Formal Reasoning 2015
Solutions Exam
(27/01/16)

1. Consider the following formula in the propositional logic:

$$\neg a \vee b \leftrightarrow a \to b$$

   (a) Write this formula according to the official grammar in the course notes. (3 points)

   According to the official grammar this formula should be written as:

$$((\neg a \vee b) \leftrightarrow (a \to b))$$

   (b) Give the truth table of this formula. (3 points)

a	b	$\neg a$	$\neg a \vee b$	$a \to b$	$(\neg a \vee b) \leftrightarrow (a \to b)$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	0	1	1	1

2. Consider the following fragment of the song *Nederwiet* by the Dutch band *Doe Maar*:[1] (6 points)

   Because the leaves, they give you a headache. [...]
   So these leaves, you throw them away, away, away!
   And you only smoke the covers of the seeds
   and... or... and/or the flowers.

   Give a formula in the propositional logic that resembles the meaning of these lyrics as well as possible. Use the dictionary:

SL	you smoke leaves
SC	you smoke covers of seeds
SF	you smoke flowers
H	you get a headache

$$(SL \to H) \wedge \neg SL \wedge (SC \vee SF)$$

3. Consider the following statement:

$$f \equiv a \to b$$

(a) What is the meaning of this statement? (3 points)

The statement says that f is logically equivalent with the formula $a \to b$. Or in other words: f has the same truth table as $a \to b$.

(b) Give a formula f in the propositional logic that complies to this statement, but which does *not* contain the connective '$\to$'. Explain your answer. (3 points)

Take $f := \neg a \vee b$. In exercise 1b we have already shown that the formulas $\neg a \vee b$ and $a \to b$ have the same truth table.

4. Give a formalization of the fragment of the lyrics by *Doe Maar* from exercise 2 in the predicate logic, and use the dictionary: (6 points)

B	domain of the members of the band *Doe Maar*
P	domain of the parts of a cannabis plant
j	you = the person about whom this song is
$L(x)$	x is a leave
$C(x)$	x is a cover of a seed
$F(x)$	x is a flower
$S(x, y)$	x smokes y
$H(x)$	x gets a headache

(For example, the formula $F(j)$ has the meaning 'you are a flower'.)

$$\begin{aligned} & \forall b \in P \left[L(b) \wedge S(j, b) \to H(j)\right] \\ \wedge \quad & \forall x \in P \left[S(j, x) \to \neg L(x) \wedge \big(C(x) \vee F(x)\big)\right] \end{aligned}$$

5. Give a formula in the predicate logic that expresses that the band *Doe Maar* has more than two members. Use the dictionary from the previous exercise. (6 points)

$$\exists e \in B \exists h \in B \left[\neg(e = h) \wedge \exists x \in B \left[\neg(e = x) \wedge \neg(h = x)\right]\right]$$

The persons e and h are two different members of the band and in addition person x is also a member, who is not person e and also not person h.

6. Give an interpretation I_6 in a model M_6 for which the following formula holds: (6 points)

$$\forall x \in D \exists y \in D \left(R(x, y) \wedge \forall y' \in D \left(R(x, y') \to y' = y\right)\right)$$

Explain your answer.

[1] This text is part of this exam because of the language construction 'and/or' that is being used and not because of the other lyrics of the song. The lecturers of this course strongly reject any kind of drug use.

The formula says that for each element x in D there is exactly one element y in D such that $R(x, y)$ holds.

Take as model M_6

Domain(s)	Natural numbers
Relation(s)	equality (=)

and as interpretation I_6

D	$\mathbb{N}$
$R(x, y)$	$y = x + 37$

The given formula holds under this interpretation I_6 in model M_6, because for each arbitrary natural number x $R(x, y)$ holds exactly if $y = x + 37$. And for each natural number x it holds that $x + 37$ is also a natural number.

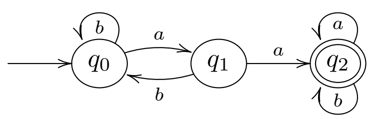
7. Give a language L_7 with alphabet $\{a, b\}$ for which $L_7^* \neq L_7$ but $L_7 L_7 = L_7$. Explain your answer. (6 points)

Take $L_7 = \emptyset$. Concatenation of $\emptyset$ with an arbitrary language gives $\emptyset$ as result. So $L_7 L_7 = \emptyset\emptyset = \emptyset = L_7$. But $\lambda \in L_7^*$ and $\lambda \notin L_7$, so indeed $L_7^* \neq L_7$.

