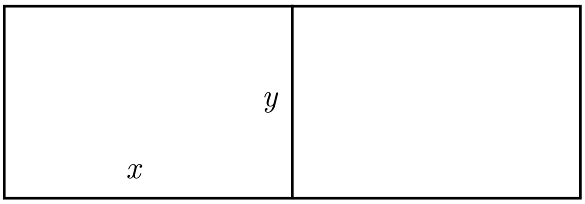
13.2. Applications to Problem Solving

In this section, we discuss how we can apply ideas in differentiation to solve various problems. We shall illustrate the techniques by discussing a few examples. Central to all of these is the crucial step where we set up the problems mathematically and in a suitable way.

EXAMPLE 13.2.1. We wish to find positive real numbers x and y such that $x + y = 6$ and the quantity xy^2 is as large as possible. In view of the restriction $x + y = 6$, the quantity $xy^2 = x(6 - x)^2$. We can therefore try to find a real number x which makes the quantity $x(6 - x)^2$ as large as possible. The idea here is to consider the function $f(x) = x(6 - x)^2$ and hope to find a local maximum. We can write $f(x) = 36x - 12x^2 + x^3$, and so $f'(x) = 36 - 24x + 3x^2 = 3(x^2 - 8x + 12) = 3(x - 2)(x - 6)$. Hence $x = 2$ and $x = 6$ are stationary points. Next, note that $f''(x) = 6x - 24$. Hence $f''(2) = -12$ and $f''(6) = 12$. It follows that the function $f(x)$ has a local maximum at the point $x = 2$. Then $y = 6 - x = 4$, with $f(2) = 32$. This choice of x and y makes xy^2 as large as possible, with value $f(2) = 32$.

EXAMPLE 13.2.2. We have 20 metres of fencing material, and wish to find the largest rectangular area that we can enclose. Suppose that the rectangular area has sides x and y in metres. Then the area is equal to xy, while the perimeter is equal to $2x + 2y$. Hence we wish to maximize the quantity xy subject to the restriction $2x + 2y = 20$. Under the restriction $2x + 2y = 20$, the quantity $xy = x(10 - x)$. We can therefore try to find a real number x which makes the quantity $x(10 - x)$ as large as possible. Consider the function $f(x) = x(10 - x) = 10x - x^2$. Then $f'(x) = 10 - 2x$, and so $x = 5$ is a stationary point. Since $f''(x) = -2$, the point $x = 5$ is a local maximum. Then $y = 10 - x = 5$, with $f(5) = 25$. This choice of x and y makes xy as large as possible, with area 25 square metres.

Example 13.2.3. We have 1200 metres of fencing material, and wish to enclose a double paddock with two equal rectangular areas as shown in the diagram below.



Suppose that each of the two rectangular areas has sides x and y in metres, as shown in the picture. Then the total area is equal to $2xy$, while the total perimeter is equal to $4x+3y$. Hence we wish to maximize the quantity $2xy$ subject to the restriction $4x+3y=1200$. Under the restriction $4x+3y=1200$, the quantity

$$2xy=2x\left(400-\frac{4x}{3}\right).$$

We can therefore try to find a real number x which makes this quantity as large as possible. Consider the function

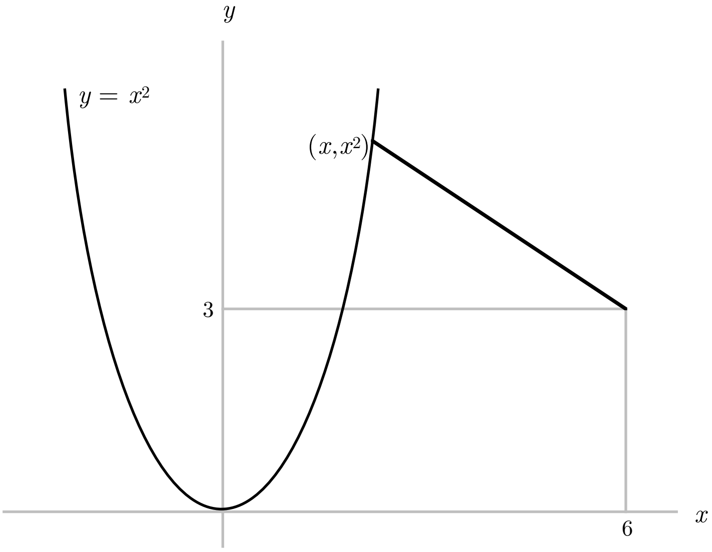
$$f(x)=2x\left(400-\frac{4x}{3}\right)=800x-\frac{8x^2}{3}.$$

Then

$$f'(x)=800-\frac{16x}{3},$$

and so $x=150$ is a stationary point. Since $f''(x)=-16/3$, the point $x=150$ is a local maximum. Then $y=200$, with $f(150)=60000$. This choice of x and y makes $2xy$ as large as possible, with total area 60000 square metres.

Example 13.2.4. We wish to find the point on the parabola $y=x^2$ which is closest to the point $(6,3)$. We begin by drawing a picture.



Note that a typical point on the parabola $y=x^2$ is given by $(x,y)=(x,x^2)$. The distance between this point and the point $(6,3)$ is given by

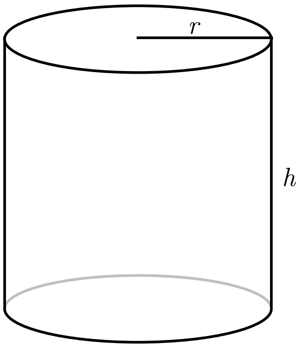
$$\sqrt{(x-6)^2+(x^2-3)^2},$$

in view of the theorem of Pythagoras. Now let $f(x) = (x-6)^2 + (x^2-3)^2$. Then $f(x)$ represents the square of this distance. We now need to find a local minimum for the function $f(x)$. Differentiating, we obtain $f'(x) = 2(x-6)+4x(x^2-3) = 4x^3-10x-12$. We observe that $x = 2$ is a root of the polynomial $4x^3-10x-12$. Hence we have

$$4x^3 - 10x - 12 = (x-2)(4x^2+8x+6) = (x-2)(4(x^2+2x+1)+2) = (x-2)(4(x+1)^2+2).$$

It follows that there is only one stationary point $x = 2$. Note next that $f''(x) = 12x^2 - 10$, so that $f''(2) > 0$. Hence $x = 2$ is a local minimum. It follows that the point $(2, 4)$ on the parabola is closest to the point $(6, 3)$, with distance $\sqrt{(2-6)^2+(4-3)^2} = \sqrt{17}$.

Example 13.2.5. A manufacturer wishes to maximize the volume of cylindrical metal cans made out of a fixed quantity of metal. To understand this problem, suppose that a typical can has radius r and height h as shown in the picture below:



Then the total surface area is equal to $2\pi r^2 + 2\pi rh = S$, where S is fixed, so that

$$h = \frac{S}{2\pi r} - r. \tag{1}$$

On the other hand, the volume of such a can is equal to $V = \pi r^2 h$. Under the restriction (1), we have

$$V = \pi r^2 h = \frac{Sr}{2} - \pi r^3.$$

Consider now the function

$$V(r) = \frac{Sr}{2} - \pi r^3.$$

Differentiating, we have

$$V'(r) = \frac{S}{2} - 3\pi r^2,$$

so that $r = \sqrt{S/6\pi}$ is the only stationary point, since negative values of r are meaningless. Furthermore, we have $V''(r) = -6\pi r$, and so this stationary point is a local maximum. For this value of r, we have

$$h = \frac{S}{2\pi r} - r = \sqrt{\frac{3S}{2\pi}} - \sqrt{\frac{S}{6\pi}} = \sqrt{\frac{9S}{6\pi}} - \sqrt{\frac{S}{6\pi}} = 3\sqrt{\frac{S}{6\pi}} - \sqrt{\frac{S}{6\pi}} = 2\sqrt{\frac{S}{6\pi}} = 2r.$$

This means that the most economical shape of a cylindrical can is when the height is twice the radius.

