Formal Reasoning 2015
Solutions Exam
(29/01/15)

1. Give a formula in the propositional logic that resembles the meaning of this sentence as well as possible: (6 points)

   If it rains I get wet, and it rains, but still I don't get wet!

   Use the following dictionary:

R	it rains
W	I get wet

   $$(R \to W) \land R \land \neg W$$

2. This exercise is about the formula

   $$a \to \neg b \leftrightarrow \neg b \to a$$

   (a) Write this formula using parentheses according to the official grammar in the course notes. (3 points)

   This is the formula according to the official grammar:

   $$((a \to \neg b) \leftrightarrow (\neg b \to a))$$

   (b) Give the truth table of this formula. (3 points)

   The truth table is:

a	b	$\neg b$	$a \to \neg b$	$\neg b \to a$	$(a \to \neg b) \leftrightarrow (\neg b \to a)$
0	0	1	1	0	0
0	1	0	1	1	1
1	0	1	1	1	1
1	1	0	0	1	0

3. This exercise is about the statement (6 points)

   $$f \to \neg g \equiv g \to \neg f$$

   where f and g are arbitrary formulas of the propositional logic.

   (a) Explain what this statement expresses. (3 points)

   This statement expresses that the formulas $f \to \neg g$ and $g \to \neg f$ are logically equivalent. In other words: their valuations are the same in each model. And in more other words: if we write down the truth tables of both formulas we see that the columns for these two formulas are exactly the same.

(b) Does this statement hold if we take $f = a$ and $g = b$? (3 points)

Yes, this statement holds in this case. We know that for arbitrary formulas h and i it holds that $h \to i \equiv \neg h \vee i$ and $h \vee i \equiv i \vee h$. Using these equivalences we can write down the following series of equivalent formulas:

$$f \to \neg g \;\equiv\; \neg f \vee \neg g \;\equiv\; \neg g \vee \neg f \;\equiv\; g \to \neg f$$

Note that this holds for all formulas f and g, so in particular it holds for the special case where $f = a$ en $g = b$.

4. Give a formula in the predicate logic that resembles the meaning of this sentence as well as possible: (6 points)

When a man loves women, then there is a woman who loves that man.

Use the dictionary:

M	the domain of men
W	the domain of women
$L(x, y)$	x loves y

Interpret this sentence as a statement about all men.

$$(\forall m \in M\ ((\exists w \in W\ L(m, w)) \to (\exists w \in W\ L(w, m))))$$

5. Give a formula in the predicate logic with equality that resembles the meaning of this sentence as well as possible: (6 points)

There exists exactly one woman that all men love.

Use the dictionary from exercise 4.

We use the additional composite predicate as abbreviation for ‘all men love w’:

$$AML(w) \;:=\; \forall m \in M\ L(m, w)$$

The original sentence can now be translated into the formula:

$$(\exists w \in W\ \ (AML\,(w) \wedge (\forall v \in W\ \ (AML\,(v) \to (v = w)))))$$

or equivalent

$$(\exists w \in women\ \ (\forall v \in W\ \ (AML\,(v) \leftrightarrow (v = w))))$$

If we do not use such an abbreviation we get the formulas:

$$(\exists w \in W\ \ ((\forall m \in M\ L\,(m, w)) \wedge (\forall v \in W\ \ ((\forall m \in M\ L\,(m, v)) \to (v = w)))))$$

$$(\exists w \in W\ \ (\forall v \in W\ \ ((\forall m \in M\ L\,(m, v)) \leftrightarrow (v = w))))$$

6. Give an interpretation I_6 in a model M_6 for which the following formula of the predicate logic with equality holds: (6 points)

$$\forall x \in D\, \exists y \in D\, (x \neq y \wedge \forall z \in D\, (R(x,z) \leftrightarrow z = y))$$

Explain your answer.

