Formal Reasoning 2014
Solutions Test 1: Propositional logic
(16/09/14)

In the first three exercises we use the following interpretation of the atomic propositions:

D	it is day
N	it is night
S	the sun shines
M	the moon shines

1. Give two propositions that respectively resemble the meaning of the following two sentences:

   (a) *Either it is day, or it is night, but when the sun shines it is not night.*

   $$(((D \lor N) \land \neg(D \land N)) \land (S \to \neg N))$$

   (b) *The sun only shines when it is day, but the sun doesn't shine now although it is day.*

   $$((S \to D) \land (\neg S \land D))$$

   (10 + 10 points)

2. Write the following formula according to the official grammar in the course notes, and give an English sentence that resembles the meaning of this formula as well as possible:

   $$\neg D \to \neg(S \lor M) \lor M$$

   (20 points)

   $$(\neg D \to (\neg(S \lor M) \lor M))$$

   'If it is not day it holds that it is not the case that the sun shines or the moon shines, or that the moon shines, or both.'

   In better English: 'If it is not day, either nothing shines or the moon shines.'

3. The night is exactly the time of the twenty-four hours when it is not day. Does this imply that the formula

   $$N \leftrightarrow \neg D$$

   is logically true? Explain your answer. (10 points)

   No, this formula is not logically true. Logically true means that the formula is true independent of the interpretation of the symbols. And in this case this interpretation is important. Put differently: below we present the truth table of this formula and in particular we see some zeroes in the last column, which indicates that the formula is not logically true.

N	D	$\neg D$	$N \leftrightarrow \neg D$
0	0	1	0
0	1	0	1
1	0	1	1
1	1	0	0

4. Give the truth table of the formula

$$(a \to b \to c) \to (a \to b) \to (a \to c)$$

and explain how you can see in this table that this formula is logically true.

(20 points)

a	b	c	$b \to c$	$a \to b \to c$	$a \to b$	$a \to c$	$(a \to b) \to (a \to c)$	$(a \to b \to c) \to (a \to b) \to (a \to c)$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	1	0	0	1	1
1	0	1	1	1	0	1	1	1
1	1	0	0	0	1	0	0	1
1	1	1	1	1	1	1	1	1

The last column contains only ones, so the formula is logically true.

5. Provide a model in which the formula

$$\neg(((a \leftrightarrow \neg b \wedge \neg c) \to \neg a) \vee b) \to c$$

does not hold. (10 points)

If such a model v exists, it must be the case that $v(c) = 0$, because otherwise the implication will automatically hold. Furthermore it must be that $v(\neg(((a \leftrightarrow \neg b \wedge \neg c) \to \neg a) \vee b)) = 1$. So $v(((a \leftrightarrow \neg b \wedge \neg c) \to \neg a) \vee b) = 0$. But this implies that $v(b) = 0$ and $v((a \leftrightarrow \neg b \wedge \neg c) \to \neg a) = 0$. Hence $v(\neg a) = 0$ and hence $v(a) = 1$. Checking this valuation gives that:

$v(a) = 1$	$v(\neg b \wedge \neg c) = 1$
$v(b) = 0$	$v(a \leftrightarrow \neg b \wedge \neg c) = 1$
$v(c) = 0$	$v((a \leftrightarrow \neg b \wedge \neg c) \to \neg a) = 0$
$v(\neg a) = 0$	$v(((a \leftrightarrow \neg b \wedge \neg c) \to \neg a) \vee b) = 0$
$v(\neg b) = 1$	$v(\neg(((a \leftrightarrow \neg b \wedge \neg c) \to \neg a) \vee b)) = 1$
$v(\neg c) = 1$	$v(\neg(((a \leftrightarrow \neg b \wedge \neg c) \to \neg a) \vee b) \to c) = 0$

So indeed, the formula does not hold in this model v.

6. Give six formulas of the propositional logic, such that for none of these formulas it holds that it is the logical consequence of one of the other formulas. Explain your answer by writing down what 'logical consequence' means in terms of models. (10 points)

Because there is no restriction on the number of atoms that can be used, this is actually not a difficult exercise. Assume that a, b, c, d, e and f are all different atoms. To prove that an arbitrary formula F is not a logical consequence of an arbitrary formula G, we have to show that there exists a model v in which $v(G) = 1$ and $v(F) = 0$. With our six atomic propositions we can make 64 models, but we will only use six of them:

model	a	b	c	d	e	f
v_1	0	1	1	1	1	1
v_2	1	0	1	1	1	1
v_3	1	1	0	1	1	1
v_4	1	1	1	0	1	1
v_5	1	1	1	1	0	1
v_6	1	1	1	1	1	0

- From model v_1 it follows that a is not a logical consequence of all other formulas, because $v_1(b) = v_1(c) = v_1(d) = v_1(e) = v_1(f) = 1$ and $v_1(a) = 0$.

- Analogously, from model v_2 it follows that a is not a logical consequence of all other formulas, because $v_2(a) = v_2(c) = v_2(d) = v_2(e) = v_2(f) = 1$ and $v_2(b) = 0$.
- Analogously, from model v_3 it follows that c is not a logical consequence of all other formulas.
- Analogously, from model v_4 it follows that d is not a logical consequence of all other formulas.
- Analogously, from model v_5 it follows that e is not a logical consequence of all other formulas.
- Analogously, from model v_6 it follows that f is not a logical consequence of all other formulas.

Hence we have shown that none of these six formulas is a logical consequence of one of the other formulas.