Example 13.2.6. A steamer travelling at constant speed due east passes a buoy at 9 am. A hydrofoil travelling at twice this speed due north passes the same buoy at 11 am. We would like to determine the time when the distance between the two vessels is smallest. To set up the problem mathematically, we consider the xy-plane, and assume that the position of the buoy is at the origin $(0, 0)$. Let s be the speed of the steamer. Then at t am, the position of the steamer is given by the point $(s(t-9), 0)$ if we relate due east to the positive horizontal axis. Furthermore, the position of the hydrofoil is given by the point $(0, 2s(t-11))$ if we relate due north to the positive vertical axis. By the theorem of Pythagoras, the distance between the two vessels at t am is given by $\sqrt{s^2(t-9)^2 + 4s^2(t-11)^2}$. Consider now the function $f(t) = s^2(t-9)^2 + 4s^2(t-11)^2$. Clearly this represents the square of the distance between the two vessels, so we need to find a local minimum for this function. Differentiating, we obtain

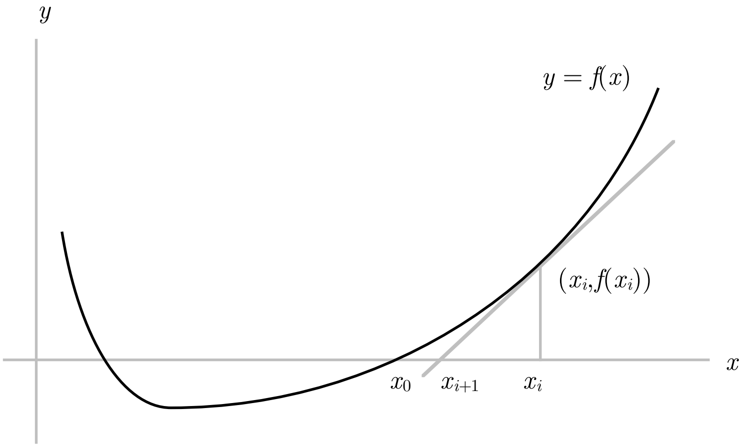
$$f'(t) = 2s^2(t-9) + 8s^2(t-11) = s^2(10t - 106),$$

with stationary point $t = 10.6$, representing the time 10:36 am. Can you convince yourself that this is a local minimum?

13.3. Newton's Method

In this section, we briefly describe a numerical technique which allows us to obtain approximations to solutions of some problems where exact answers may be hard or even impossible to calculate.

Let us consider an equation of the form $f(x) = 0$. Suppose that we wish to find some real number x for which the equation is satisfied. Consider the picture below:



Here x_0 represents a solution of the equation $f(x) = 0$. Unfortunately, we are unable to calculate the value of x_0 precisely. We now take some number x_i close to x_0, and consider the tangent to the curve $y = f(x)$ at the point $(x_i, f(x_i))$. Clearly the tangent has slope $f'(x_i)$. It follows that the equation of the tangent is given by

$$\frac{y - f(x_i)}{x - x_i} = f'(x_i).$$

Let x_{i+1} be the x-intercept of this tangent line. Then it is easy to see that

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \tag{2}$$

From the picture, this new point x_{i+1} gives a better approximation to x_0 than the point x_i does.

Newton's method is now to start with some point x_1 close to a solution x_0 of $f(x) = 0$, and then obtain a sequence of successive approximations $x_2, x_3, x_4, \ldots$ by using the formula (2).

EXAMPLE 13.3.1. We shall try to obtain some approximation for $\sqrt{2}$. To do so, we consider the equation $f(x) = 0$, where $f(x) = x^2 - 2$. Then $f'(x) = 2x$, and so equation (2) becomes

$$x_{i+1} = x_i - \frac{x_i^2 - 2}{2x_i}.$$

Using this, and taking $x_1 = 2$, we obtain $x_2 = 1.5$, $x_3 = 1.41667$, $x_4 = 1.41422$, $x_5 = 1.41421$, and so on, all to 5 decimal places. On the other hand, if we take $x_1 = -2$, then we obtain $x_2 = -1.5$, $x_3 = -1.41667$, $x_4 = -1.41422$, $x_5 = -1.41421$, and so on, all to 5 decimal places.

EXAMPLE 13.3.2. We shall try to obtain some approximation to a solution of the equation $x^3+x-1 = 0$. To do so, we consider the function $f(x) = x^3 + x - 1$. Then $f'(x) = 3x^2 + 1$, and so equation (2) becomes

$$x_{i+1} = x_i - \frac{x_i^3 + x_i - 1}{3x_i^2 + 1}.$$

Using this, and taking $x_1 = 1$, we obtain $x_2 = 0.75$, $x_3 = 0.68605$, $x_4 = 0.68234$, $x_5 = 0.68233$, and so on, all to 5 decimal places.

REMARK. As is the case for much of numerical mathematics, Newton's method is imprecise. It may fail to work in some instances. If there are many possible solutions, then it is sometimes unclear which solution the method will give.

EXAMPLE 13.3.3. Let us return to Example 13.1.7, and consider the function

$$f(x) = \frac{x}{x^2 + 1}, \qquad \text{with} \qquad f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}.$$

It is easy to see that the equation $f(x) = 0$ has precisely one solution, namely $x = 0$. Nevertheless, let us apply Newton's method to this function. Then equation (2) becomes

$$x_{i+1} = x_i - \frac{x_i(x_i^2 + 1)}{1 - x_i^2}.$$

Using this, and taking $x_1 = 0.5$, we obtain $x_2 = -0.33333$, $x_3 = 0.08333$, $x_4 = -0.00117$, $x_5 = 0.00000$, and so on, all to 5 decimal places. On the other hand, if we take $x_1 = 2$, then we obtain $x_2 = 5.33333$, $x_3 = 11.05533$, $x_4 = 22.29306$, $x_5 = 44.67602$, and so on, all to 5 decimal places. The method clearly fails in this second case.

PROBLEMS FOR CHAPTER 13

1. For each of the following functions, find all of the stationary points. For each such stationary point, determine whether it is a local maximum, a local minimum or another type of stationary point:
   a) $f(x) = 3x^2 + 6x + 9$
   b) $f(x) = 6x - x^2$
   c) $f(x) = 2 - 3x - 3x^2$
   d) $f(x) = 6 + 9x - 3x^2 - x^3$
   e) $f(x) = x + \dfrac{4}{x+1}$
   f) $f(x) = 4x - 1 + \dfrac{36}{x-1}$
   g) $f(x) = (x+1)^2 - (x-1)^2$
   h) $f(x) = 6 - \dfrac{2}{x} - x^2$
2. A bullet is shot upwards at time $t = 0$ from the top of a building 176 metres tall, with an initial speed of 160 metres per second. The height of the bullet is given by $h(t) = -16t^2 + 160t + 176$ after t seconds. At what time is the bullet at maximum height above the ground? What is this height?
3. What number, when squared and added to 16 times its reciprocal, gives a minimum value for this sum?

4. A piece of wire is to be cut into two pieces to form a circle and a square. How should the wire be cut to minimize the total area of the two pieces?

5. Find two real numbers whose sum is 16 and whose product is a maximum.

6. Find two positive real numbers whose product is 81 and the sum of whose squares is a minimum.

7. What positive real number is exceeded by its square root by the greatest amount?

8. Find the dimension of a right circular cylinder of volume 1 cubic metre and having the minimum surface area.

9. A closed box is to be constructed with a square base and volume of 1500 cubic metres. The material used for the base costs twice as much as for the top and sides. What dimension should the box have to keep the cost to a minimum?

