Formal Reasoning 2014
Solutions Test Block 4: Discrete mathematics
(2/12/14)

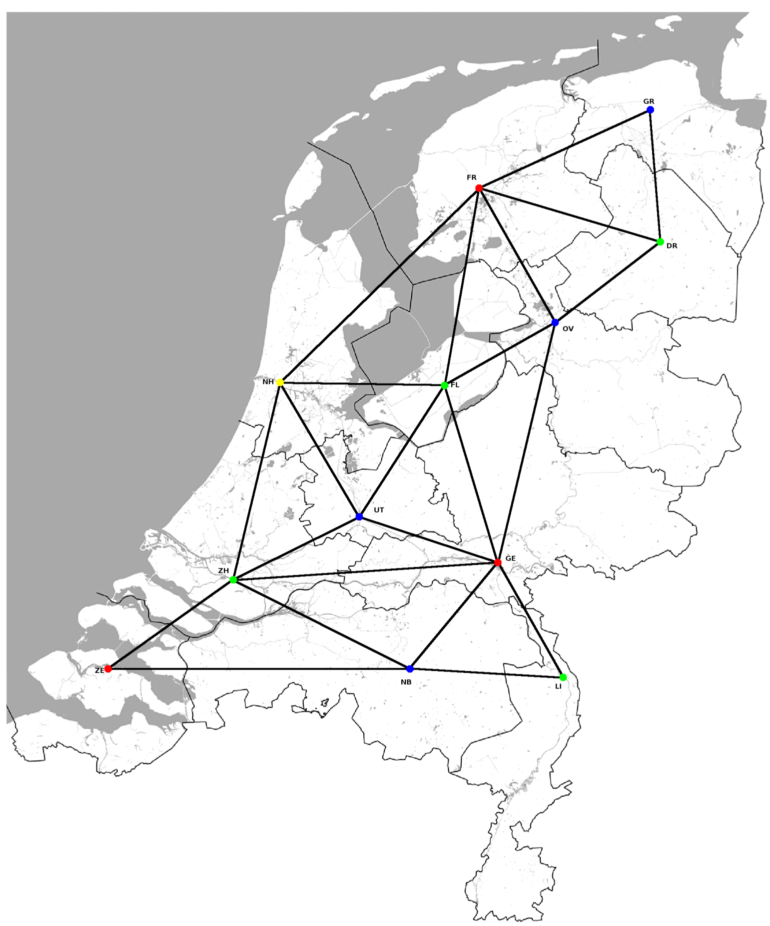
1. We define the graph

$$G_1 = \langle \{x \mid x \text{ is a province of the Netherlands}\},$$
$$\{(x, y) \mid x \text{ shares a border with } y\}\rangle$$

(see the map in the appendix) This graph has 12 vertices and 23 edges.

(a) Draw the graph G_1 on the appendix. (Place the vertices into the provinces.) (10 points)

This is graph G_1:



(b) Give the chromatic number of G_1. Explain your answer. (10 points)

Because it is a planar graph by applying the four color theorem, we know that the chromatic number is at most four. However, it is not possible to color the graph with only three colors. Look at the subgraph $\langle P_2, L_2 \rangle$ where

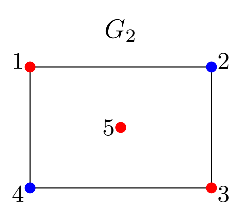
$$\begin{aligned} P_2 &= \{FR, NH, UT, GE, OV, FL\} \\ L_2 &= \{(FR, NH), (NH, UT), (UT, GE), (GE, OV), (OV, FR), \\ &\quad (FR, FL), (NH, FL), (UT, FL), (GE, FL), (OV, FL)\} \end{aligned}$$

If we look carefully, we see five triangles next to each other with FL as center. (This graph is known in the literature as W_6, the wheel graph with six vertices.) It is immediately clear that this center vertex needs to have a different color than the ones on the outer boundary. So without loss of generality we may assume that FL is green and FR is red. If we follow the outer boundary clockwise, then OV must be blue, GE red, UT blue and NH red. But because (NH, FR) is also an edge, NH and FR can't be both red. Hence it is not possible to color this graph with only three colors and hence the chromatic number is four.

(c) Does this graph contain a subgraph isomorphic to $K_{2,2}$? Explain your answer. (10 points)

Yes, because $K_{2,2}$ is basically nothing but a cycle of length four and it contains plenty of those. Take for instance $GE \to NB \to ZE \to ZH \to GE$.