This formula means that 'for all x in D there exists exactly one y in D such that $R(x,y)$ holds and this y is not equal to x'. So take for instance model $M_6 := (\mathbb{N}, 1, +)$ and interpretation I_6;

$$\begin{array}{rcl} D & \longrightarrow & \mathbb{N} \\ R(x,y) & \longrightarrow & y = x+1 \end{array}$$

It is clear that within the set of natural numbers it holds that for each natural number there is precisely one element which is exactly one larger and, obviously, this larger element is not equal to the original element.

7. Give a language L_7 with alphabet $\Sigma = \{a, b\}$ for which (6 points)

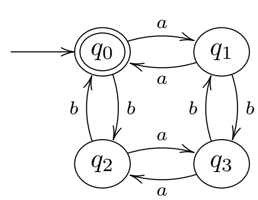
$$L_7{}^* \cap \overline{L_7}{}^* \neq \{\lambda\}$$

Explain your answer.

Take for instance $L_7 = \{a\}$. Obviously $a \in L_7$ and hence also $aaaa \in L_7^*$. In addition we know that $aa \notin L_7$, so $aa \in \overline{L_7}$. But from this it follows that $aaaa \in \overline{L_7}^*$. So $aaaa \in L_7 \cap \overline{L_7}^*$ and $aaaa \neq \lambda$.

8. Give a finite automaton with a minimal number of states that recognizes the language (6 points)

$$L_8 := \mathcal{L}((aa \cup bb \cup (ab \cup ba)(aa \cup bb)^*(ab \cup ba))^*)$$



State q_0 represents the situations in which an even number of a's and an even number of b's have been processed. State q_1 represents the situations in which an odd number of a's and an even number of b's have been processed. State q_2 represents the situations in which an even number of a's and an odd number of b's have been processed. State q_3 represents the situations in which an odd number of a's and an odd number of b's have been processed. Because in all of these four situations different actions need to be taken, there cannot be such an automaton with less states.

9. Consider the following context-free grammar G_9:

$$
\begin{aligned}
S &\to AS \mid \lambda \\
A &\to ab
\end{aligned}
$$

(a) Is G_9 right-linear? Explain your answer. (1 punt)

No, it is not because in the rule $S \to AS$ the non-terminal A is not standing at the far right.

(b) Is $\mathcal{L}(G_9)$ regular? Explain your answer. (1 punt)

Yes, because the language $\mathcal{L}(G_9) = \mathcal{L}((ab)^*)$ and this last one is by definition a regular language.

(c) Give an invariant that proves that (4 points)

$$ba \notin \mathcal{L}(G_9)$$

Explain your answer.

Take

$$P(w) \;:=\; \text{there is an } a \text{ or } A \text{ before the first } b$$

Now $P(S)$ vacuously holds, because there is no (first) b in S hence there is nothing left to prove.

Now assume that v is a word such that $P(v)$ holds and v' is a word such that $v \to v'$. Then either in v an S is being replaced by AS or by λ, or an A is replaced by ab.

- In the case that S is being replaced no A's or a's are removed and no b's are added. Hence any A or a that was before the first b will still be before the first b. So $P(v')$ holds in this case.
- In the case that an A is replaced that is standing before the first b, a 'new first' b appears, but directly in front of this b stands an a. So $P(v')$ also holds in this case.
- In the case that an A is replaced that is not standing before the first b, nothing before the first b changes. And because of $P(v)$ there must be an A or a before this first b, so this will also be the case for v'. Hence $P(v')$ also holds in this case.

So $P(v')$ always holds if $P(v)$ holds and $v \to v'$. This means that P is indeed an invariant of G_9.

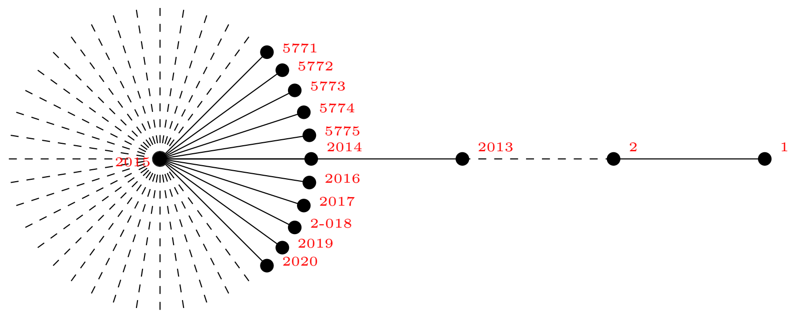
And because $P(ba)$ does not hold, it is immediately clear that $ba \notin \mathcal{L}(G_9)$.