10. Find the maximum area of a rectangle inscribed in a semicircle of radius 12 metres.

11. A rectangular beam, of width w and depth d, is cut from a circular log of diameter $a = 25$ centimetres. The beam has strength S given by $S = 2wd^2$. Find the dimension that will give the strongest beam.
    [HINT: Use $d^2 + w^2 = a^2$ to relate the variables d and w.]

12. For each of the following, use a calculator and Newton's method to obtain estimates for the desired quantity to at least 4 decimal places:
    a) The number $\sqrt{5}$, starting with an initial estimate of $x_0 = 2$.
    b) The number $\sqrt{7}$, starting with an initial estimate of $x_0 = 3$.
    c) The number $\sqrt[3]{2}$.
    d) A solution to the equation $x^3 - 2x - 5 = 0$.
    e) A solution to the equation $\cos x = x$.
    f) The largest real root of the polynomial $x^3 + x - 1$.
    g) The largest solution to the equation $\sin x = \mathrm{e}^x$.

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ELEMENTARY MATHEMATICS

W W L CHEN and X T DUONG

Chapter 14

INTRODUCTION TO INTEGRATION

14.1. Antiderivatives

In this chapter, we discuss the inverse process of differentiation. In other words, given a function $f(x)$, we wish to find a function $F(x)$ such that $F'(x) = f(x)$. Any such function $F(x)$ is called an antiderivative, or indefinite integral, of the function $f(x)$, and we write

$$F(x) = \int f(x)\,\mathrm{d}x.$$

A first observation is that the antiderivative, if it exists, is not unique. Suppose that the function $F(x)$ is an antiderivative of the function $f(x)$, so that $F'(x) = f(x)$. Let $G(x) = F(x) + C$, where C is any fixed real number. Then it is easy to see that $G'(x) = F'(x) = f(x)$, so that $G(x)$ is also an antiderivative of $f(x)$. A second observation, somewhat less obvious, is that for any given function $f(x)$, any two distinct antiderivatives of $f(x)$ must differ only by a constant. In other words, if $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $F(x) - G(x)$ is a constant. In this chapter, we shall denote any such constant by C, with or without subscripts.

An immediate consequence of this second observation is the following simple result related to the derivatives of constants in Section 11.1.

ANTIDERIVATIVES OF ZERO. *We have*

$$\int 0\,\mathrm{d}x = C.$$

In other words, the antiderivatives of the zero function are precisely all the constant functions.

Indeed, many antiderivatives can be obtained simply by referring to various rules concerning derivatives. We list here a number of such results. The first of these is related to the constant multiple rule for differentiation in Section 11.2.

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† This chapter was written at Macquarie University in 1999.

CONSTANT MULTIPLE RULE. *Suppose that a function $f(x)$ has antiderivatives. Then for any fixed real number c, we have*

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x.$$

ANTIDERIVATIVES OF POWERS.

(a) Suppose that n is a fixed real number such that $n \neq -1$. Then

$$\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C.$$

(b) We have

$$\int x^{-1}\,\mathrm{d}x = \log|x| + C.$$

Proof. Part (a) is a consequence of the rule concerning derivatives of powers in Section 11.1. If $x > 0$, then part (b) is a consequence of the rule concerning the derivative of the logarithmic function in Section 12.3. If $x < 0$, we can write $|x| = u$, where $u = -x$. It then follows from the Chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log|x|) = \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}}{\mathrm{d}u}(\log u) = -\frac{1}{u} = \frac{1}{x} \tag{1}$$

again. ♣

Corresponding to the sum rule for differentiation in Section 11.2, we have the following.

SUM RULE. *Suppose that functions $f(x)$ and $g(x)$ have antiderivatives. Then*

$$\int (f(x) + g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

We next consider trigonometric functions.

ANTIDERIVATIVES OF TRIGONOMETRIC FUNCTIONS.

(a) We have

$$\int \cos x\,\mathrm{d}x = \sin x + C \qquad \textit{and} \qquad \int \sin x\,\mathrm{d}x = -\cos x + C.$$

(b) We have

$$\int \sec^2 x\,\mathrm{d}x = \tan x + C \qquad \textit{and} \qquad \int \csc^2 x\,\mathrm{d}x = -\cot x + C.$$

(c) We have

$$\int \tan x \sec x\,\mathrm{d}x = \sec x + C \qquad \textit{and} \qquad \int \cot x \csc x\,\mathrm{d}x = -\csc x + C.$$

(d) We have

$$\int \sec x\,\mathrm{d}x = \log|\tan x + \sec x| + C \qquad \textit{and} \qquad \int \csc x\,\mathrm{d}x = -\log|\cot x + \csc x| + C.$$

Proof. Parts (a)–(c) follow immediately from the rules concerning derivatives of the trigonometric functions in Section 11.3. Part (d) follows from Example 12.3.12 and Example 12.3.13 if we note (1). ♣

Corresponding to the rule concerning the derivative of the exponential function in Section 12.3, we have the following.

ANTIDERIVATIVES OF THE EXPONENTIAL FUNCTION. *We have*

$$\int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x + C.$$

EXAMPLE 14.1.1. Using the sum rule, the constant multiple rule and the rule concerning antiderivatives of powers, we have

$$\int (x^2+3x+1)\,\mathrm{d}x = \int x^2\,\mathrm{d}x + 3\int x\,\mathrm{d}x + \int x^0\,\mathrm{d}x = \frac{1}{3}x^3 + \frac{3}{2}x^2 + x + C.$$

EXAMPLE 14.1.2. Using the sum rule and the rules concerning antiderivatives of powers and of trigonometric functions, we have

$$\int (x^3+\sin x)\,\mathrm{d}x = \int x^3\,\mathrm{d}x + \int \sin x\,\mathrm{d}x = \frac{1}{4}x^4 - \cos x + C.$$

EXAMPLE 14.1.3. We have

$$\int (\sin x+\sec x)\,\mathrm{d}x = \int \sin x\,\mathrm{d}x + \int \sec x\,\mathrm{d}x = -\cos x + \log|\tan x+\sec x| + C.$$

EXAMPLE 14.1.4. We have

$$\int (\mathrm{e}^x+3\cos x)\,\mathrm{d}x = \int \mathrm{e}^x\,\mathrm{d}x + 3\int \cos x\,\mathrm{d}x = \mathrm{e}^x + 3\sin x + C.$$

EXAMPLE 14.1.5. To find

$$\int \frac{1-\sin x}{1+\sin x}\,\mathrm{d}x,$$

note first of all that

$$\begin{aligned}\frac{1-\sin x}{1+\sin x} &= \frac{(1-\sin x)(1-\sin x)}{(1+\sin x)(1-\sin x)} = \frac{1-2\sin x+\sin^2 x}{1-\sin^2 x} = \frac{1-2\sin x+\sin^2 x}{\cos^2 x}\\ &= \sec^2 x - 2\tan x\sec x + \tan^2 x = 2\sec^2 x - 2\tan x\sec x - 1.\end{aligned}$$

It follows that

$$\int \frac{1-\sin x}{1+\sin x}\,\mathrm{d}x = 2\int \sec^2 x\,\mathrm{d}x - 2\int \tan x\sec x\,\mathrm{d}x - \int \mathrm{d}x = 2\tan x - 2\sec x - x + C.$$

14.2. Integration by Substitution

We now discuss how we can use the chain rule in differentiation to help solve problems in integration. This technique is usually called integration by substitution. As we shall not prove any result here, our discussion will be only heuristic.

We emphasize that the technique does not always work. First of all, we have little or no knowledge of the antiderivatives of many functions. Secondly, there is no simple routine that we can describe to help us find a suitable substitution even in the cases where the technique works. On the other hand, when the technique does work, there may well be more than one suitable substitution!

REMARK. It is imperative that one does not give up when one's effort does not seem to yield results. We learn far more from indefinite integrals that we cannot find than from those than we can.

