Formal Reasoning 2014
Solutions Test Block 3: Languages & Automata
(22/10/14)

1. Give a regular expression for the language (15 points)

$$L_1 := \{w \in \{a,b\}^* \mid w \text{ contains } ab \text{ and } w \text{ contains } ba\}$$

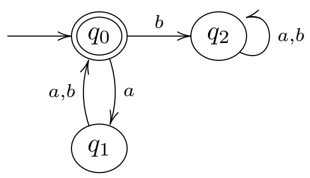
If every word in L_1 has to contain the string ab and the string ba, then somewhere in this word there has to be a sequence of b's between two a's. An expression representing this property is:

$$(a \cup b)^*(abb^*a \cup baa^*b)(a \cup b)^*$$

Note that ab and ba may overlap, hence $aba \in L_1$. (Instead of $(a \cup b)^*$ it is possible to use $(a^*b^*)^*$, but we consider this a strange way to write 'an arbitrary word over the alphabet $\{a,b\}$' using a regular expression.)

2. We define the language $L_2 := \mathcal{L}((aa \cup ab)^*)$.

   (a) Give a state transition diagram for a (deterministic) finite automaton that accepts the language L_2. (15 points)



   (b) Write the same automaton also as a quintuple $\langle \Sigma, Q, q_0, F, \delta \rangle$. (5 points)

   We have $L_2 = L(M_2)$ if $M_2 = \langle \Sigma, Q, q_0, F, \delta \rangle$ where

$$\Sigma = \{a, b\}$$
$$Q = \{q_0, q_1, q_2\}$$
$$F = \{q_0\}$$

   and $\delta : Q \times \Sigma \to Q$ the transition function defined by

$$\delta(q_0, a) = q_1$$
$$\delta(q_0, b) = q_2$$
$$\delta(q_1, a) = q_0$$
$$\delta(q_1, b) = q_0$$
$$\delta(q_2, a) = q_2$$
$$\delta(q_2, b) = q_2$$

3. Give a context-free grammar for the language (15 points)

$$L_3 := \{uvcv^Ru^R \mid u \in \{a,c\}^*, v \in \{b,c\}^*\}$$

(Note that for instance $accbbcbbcca \in L_3$, where $u = ac$ and $v = cbb$.)

We use the non-terminal A to produce the part vcv^R. Both for S and for A we ensure that the left hand side and the right hand side correspond with each other by letting them grow from the central character to the outer characters.

$$S \to aSa \mid cSc \mid A$$
$$A \to bAb \mid cAc \mid c$$

We can produce the word $accbbcbbcca$ from the example like this:

$$S \to aSa \to acSca \to acAca \to accAcca \to accbAbcca \to accbbAbbcca \to accbbcbbcca$$

4. Let G_4 be this context-free grammar:

$$S \to aB \mid aaa \mid bS$$
$$B \to abS \mid bB \mid \lambda$$

   (a) Is G_4 right-linear? Explain your answer. (10 points)

   Yes, the grammar is right-linear. There is no nonterminal on the right hand side of the arrows which is not completely to the right.

   (b) Somebody claims that

   $$P(w) := w \text{ does not contain } aaaa$$

   is an invariant that shows that $aaaa \notin \mathcal{L}(G_4)$. Is this claim correct? Explain your answer. (Note: if it is not a proper invariant, you don't have to provide a proper one.) (10 points)

   This is not an invariant. Consider the word $aaaS$. Then it is clear that $P(aaaS)$ holds. However, $aaaS \to aaaaB$, but $P(aaaaB)$ does not hold. Hence this property is not retained by all transitions.

   A property that is an invariant is

$$P(w) := \text{either } w \in \{S, aB, a, aaa\}, \text{ or } w \text{ ends on a word in } \{bS, bB, baB, b, ba, baaa\}$$

5. Let L be a regular language.

   (a) Does it hold that if $L \subseteq L'$, then L' also has to be a regular language? Explain your answer. (5 points)

   No, this does not hold. Take $L = \emptyset$. The empty language is regular because $\emptyset = \mathcal{L}(\emptyset)$. Furthermore we know that language $L' = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular. However, it does hold that $L \subseteq L'$.

   (b) Does it hold that if $L' \subseteq L$, then L' also has to be a regular language? Explain your answer. (5 points)

   Again, no, this does not hold. Take $L = \{a, b\}^*$. Then this language is regular because $L = \mathcal{L}((a \cup b)^*)$. Furthermore we know that language $L' = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular. However, it does hold that $L' \subseteq L$.

6. Does the equality below holds for any language L? (10 points)

$$L^* = \{\lambda\} \cup LL^*$$

Explain your answer.

Yes, this equality holds for any language L. The language L^* consists of zero or more concatenations of words from L. The part with zero concatenations is exactly equal to $\{\lambda\}$; the part with one or more concatenations is exactly equal to LL^*, where the L stands for the obligatory concatenation and the L^* for all other concatenations.

Written down a bit more formally:

$$\begin{aligned}
L &= \{w_1 w_2 \cdots w_k \mid k \in \mathbb{N} \text{ and } w_i \in L \text{ for all } i \in \{1, 2, \ldots, k\}\} \\
&= \{w_1 w_2 \cdots w_k \mid k \in \mathbb{N} \text{ and } k = 0 \text{ and } w_i \in L \text{ for all } i \in \{1, 2, \ldots, k\}\} \\
&\quad \cup \\
&\quad \{w_1 w_2 \cdots w_k \mid k \in \mathbb{N} \text{ and } k > 0 \text{ and } w_i \in L \text{ for all } i \in \{1, 2, \ldots, k\}\} \\
&= \{\lambda\} \cup \{w_1 v_1 \cdots v_{k-1} \mid k \in \mathbb{N} \text{ and } k > 0 \text{ and } w_1 \in L \text{ and } v_i \in L \text{ for all } i \in \{1, 2, \ldots, k-1\}\} \\
&= \{\lambda\} \cup L\{v_1 \cdots v_{k-1} \mid k \in \mathbb{N} \text{ and } k > 0 \text{ and } v_i \in L \text{ for all } i \in \{1, 2, \ldots, k-1\}\} \\
&= \{\lambda\} \cup L\{v_1 \cdots v_{k'} \mid k' \in \mathbb{N} \text{ and } v_i \in L \text{ for all } i \in \{1, 2, \ldots, k'\}\} \\
&= \{\lambda\} \cup LL^*
\end{aligned}$$