10. Define a tree with 5775 vertices in which the length of the longest path is 2015. Write your answer as a pair $\langle P_{10}, L_{10}\rangle$. (6 points)

Take $\langle P_{10}, L_{10}\rangle$ where

$$
\begin{aligned}
P_{10} &:= \{1, 2, 3, \ldots, 5775\} \\
L_{10} &:= \{(i, i+1) \mid 1 \le i \le 2014\} \cup \{(2015, i) \mid 2016 \le i \le 5775\}
\end{aligned}
$$

This gives a kind of star with a tail: vertex 2015 is the center of the star, vertices 2016 till 5775 (including 5775) around 2015 and a tail from vertex 2015 till vertex 1.



- For each i with $2016 \leq i \leq 5775$ there is exactly one path from vertex i to vertex 2015 of length 1, namely $i \to 2015$.
- For each i with $2016 \leq i \leq 5775$ and each j with $2016 \leq j \leq 5775$ where $i \neq j$ there is exactly one path from vertex i to vertex j of length 2, namely $i \to 2015 \to j$.
- For each i with $2016 \leq i \leq 5775$ and each j with $1 \leq j \leq 2014$ there is exactly one path from vertex i to vertex j of length $2015 - j + 1$, namely $i \to 2015 \to 2014 \to \cdots \to j$.
- Hence the graph is connected and the longest paths have length 2015.
- For each i with $2016 \leq i \leq 5775$ we have that the degree of vertex i is 1. This implies that vertex i is not part of a cycle.
- The degree of vertex 1 is 1, so vertex 1 is also not part of a cycle.
- For each i with $2 \leq i \leq 2014$ the degree of vertex i is 2 By an induction argument we can show that vertex i is not part of a cycle. We start with the base case: if vertex 2 would be part of a cycle, then it must be via the edge $(2, 1)$. But then, vertex 1 would also be in a cycle and we have already seen that this is not the case. In the general case we assume that vertex $i - 1$ is not part of a cycle. Now if vertex i is in a cycle, this must be via the edge $(i, i-1)$. But then $i-1$ is also in a cycle which contradicts the assumption. This proves that such a vertex i is not in a cycle.
- The degree of vertex 2015 is 3761. But for all neighbors of 2015 we know that they are not part of a cycle. Hence vertex 2015 is also not in a cycle.
- So we have established that no vertex is part of a cycle, so the graph does not contain cycles.
- Hence this graph is indeed a tree.

11. This exercise is about the recurrence relations:

$$f(m, 0) = 0$$
$$f(m, n + 1) = f(m, n) + m$$

(a) Show how $f(3, 3)$ is computed using these recurrence relations. (2 points)

$$\begin{aligned} f(3,3) &= f(3, 2+1) \\ &= f(3,2) + 3 \\ &= f(3, 1+1) + 3 \\ &= f(3,1) + 3 + 3 \\ &= f(3, 0+1) + 3 + 3 \\ &= f(3,0) + 3 + 3 + 3 \\ &= 0 + 3 + 3 + 3 \\ &= 9 \end{aligned}$$

(b) Prove with induction on n that (4 points)

$$f(m, n) = m \cdot n$$

for all $m, n \geq 0$.

If we can prove this for arbitrary m without any extra assumptions about this m, we may generalize that this theorem holds for all $m \in \mathbb{N}$. So let m be such an arbitrary element of $\mathbb{N}$. **Proposition:** 0

$f(m, n) = m \cdot n$ for all $n \geq 0$.