INTEGRATION BY SUBSTITUTION – VERSION 1. *If we make a substitution* $x = g(u)$, *then* $\mathrm{d}x = g'(u)\,\mathrm{d}u$, *and*

$$\int f(x)\,\mathrm{d}x = \int f(g(u))g'(u)\,\mathrm{d}u.$$

Example 14.2.1. Consider the indefinite integral

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

If we make a substitution $x = \sin u$, then $\mathrm{d}x = \cos u\,\mathrm{d}u$, and

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\,\mathrm{d}u = \int \mathrm{d}u = u + C = \sin^{-1} x + C.$$

On the other hand, if we make a substitution $x = \cos v$, then $\mathrm{d}x = -\sin v\,\mathrm{d}v$, and

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int \frac{\sin v}{\sqrt{1-\cos^2 v}}\,\mathrm{d}v = -\int \mathrm{d}v = -v + C = -\cos^{-1} x + C.$$

See Section 12.4 concerning derivatives of inverse trigonometric functions.

Example 14.2.2. Consider the indefinite integral

$$\int \frac{1}{1+x^2}\,\mathrm{d}x.$$

If we make a substitution $x = \tan u$, then $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$, and

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int \frac{\sec^2 u}{1+\tan^2 u}\,\mathrm{d}u = \int \mathrm{d}u = u + C = \tan^{-1} x + C.$$

On the other hand, if we make a substitution $x = \cot v$, then $\mathrm{d}x = -\csc^2 v\,\mathrm{d}v$, and

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = -\int \frac{\csc^2 v}{1+\cot^2 v}\,\mathrm{d}v = -\int \mathrm{d}v = -v + C = -\cot^{-1} x + C.$$

Example 14.2.3. Consider the indefinite integral

$$\int x\sqrt{x+1}\,\mathrm{d}x.$$

If we make a substitution $x = u^2 - 1$, then $\mathrm{d}x = 2u\,\mathrm{d}u$, and

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int 2(u^2-1)u^2\,\mathrm{d}u = 2\int u^4\,\mathrm{d}u - 2\int u^2\,\mathrm{d}u\\ &= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

On the other hand, if we make a substitution $x = v - 1$, then $\mathrm{d}x = \mathrm{d}v$, and

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int (v-1)v^{1/2}\,\mathrm{d}v = \int v^{3/2}\,\mathrm{d}v - \int v^{1/2}\,\mathrm{d}v\\ &= \frac{2}{5}v^{5/2} - \frac{2}{3}v^{3/2} + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

We can confirm that the indefinite integral is correct by checking that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C\right) = x\sqrt{x+1}.$$

INTEGRATION BY SUBSTITUTION – VERSION 2. *Suppose that a function $f(x)$ can be written in the form $f(x)=g(h(x))h'(x)$. If we make a substitution $u=h(x)$, then $\mathrm{d}u=h'(x)\,\mathrm{d}x$, and*

$$\int f(x)\,\mathrm{d}x=\int g(h(x))h'(x)\,\mathrm{d}x=\int g(u)\,\mathrm{d}u.$$

Remark. Note that in Version 1, the variable x is initially written as a function of the new variable u, whereas in Version 2, the new variable u is written as a function of x. The difference, however, is minimal, as the substitution $x=g(u)$ in Version 1 has to be invertible to enable us to return from the new variable u to the original variable x at the end of the process.

Example 14.2.4. Consider the indefinite integral

$$\int x(x^2+3)^4\,\mathrm{d}x.$$

Note first of all that the derivative of the function x^2+3 is equal to $2x$, so it is convenient to make the substitution $u=x^2+3$. Then $\mathrm{d}u=2x\,\mathrm{d}x$, and

$$\int x(x^2+3)^4\,\mathrm{d}x=\frac{1}{2}\int 2x(x^2+3)^4\,\mathrm{d}x=\frac{1}{2}\int u^4\,\mathrm{d}u=\frac{1}{10}u^5+C=\frac{1}{10}(x^2+3)^5+C.$$

Example 14.2.5. Consider the indefinite integral

$$\int \frac{1}{x\log x}\,\mathrm{d}x.$$

Note first of all that the derivative of the function $\log x$ is equal to $1/x$, so it is convenient to make the substitution $u=\log x$. Then $\mathrm{d}u=(1/x)\,\mathrm{d}x$, and

$$\int \frac{1}{x\log x}\,\mathrm{d}x=\int \frac{1}{u}\,\mathrm{d}u=\log|u|+C=\log|\log x|+C.$$

Example 14.2.6. Consider the indefinite integral

$$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x.$$

Note first of all that the derivative of the function x^3 is equal to $3x^2$, so it is convenient to make the substitution $u=x^3$. Then $\mathrm{d}u=3x^2\,\mathrm{d}x$, and

$$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x=\frac{1}{3}\int 3x^2\mathrm{e}^{x^3}\,\mathrm{d}x=\frac{1}{3}\int \mathrm{e}^u\,\mathrm{d}u=\frac{1}{3}\mathrm{e}^u+C=\frac{1}{3}\mathrm{e}^{x^3}+C.$$

A somewhat more complicated alternative is to note that the derivative of the function e^{x^3} is equal to $3x^2\mathrm{e}^{x^3}$, so it is convenient to make the substitution $v=\mathrm{e}^{x^3}$. Then $\mathrm{d}v=3x^2\mathrm{e}^{x^3}\,\mathrm{d}x$, and

$$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x=\frac{1}{3}\int 3x^2\mathrm{e}^{x^3}\,\mathrm{d}x=\frac{1}{3}\int \mathrm{d}v=\frac{1}{3}v+C=\frac{1}{3}\mathrm{e}^{x^3}+C.$$

Example 14.2.7. Consider the indefinite integral

$$\int \tan^3 x\sec^2 x\,\mathrm{d}x.$$

Note first of all that the derivative of the function $\tan x$ is equal to $\sec^2 x$, so it is convenient to make the substitution $u=\tan x$. Then $\mathrm{d}u=\sec^2 x\,\mathrm{d}x$, and

$$\int \tan^3 x\sec^2 x\,\mathrm{d}x=\int u^3\,\mathrm{d}u=\frac{1}{4}u^4+C=\frac{1}{4}\tan^4 x+C.$$

Occasionally, the possibility of substitution may not be immediately obvious, and a certain amount of trial and error does occur. The fact that one substitution does not appear to work does not mean that the method fails. It may very well be the case that we have used a bad substitution. Or perhaps we may slightly modify the problem first. We illustrate this point by looking at two more examples.

Example 14.2.8. Consider the indefinite integral

$$\int \tan x \,\mathrm{d}x.$$

Here it does not appear that any substitution will work. However, if we write

$$\int \tan x \,\mathrm{d}x = \int \frac{\sin x}{\cos x} \,\mathrm{d}x,$$

then we observe that the derivative of the function $\cos x$ is equal to $-\sin x$, so it is convenient to make the substitution $u = \cos x$. Then $\mathrm{d}u = -\sin x \,\mathrm{d}x$, and

$$\int \tan x \,\mathrm{d}x = -\int \frac{-\sin x}{\cos x} \,\mathrm{d}x = -\int \frac{1}{u} \,\mathrm{d}u = -\log |u| + C = -\log |\cos x| + C.$$

Example 14.2.9. The indefinite integral

$$\int \frac{9 + 6x + 2x^2 + x^3}{4 + x^2} \,\mathrm{d}x$$

is rather daunting at first sight, but we have enough technique to study it. Note first of all that

$$\begin{aligned} 9 + 6x + 2x^2 + x^3 &= 9 + 2x + 2x^2 + 4x + x^3 = 9 + 2x + 2x^2 + x(4 + x^2) \\ &= 1 + 2x + 8 + 2x^2 + x(4 + x^2) = 1 + 2x + 2(4 + x^2) + x(4 + x^2). \end{aligned}$$