2. Give a planar bipartite graph that does have an Euler circuit, but does not have a Hamilton path. Explain your answer. (10 points)



- This graph is planar because it is drawn without crossing edges.
- This graph is bipartite because it can be divided into the sets $\{1, 3, 5\}$ of red vertices and $\{2, 4\}$ of blue vertices, where all edges have one red endpoint and one blue endpoint.
- This graph does have an Euler circuit, because the cycle $1 \to 2\ to 3 \to 4 \to 1$ uses every edge exactly once.
- This graph has no Hamilton path since there is no path that visits vertex 5.

3. We define a sequence of numbers $(a_n)_{n\in\mathbb{N}}$ via these recurrence relations:

$$a_0 = 1$$
$$a_{n+1} = 3a_n - 1 \qquad \text{for } n \geq 0$$

(a) Compute the value a_6 (without using the formula below). Explain how you derived this value. (15 points)

$$\begin{array}{lllllll}
a_0 & = & 1 & & & & \\
a_1 & = & 3\cdot a_0 - 1 & = & 3\cdot 1 - 1 & = & 2 \\
a_2 & = & 3\cdot a_1 - 1 & = & 3\cdot 2 - 1 & = & 5 \\
a_3 & = & 3\cdot a_2 - 1 & = & 3\cdot 5 - 1 & = & 14 \\
a_4 & = & 3\cdot a_3 - 1 & = & 3\cdot 14 - 1 & = & 41 \\
a_5 & = & 3\cdot a_4 - 1 & = & 3\cdot 41 - 1 & = & 122 \\
a_6 & = & 3\cdot a_5 - 1 & = & 3\cdot 122 - 1 & = & 365
\end{array}$$

(b) Prove by induction that

$$a_n = \frac{1}{2}(3^n + 1)$$

holds for all $n \geq 0$. (15 points)

Proposition:

$a_n = \frac{1}{2}(3^n + 1)$ for all $n \geq 0$.

Proof by induction on n. 1

We first define our predicate P as:

$$P(n) \quad := \quad a_n = \tfrac{1}{2}(3^n + 1)$$ 2

Base Case. We show that $P(0)$ holds, i.e. we show that 3

$a_0 = \frac{1}{2}(3^0 + 1)$

This indeed holds, because $a_0 = 1$ and $\frac{1}{2}(3^0 + 1) = \frac{1}{2}(1 + 1) = \frac{1}{2}\cdot 2 = 1$. 4

Induction Step. Let k be any natural number such that $k \geq 0$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6

$a_k = \frac{1}{2}(3^k + 1)$ (Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7

$a_{k+1} = \frac{1}{2}(3^{k+1} + 1)$

This indeed holds, because 8

$$
\begin{aligned}
a_{k+1} &= 3a_k - 1 \\
&\overset{\text{IH}}{=} 3 \cdot \frac{1}{2}(3^k + 1) - 1 \\
&= \frac{1}{2} \cdot 3(3^k + 1) - 1 \\
&= \frac{1}{2}(3 \cdot 3^k + 3) - \frac{1}{2} \cdot 2 \\
&= \frac{1}{2}(3^{k+1} + 3) - \frac{1}{2} \cdot 2 \\
&= \frac{1}{2}(3^{k+1} + 3 - 2) \\
&= \frac{1}{2}(3^{k+1} + 1)
\end{aligned}
$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$. 9

4. A group of twelve students want to play volleyball. In how many ways can they divide this group into two teams of six players each? Explain how you computed your answer. (20 points)

   We assume it doesn't matter who plays on which position within the team, so the question is purely about who will be together with whom in the same team. For the first team we have to choose six players. This can be done in

$$
\begin{aligned}
\binom{12}{6} &= \frac{12!}{6! \cdot 6!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{6 \cdot 5 \cdot 2} \\
&= \frac{11 \cdot 9 \cdot 8 \cdot 7}{6} = \frac{11 \cdot 3 \cdot 8 \cdot 7}{2} = 11 \cdot 3 \cdot 4 \cdot 7 \\
&= 924
\end{aligned}
$$

   ways.

   However, in this computation it is not taken into account that choosing six players for the first team, gives exactly the same teams as choosing the other six members for the first team. So every possibility is counted twice, which means that there are in fact only $\frac{924}{2} = 462$ possibilities to create two teams.