Proof by induction on n. 1

We first define our predicate P as:

$$P(n) \quad := \quad f(m, n) = m \cdot n$$ 2

Base Case. We show that $P(0)$ holds, i.e. we show that 3

$f(m, 0) = m \cdot 0$

This indeed holds, because 4

$$f(m, 0) = 0 = m \cdot 0$$

Induction Step. Let k be any natural number such that $k \geq 1$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6

$f(m, k) = m \cdot k$ (Induction Hypothesis IH)

We now show that $P(k + 1)$ also holds, i.e. we show that 7

$f(m, k + 1) = m \cdot (k + 1)$

This indeed holds, because 8

$$\begin{aligned} f(m, k+1) &= f(m,k) + m \\ &\overset{\text{IH}}{=} m \cdot k + m \\ &= m \cdot (k+1) \end{aligned}$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$. 9

12. Jan, Piet and Klaas want to divide six different objects among each other in such a way that everybody gets two objects. In how many ways can they do this? Explain how you calculated this and which binomial coefficients you have used for this. (6 points)

From the text in the exercise it follows that the order in which the persons get the objects doesn't matter, but only which objects. We give an algorithm:

- Jan gets two out of the six objects. This can be done in $\binom{6}{2} = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5}{2} = 15$ ways.
- Piet gets two out of the four remaining objects. This can be done in $\binom{4}{2} = \frac{4!}{2! \cdot 2!} = \frac{4 \cdot 3}{2} = 6$ ways.
- Klaas gets the remaining two objects. This can be done in only 1 way.

Using the product rule this means that the six objects can be divided in $15 \cdot 6 \cdot 1 = 90$ ways.

13. Give a formula in the epistemic logic that formalizes the following sentence as well as possible: (6 points)

> *I know that I won't get wet when it rains, because if I know that it rains I bring an umbrella and if I bring an umbrella I know that I won't get wet.*

Use as dictionary:

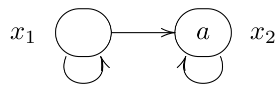
R	it rains
W	I get wet
U	I bring an umbrella

$$\Box(R \to \neg W) \land (\Box R \to U) \land (U \to \Box\neg W)$$

14. Give a serial Kripke model $\mathcal{M}_{14}$ in which the formula of the modal logic $(\Box a \to \Box b) \to \Box(a \to b)$ is not true. Explain your answer. (6 points)

A serial Kripke model is a model in which each world has at least one outgoing arrow. If $(\Box a \to \Box b) \to \Box(a \to b)$ is not true, there must be a world in the model in which it holds that $\Box a \to \Box b$ is true, but $\Box(a \to b)$ is not true. For this last part, there must be a reachable world in which a is true but b is not true. The first part can be achieved by making $\Box a$ not true; in that case it is no longer important whether $\Box b$ is true or not.

Take for instance the model M_{14}:



This model is clearly serial. And the following table shows which formulas are true or false in which worlds,

$\Vdash$	a	b	$a \to b$	$\Box a$	$\Box b$	$\Box a \to \Box b$	$\Box(a \to b)$	$(\Box a \to \Box b) \to \Box(a \to b)$
x_1	0	0	1	0	0	1	0	0
x_2	1	0	0	1	0	0	0	1

Because $x_1 \nVdash (\Box a \to \Box b) \to \Box(a \to b)$ holds, we automatically also have that $M_{14} \nVdash (\Box a \to \Box b) \to \Box(a \to b)$ holds.

15. Give an LTL formula that expresses that a always becomes true again and again, but that between the moments where a is true, always b needs to be true at least once. (6 points)

$$\mathcal{G}\mathcal{F}a \wedge \mathcal{G}(a \to \mathcal{X}(\neg a\, \mathcal{U}\ (b \wedge \neg a)))$$

The first part of the formula represents the fact that a is infinitely often true, because on each moment there will be another moment where a will be true.

The second part of the formula represents that always if a is true, then from the next moment it holds that there will be a moment where b is true and a is not true on that moment and all moments until the moment b becomes true. Hence b must become true at least once before a becomes true again.