It follows that

$$\int \frac{9 + 6x + 2x^2 + x^3}{4 + x^2} \,\mathrm{d}x = \int \frac{1}{4 + x^2} \,\mathrm{d}x + \int \frac{2x}{4 + x^2} \,\mathrm{d}x + \int (2 + x) \,\mathrm{d}x. \tag{2}$$

To study the first integral on the right hand side of (2), we can make a substitution $x = 2 \tan u$. Then $\mathrm{d}x = 2 \sec^2 u \,\mathrm{d}u$, and

$$\int \frac{1}{4 + x^2} \,\mathrm{d}x = \int \frac{2 \sec^2 u}{4 + 4 \tan^2 u} \,\mathrm{d}u = \frac{1}{2} \int \mathrm{d}u = \frac{1}{2} u + C_1 = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + C_1. \tag{3}$$

To study the second integral on the right hand side of (2), we note that the derivative of the function $4 + x^2$ is equal to $2x$. If we make a substitution $v = 4 + x^2$, then $\mathrm{d}v = 2x \,\mathrm{d}x$, and

$$\int \frac{2x}{4 + x^2} \,\mathrm{d}x = \int \frac{1}{v} \,\mathrm{d}v = \log |v| + C_2 = \log(4 + x^2) + C_2. \tag{4}$$

The third integral on the right hand side of (2) is easy to evaluate. We have

$$\int (2 + x) \,\mathrm{d}x = 2x + \frac{1}{2} x^2 + C_3. \tag{5}$$

Substituting (3)–(5) into (2) and writing $C = C_1 + C_2 + C_3$, we obtain

$$\int \frac{9 + 6x + 2x^2 + x^3}{4 + x^2} \,\mathrm{d}x = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + \log(4 + x^2) + 2x + \frac{1}{2} x^2 + C.$$

It may be worth checking that

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + \log(4 + x^2) + 2x + \frac{1}{2} x^2 + C \right) = \frac{9 + 6x + 2x^2 + x^3}{4 + x^2}.$$

14.3. Definite Integrals

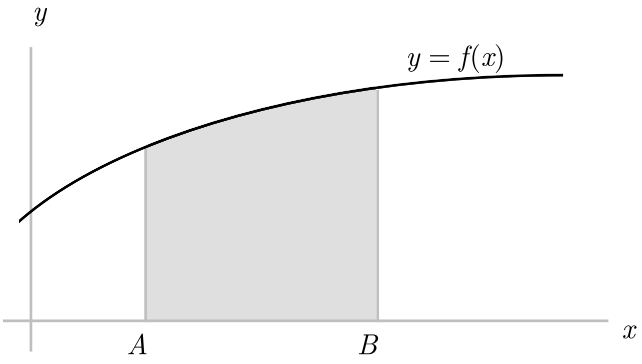
The formal definition of a definite integral is rather complicated, and we do not propose to discuss it here. Instead, we shall only give some geometric motivation, and then relate the definite integral to indefinite integrals we have discussed earlier.

Suppose that $f(x)$ is a real valued function, defined on an interval $[A, B] = \{x \in \mathbb{R} : A \leq x \leq B\}$. We shall suppose also that $f(x)$ has an antiderivative $F(x)$ for every $x \in [A, B]$.

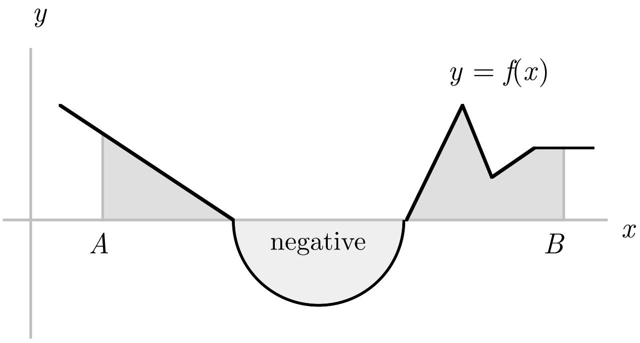
Consider first of all the special case that $f(x) \geq 0$ for every $x \in [A, B]$. By the definite integral

$$\int_A^B f(x)\,\mathrm{d}x,$$

we mean the area below the curve $y = f(x)$ and above the horizontal axis $y = 0$, bounded between the vertical lines $x = A$ and $x = B$, as shown in the picture below.



In general, we take the area between the curve $y = f(x)$ and the horizontal axis $y = 0$, bounded between the vertical lines $x = A$ and $x = B$, with the convention that the area below the horizontal axis $y = 0$ is taken to be negative, as shown in the picture below.



We now need a way of calculating this area. In some very special cases, this is very simple.

EXAMPLE 14.3.1. If we examine the graph of the trigonometric functions in Chapter 3, then it is easy to see that

$$\int_0^{2\pi} \sin x\,\mathrm{d}x = 0 \qquad \text{and} \qquad \int_0^{\pi} \cos x\,\mathrm{d}x = 0.$$

In each case, it is easy to see that the area in question above the horizontal axis $y = 0$ is equal to the area in question below this axis.

EXAMPLE 14.3.2. It is easy to see that the area between the line $y = x$ and the horizontal axis $y = 0$, bounded between the vertical lines $x = 0$ and $x = 1$, is the area of a triangle with base 1 and height 1. Hence

$$\int_0^1 x \, \mathrm{d}x = \frac{1}{2}.$$

In many instances, we do not have such geometric information to help us calculate the area in question. Instead, we can use the indefinite integral.

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS. *Suppose that a function* $F(x)$ *satisfies* $F'(x) = f(x)$ *for every* $x \in [A, B]$. *Then*

$$\int_A^B f(x) \, \mathrm{d}x = \Big[F(x) \Big]_A^B = F(B) - F(A).$$

REMARK. A simple consequence of the above is that the constant multiple rule and sum rule for indefinite integrals extend to definite integrals. For any fixed real number c, we have

$$\int_A^B c f(x) \, \mathrm{d}x = c \int_A^B f(x) \, \mathrm{d}x.$$

We also have

$$\int_A^B (f(x) + g(x)) \, \mathrm{d}x = \int_A^B f(x) \, \mathrm{d}x + \int_A^B g(x) \, \mathrm{d}x.$$

A further consequence of the Fundamental theorem of integral calculus is a rule concerning splitting up an interval $[A, B]$ into two. Suppose that $A < A^* < B$. Then

$$\int_A^B f(x) \, \mathrm{d}x = \int_A^{A^*} f(x) \, \mathrm{d}x + \int_{A^*}^B f(x) \, \mathrm{d}x.$$

EXAMPLE 14.3.3. Returning to Example 14.3.1, we have

$$\int_0^{2\pi} \sin x \, \mathrm{d}x = \Big[-\cos x \Big]_0^{2\pi} = -\cos 2\pi + \cos 0 = 0$$

and

$$\int_0^{\pi} \cos x \, \mathrm{d}x = \Big[\sin x \Big]_0^{\pi} = \sin \pi - \sin 0 = 0.$$

EXAMPLE 14.3.4. Returning to Example 14.3.2, we have

$$\int_0^1 x \, \mathrm{d}x = \left[\frac{1}{2} x \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

EXAMPLE 14.3.5. We have

$$\int_0^{\pi} \sin x \, \mathrm{d}x = \Big[-\cos x \Big]_0^{\pi} = -\cos \pi + \cos 0 = 2.$$

EXAMPLE 14.3.6. We have

$$\int_1^2 \frac{1}{x} \, \mathrm{d}x = \Big[\log |x| \Big]_1^2 = \log 2 - \log 1 = \log 2.$$

EXAMPLE 14.3.7. We have

$$\int_0^1 \mathrm{e}^x\,\mathrm{d}x = \Big[\mathrm{e}^x\Big]_0^1 = \mathrm{e}^1 - \mathrm{e}^0 = \mathrm{e} - 1.$$

EXAMPLE 14.3.8. We have

$$\int_0^{\pi/4} \sec^2 x\,\mathrm{d}x = \Big[\tan x\Big]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1.$$

EXAMPLE 14.3.9. We have

$$\int_{-1}^1 (x^3+x^2)\,\mathrm{d}x = \left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^1 = \left(\frac{1}{4}+\frac{1}{3}\right) - \left(\frac{1}{4}-\frac{1}{3}\right) = \frac{2}{3}.$$

EXAMPLE 14.3.10. Recall Example 14.2.1. Since

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1} x + C, \tag{6}$$

we have

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \Big[\sin^{-1} x\Big]_0^{1/2} = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6}.$$

To obtain (6), recall that we can use the substitution $x = \sin u$ to show that

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \mathrm{d}u = u + C,$$

followed by an inverse substitution $u = \sin^{-1} x$. Here, we need to make the extra step of substituting the values $x = 0$ and $x = 1/2$ to the indefinite integral $\sin^{-1} x$. Observe, however, that with the substitution $x = \sin u$, the variable x increases from 0 to $1/2$ as the variable u increases from 0 to $\pi/6$. But then

$$\int_0^{\pi/6} \mathrm{d}u = \Big[u\Big]_0^{\pi/6} = \frac{\pi}{6} = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x,$$

so it appears that we do not need the inverse substitution $u = \sin^{-1} x$. Perhaps we can directly substitute $u = 0$ and $u = \pi/6$ to the indefinite integral u.

DEFINITE INTEGRAL BY SUBSTITUTION – VERSION 1. *Suppose that a substitution $x = g(u)$ satisfies the following conditions:*
(a) There exist $\alpha, \beta \in \mathbb{R}$ such that $g(\alpha) = A$ and $g(\beta) = B$.
(b) The derivative $g'(u) > 0$ for every u satisfying $\alpha < u < \beta$.
Then $\mathrm{d}x = g'(u)\,\mathrm{d}u$, and

$$\int_A^B f(x)\,\mathrm{d}x = \int_\alpha^\beta f(g(u))g'(u)\,\mathrm{d}u.$$

REMARK. If condition (b) above is replaced by the condition that the derivative $g'(u) < 0$ for every u satisfying $\beta < u < \alpha$, then the same conclusion holds if we adopt the convention that

$$\int_\alpha^\beta f(g(u))g'(u)\,\mathrm{d}u = -\int_\beta^\alpha f(g(u))g'(u)\,\mathrm{d}u.$$

Example 14.3.11. To calculate the definite integral

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x,$$

we can use the substitution $x = \tan u$, so that $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$. Note that $\tan 0 = 0$ and $\tan(\pi/4) = 1$, and that $\sec^2 u > 0$ whenever $0 < u < \pi/4$. It follows that

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \int_0^{\pi/4} \frac{\sec^2 u}{1+\tan^2 u}\,\mathrm{d}u = \int_0^{\pi/4} \mathrm{d}u = \Big[u\Big]_0^{\pi/4} = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

We can compare this to first observing Example 14.2.2, so that

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \Big[\tan^{-1} x\Big]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

Example 14.3.12. To calculate the definite integral

$$\int_0^3 x\sqrt{x+1}\,\mathrm{d}x,$$

we can use the substitution $x = g(u) = u^2 - 1$, so that $\mathrm{d}x = 2u\,\mathrm{d}u$. Note that $g(1) = 0$ and $g(2) = 3$, and that $g'(u) = 2u > 0$ whenever $1 < u < 2$. It follows that

$$\int_0^3 x\sqrt{x+1}\,\mathrm{d}x = \int_1^2 2(u^2-1)u^2\,\mathrm{d}u = \left[\frac{2}{5}u^5 - \frac{2}{3}u^3\right]_1^2 = \left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{62}{5} - \frac{14}{3} = \frac{116}{15}.$$

DEFINITE INTEGRAL BY SUBSTITUTION – VERSION 2. *Suppose that a substitution* $u = h(x)$ *satisfies the following conditions:*
(a) There exists a function $g(u)$ *such that* $f(x) = g(h(x))h'(x)$ *for every* $x \in [A,B]$.
(b) The derivative $h'(x) > 0$ *for every* x *satisfying* $A < x < B$.
Then $\mathrm{d}u = h'(x)\,\mathrm{d}x$, *and*

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^B g(h(x))h'(x)\,\mathrm{d}x = \int_{h(A)}^{h(B)} g(u)\,\mathrm{d}u.$$

Remark. If condition (b) above is replaced by the condition that the derivative $h'(x) < 0$ for every x satisfying $A < x < B$, then the same conclusion holds if we adopt the convention that

$$\int_{h(A)}^{h(B)} g(u)\,\mathrm{d}u = -\int_{h(B)}^{h(A)} g(u)\,\mathrm{d}u.$$

Example 14.3.13. To calculate the definite integral

$$\int_0^1 x(x^2+3)^4\,\mathrm{d}x,$$

we can use the substitution $u = h(x) = x^2 + 3$, so that $\mathrm{d}u = 2x\,\mathrm{d}x$. Note that $h(0) = 3$ and $h(1) = 4$, and that $h'(x) = 2x > 0$ whenever $0 < x < 1$. It follows that

$$\int_0^1 x(x^2+3)^4\,\mathrm{d}x = \frac{1}{2}\int_3^4 u^4\,\mathrm{d}u = \frac{1}{2}\left[\frac{u^5}{5}\right]_3^4 = \frac{1}{2}\left(\frac{1024}{5} - \frac{243}{5}\right) = \frac{781}{10}.$$

Example 14.3.14. To calculate the definite integral

$$\int_2^4 \frac{1}{x\log x}\,\mathrm{d}x,$$

we can use the substitution $u = h(x) = \log x$, so that $\mathrm{d}u = h'(x)\,\mathrm{d}x$, where $h'(x) = 1/x > 0$ whenever $2 < x < 4$. Note also that $h(2) = \log 2$ and $h(4) = \log 4$. It follows that

$$\int_2^4 \frac{1}{x\log x}\,\mathrm{d}x = \int_{\log 2}^{\log 4} \frac{1}{u}\,\mathrm{d}u = \Big[\log|u|\Big]_{\log 2}^{\log 4} = \log\log 4 - \log\log 2 = \log\left(\frac{\log 4}{\log 2}\right) = \log 2.$$

Example 14.3.15. To calculate the definite integral

$$\int_0^\pi \sin^2 x\cos x\,\mathrm{d}x,$$

we can use the substitution $u = h(x) = \sin x$, so that $\mathrm{d}u = \cos x\,\mathrm{d}x$. Now $h(0) = 0$ and $h(\pi) = 0$, so something is funny here! The problem is that

$$h'(x) = \cos x \begin{cases} > 0 & \left(0 < x < \dfrac{\pi}{2}\right), \\ < 0 & \left(\dfrac{\pi}{2} < x < \pi\right). \end{cases}$$

It follows that we must first write

$$\int_0^\pi \sin^2 x\cos x\,\mathrm{d}x = \int_0^{\pi/2} \sin^2 x\cos x\,\mathrm{d}x + \int_{\pi/2}^\pi \sin^2 x\cos x\,\mathrm{d}x \qquad (7)$$

before we can make any substitution. Consider now the first integral on the right hand side of (7). Using the substitution $u = h(x) = \sin x$, we note that $h(0) = 0$ and $h(\pi/2) = 1$, and that $h'(x) > 0$ whenever $0 < x < \pi/2$. Hence

$$\int_0^{\pi/2} \sin^2 x\cos x\,\mathrm{d}x = \int_0^1 u^2\,\mathrm{d}u = \frac{1}{3}.$$

Consider next the second integral on the right hand side of (7). Using the substitution $u = h(x) = \sin x$, we note that $h(\pi/2) = 1$ and $h(\pi) = 0$, and that $h'(x) < 0$ whenever $\pi/2 < x < \pi$. Hence

$$\int_{\pi/2}^\pi \sin^2 x\cos x\,\mathrm{d}x = \int_1^0 u^2\,\mathrm{d}u = -\int_0^1 u^2\,\mathrm{d}u = -\frac{1}{3}.$$

Combining the two parts, we conclude that

$$\int_0^\pi \sin^2 x\cos x\,\mathrm{d}x = 0.$$

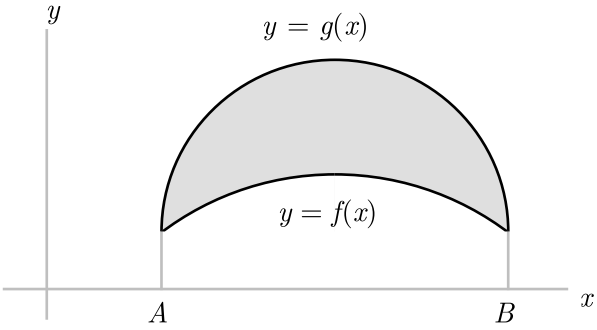
Alternatively, we can make the substitution $x = g(v) = \pi - v$ to the second integral on the right hand side of (7). Then $g(\pi/2) = \pi/2$ and $g(0) = \pi$, and $g'(v) = -1 < 0$ for every v satisfying $0 < v < \pi/2$. It follows that

$$\int_{\pi/2}^\pi \sin^2 x\cos x\,\mathrm{d}x = \int_{\pi/2}^0 \sin^2 v\cos v\,\mathrm{d}v = -\int_0^{\pi/2} \sin^2 v\cos v\,\mathrm{d}v.$$

This, combined with (7), gives the same conclusion.

14.4. Areas

We conclude this chapter by describing how we may use definite integrals to evaluate areas. Suppose that the boundary of a region on the xy-plane can be described by a top edge $y = g(x)$ and a bottom edge $y = f(x)$ bounded between two vertical lines $x = A$ and $x = B$, as shown in the picture below.



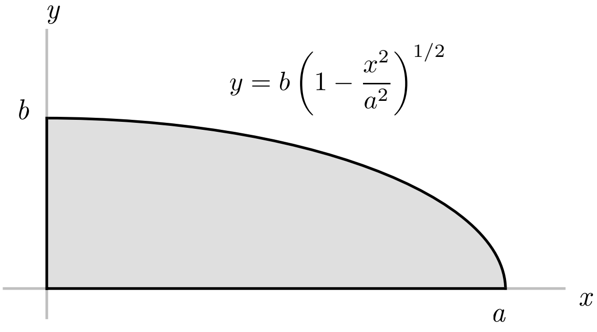
Then the area of the region is given by the definite integral

$$\int_A^B (g(x) - f(x))\,\mathrm{d}x.$$

Example 14.4.1. We wish to show that the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b \in \mathbb{R}$ are positive, is equal to πab. To do this, we may consider the quarter of the ellipse in the first quadrant, as shown in the picture below.



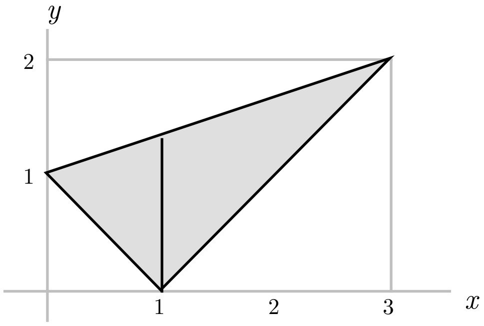
It follows that the shaded region has area

$$\int_0^a b\left(1 - \frac{x^2}{a^2}\right)^{1/2} \mathrm{d}x.$$

We can use the substitution $x = g(u) = a \sin u$. Then $g(0) = 0$ and $g(\pi/2) = a$. Furthermore, we have $\mathrm{d}x = g'(u)\,\mathrm{d}u$, where $g'(u) = a \cos u > 0$ whenever $0 < u < \pi/2$. It follows that

$$\begin{aligned}\int_0^a b\left(1 - \frac{x^2}{a^2}\right)^{1/2} \mathrm{d}x &= \int_0^{\pi/2} ab(1 - \sin^2 u)^{1/2} \cos u\,\mathrm{d}u = ab\int_0^{\pi/2} \cos^2 u\,\mathrm{d}u \\ &= ab\int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2u\right) \mathrm{d}u = ab\left[\frac{1}{2}u + \frac{1}{4}\sin 2u\right]_0^{\pi/2} = \frac{\pi ab}{4}.\end{aligned}$$

EXAMPLE 14.4.2. We wish to evaluate the area of the triangle with vertices $(0,1)$, $(1,0)$ and $(3,2)$. To do this, we split the triangle into two regions as shown in the picture below.



The triangle on the left is bounded between the vertical lines $x = 0$ and $x = 1$, and the top edge and the bottom edge are given respectively by

$$y = \frac{1}{3}x + 1 \qquad \text{and} \qquad y = 1 - x.$$

The triangle on the right is bounded between the vertical lines $x = 1$ and $x = 3$, and the top edge and the bottom edge are given respectively by

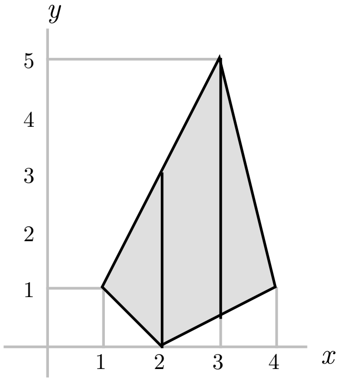
$$y = \frac{1}{3}x + 1 \qquad \text{and} \qquad y = x - 1.$$

It follows that the area of the original triangle is given by

$$\int_0^1 \left(\left(\frac{1}{3}x + 1\right) - (1 - x)\right) \mathrm{d}x + \int_1^3 \left(\left(\frac{1}{3}x + 1\right) - (x - 1)\right) \mathrm{d}x$$

$$= \int_0^1 \frac{4}{3}x \,\mathrm{d}x + \int_1^3 \left(2 - \frac{2}{3}x\right) \mathrm{d}x = \left[\frac{2}{3}x^2\right]_0^1 + \left[2x - \frac{1}{3}x^2\right]_1^3 = 2.$$

EXAMPLE 14.4.3. We wish to evaluate the area of the quadrilateral with vertices $(1,1)$, $(2,0)$, $(4,1)$ and $(3,5)$. To do this, we split the quadrilateral into three regions as shown in the picture below.



The triangle on the left is bounded between the vertical lines $x = 1$ and $x = 2$, and the top edge and the bottom edge are given respectively by

$$y = 2x - 1 \qquad \text{and} \qquad y = 2 - x.$$

The quadrilateral in the middle is bounded between the vertical lines $x = 2$ and $x = 3$, and the top edge and the bottom edge are given respectively by

$$y = 2x - 1 \qquad \text{and} \qquad y = \frac{1}{2}x - 1.$$

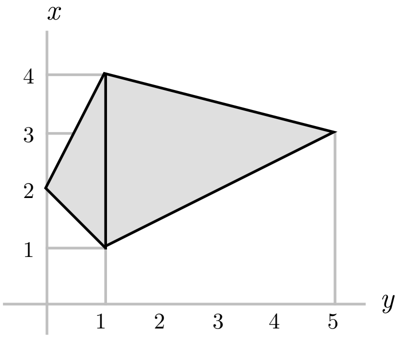
The triangle on the right is bounded between the vertical lines $x = 3$ and $x = 4$, and the top edge and the bottom edge are given respectively by

$$y = 17 - 4x \qquad \text{and} \qquad y = \frac{1}{2}x - 1.$$

It follows that the area of the original quadrilateral is given by

$$\begin{aligned}
&\int_1^2 ((2x-1)-(2-x))\,\mathrm{d}x + \int_2^3 \left((2x-1) - \left(\frac{1}{2}x - 1\right)\right)\mathrm{d}x + \int_3^4 \left((17-4x) - \left(\frac{1}{2}x - 1\right)\right)\mathrm{d}x \\
&\qquad = \int_1^2 (3x-3)\,\mathrm{d}x + \int_2^3 \frac{3}{2}x\,\mathrm{d}x + \int_3^4 \left(18 - \frac{9}{2}x\right)\mathrm{d}x \\
&\qquad = \left[\frac{3}{2}x^2 - 3x\right]_1^2 + \left[\frac{3}{4}x^2\right]_2^3 + \left[18x - \frac{9}{4}x^2\right]_3^4 = \frac{15}{2}.
\end{aligned}$$

Alternatively, we can transpose the picture above and split the quadrilateral into two regions as shown in the picture below:



Note that the roles of x and y are now interchanged. The triangle on the left is bounded between the vertical lines $y = 0$ and $y = 1$, and the top edge and the bottom edge are given respectively by

$$x = 2y + 2 \qquad \text{and} \qquad x = 2 - y.$$

The triangle on the right is bounded between the vertical lines $y = 1$ and $y = 5$, and the top edge and the bottom edge are given respectively by

$$x = \frac{17}{4} - \frac{1}{4}y \qquad \text{and} \qquad x = \frac{1}{2}y + \frac{1}{2}.$$

It follows that the area of the original quadrilateral is given by

$$\begin{aligned}
&\int_0^1 ((2y+2)-(2-y))\,\mathrm{d}y + \int_1^5 \left(\left(\frac{17}{4} - \frac{1}{4}y\right) - \left(\frac{1}{2}y + \frac{1}{2}\right)\right)\mathrm{d}y \\
&\qquad = \int_0^1 3y\,\mathrm{d}y + \int_1^5 \left(\frac{15}{4} - \frac{3}{4}y\right)\mathrm{d}y = \left[\frac{3}{2}y^2\right]_0^1 + \left[\frac{15}{4}y - \frac{3}{8}y^2\right]_1^5 = \frac{15}{2}
\end{aligned}$$

as before.

Problems for Chapter 14

1. Find each of the following indefinite integrals:

a) $\displaystyle\int \sqrt{3}\,\mathrm{d}x$ b) $\displaystyle\int (5x+3)\,\mathrm{d}x$ c) $\displaystyle\int (2x^2-3x+1)\,\mathrm{d}x$

d) $\displaystyle\int x^3\,\mathrm{d}x$ e) $\displaystyle\int (x-2)(x+3)\,\mathrm{d}x$ f) $\displaystyle\int (1-2\cos x)\,\mathrm{d}x$

g) $\displaystyle\int (5\cos x+4x)\,\mathrm{d}x$ h) $\displaystyle\int 8\mathrm{e}^x\,\mathrm{d}x$ i) $\displaystyle\int \frac{1}{x}\,\mathrm{d}x$

2. Evaluate each of the following indefinite integrals using the given substitution:

a) $\displaystyle\int x(x^2-1)^{99}\,\mathrm{d}x$ (use the substitution $u=x^2-1$)

b) $\displaystyle\int \frac{x^2}{\sqrt{2+x^3}}\,\mathrm{d}x$ (use the substitution $u=x^3+2$)

c) $\displaystyle\int \sin 4x\,\mathrm{d}x$ (use the substitution $u=4x$)

d) $\displaystyle\int \frac{\mathrm{d}x}{(2x+1)^2}$ (use the substitution $u=2x+1$)

e) $\displaystyle\int \frac{x+3}{(x^2+6x)^2}\,\mathrm{d}x$ (use the substitution $u=x^2+6x$)

f) $\displaystyle\int \sec ax\,\tan ax\,\mathrm{d}x$ (use the substitution $u=ax$)

3. Evaluate each of the following indefinite integrals:

a) $\displaystyle\int \cos 2x\,\mathrm{d}x$ b) $\displaystyle\int \sqrt{x-1}\,\mathrm{d}x$ c) $\displaystyle\int x^2\cos(1-x^3)\,\mathrm{d}x$

d) $\displaystyle\int x\sin(x^2)\,\mathrm{d}x$ e) $\displaystyle\int \frac{1}{(1-3x)^4}\,\mathrm{d}x$ f) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x$

g) $\displaystyle\int \sec^2(3x)\,\mathrm{d}x$ h) $\displaystyle\int \sin^3 x\cos x\,\mathrm{d}x$ i) $\displaystyle\int x(x^2+16)^2\,\mathrm{d}x$

j) $\displaystyle\int x^2\sqrt{x^3+8}\,\mathrm{d}x$ k) $\displaystyle\int \frac{1}{\sqrt{2x+5}}\,\mathrm{d}x$ l) $\displaystyle\int \left(x-\frac{1}{x}\right)\,\mathrm{d}x$

m) $\displaystyle\int \frac{2x+1}{x^2+x+3}\,\mathrm{d}x$ n) $\displaystyle\int \frac{1}{x^2-4x+4}\,\mathrm{d}x$ o) $\displaystyle\int \frac{\log x}{x}\,\mathrm{d}x$

p) $\displaystyle\int \frac{\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x$ q) $\displaystyle\int x\mathrm{e}^{x^2}\,\mathrm{d}x$ r) $\displaystyle\int \mathrm{e}^{2x-1}\,\mathrm{d}x$

s) $\displaystyle\int \sec(4x)\tan(4x)\,\mathrm{d}x$ t) $\displaystyle\int x^3\cos(5x^4)\,\mathrm{d}x$ u) $\displaystyle\int \sec^2(2x+1)\,\mathrm{d}x$

v) $\displaystyle\int \mathrm{e}^x\cos(\mathrm{e}^x)\,\mathrm{d}x$ w) $\displaystyle\int \frac{(\log x)^2}{x}\,\mathrm{d}x$ x) $\displaystyle\int \tan x\sec^3 x\,\mathrm{d}x$

4. Evaluate each of the following definite integrals:

a) $\displaystyle\int_1^2 2x\,\mathrm{d}x$ b) $\displaystyle\int_1^3 \frac{1}{x}\,\mathrm{d}x$ c) $\displaystyle\int_0^2 \mathrm{e}^{-x}\,\mathrm{d}x$

d) $\displaystyle\int_2^3 (3x+1)\,\mathrm{d}x$ e) $\displaystyle\int_0^\pi \sin x\,\mathrm{d}x$ f) $\displaystyle\int_3^6 (x-3)^2\,\mathrm{d}x$

g) $\displaystyle\int_0^2 \frac{1}{4+x^2}\,\mathrm{d}x$ h) $\displaystyle\int_0^1 x\mathrm{e}^{x^2}\,\mathrm{d}x$ i) $\displaystyle\int_0^a (x^2+a^2)\,\mathrm{d}x$

j) $\displaystyle\int_0^1 (1+x+3x^2)\,\mathrm{d}x$ k) $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$ l) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\,\mathrm{d}x$

5. a) Draw the graphs of the line $y = x$ and the parabola $y = x^2$.
   b) Find the two points of intersection of the two curves.
   c) Use definite integrals to find the area bounded between the two curves.

6. Use definite integrals to find the area between the curves $y = \mathrm{e}^x$ and $y = \mathrm{e}^{2x}$, bounded between the lines $x = 0$ and $x = 1$.

7. Find the area of the triangle with vertices $(0, 0)$, $(4, 3)$ and $(1, 5)$.

8. Find the area of the quadrilateral with vertices $(1, 1)$, $(5, 2)$, $(2, 3)$ and $(4, 3)$.

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