8. Give a finite automaton that recognizes the language (6 points)

$$L_8 := \mathcal{L}((a \cup b)^* aa(a \cup b)^*)$$

Language L_8 consists of all words containing the string aa. So only after reading two consecutive a's, we may end up in a final state.



9. Consider the following context-free grammar G_9:

$$S \to Sa \mid \lambda$$

(a) Is this grammar right-linear? Explain your answer. (2 points)

A grammar is right-linear if all nonterminals on the right side of the arrow are on the rightmost position. This is not the case for $S \to Sa$, so this grammar is not right-linear.

(b) Is the language $\mathcal{L}(G_9)$ regular? Explain your answer. (2 points)

$\mathcal{L}(G_9)$ is the language where each word consists of zero or more a's. So $\mathcal{L}(G_9) = \mathcal{L}(a^*)$ and hence the language is regular.

(c) Somebody claims that (2 points)

$$P(w) := w \text{ does not contain } b$$

is an invariant of G_9 that can be used to prove that

$$abba \notin \mathcal{L}(G_9)$$

Is this a correct claim? Explain your answer.

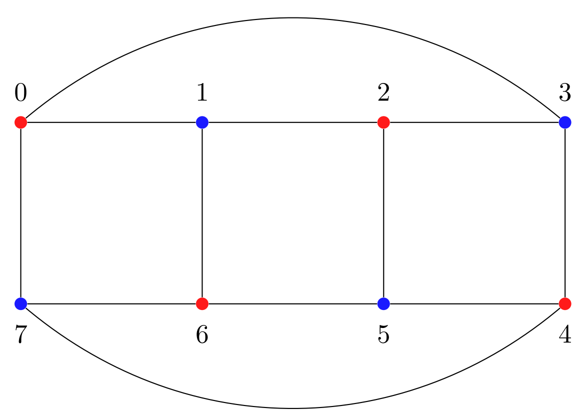
Yes, this is a correct claim.

- S does not contain b, so $P(S)$ holds.
- Now assume that v and w are words such that $P(v)$ holds and $v \to w$. We will show that it then follows that $P(w)$ also holds. If $v \to w$ then there must be an S in v that is being rewritten. So we know that $v = v_1 S v_2$ where $v_1, v_2 \in \{a, S\}^*$. And if $P(v)$ holds, we know that v does not contain any b. But this automatically implies that $P(v_1)$ and $P(v_2)$ also hold. Now we have two possibilities for $v \to w$:
  - $v_1 S v_2 \to v_1 S a v_2$. Because $P(v_1)$ and $P(v_2)$ hold and the string Sa also does not contain any b it follows that $P(v_1 S a v_2)$ holds and hence $P(w)$ also.
  - $v_1 S v_2 \to v_1 v_2$. Because $P(v_1)$ and $P(v_2)$ hold, it follows that $P(v_1 v_2)$ holds and hence $P(w)$ also.

  So P is indeed an invariant of the grammar G_9.
- In addition $P(abba)$ clearly does not hold, so $abba$ cannot have been produced with the grammar G_9. And hence $abba \notin \mathcal{L}(G_9)$.

10. (a) Give a connected planar bipartite graph G_{10} with eight vertices, such that each vertex has degree three. Explain your answer. (Make sure that you do not forget any properties!) (2 points)

Take for instance G_{10}:



This graph complies to all requirements:

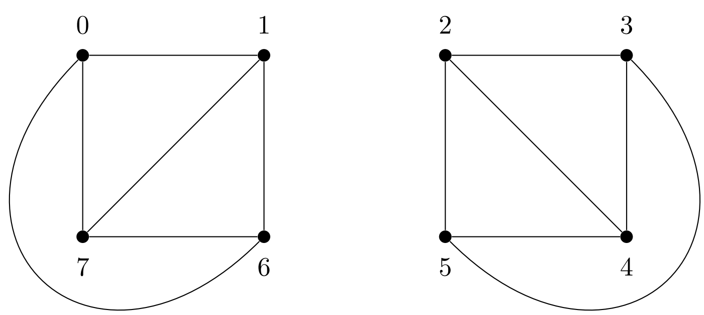
- It has eight vertices, namely $\{0, 1, 2, 3, 4, 5, 6, 7\}$.
- It is connected, because you can get from each point i to each point j via the path $i \to i+1 \to i+2 \to \cdots \to j-1 \to j$ if we compute modulo 8.
- The graph is planar because there are no crossing edges.
- The graph is bipartite because there exists a vertex coloring that uses only two colors: red for all even numbered vertices and blue for all odd numbered vertices.
- Each vertex has degree three. Here we present a table of the vertices each vertex is connected to by an edge. And apparently every vertex is connected to three vertices.

vertex	connected to
0	$1, 3, 7$
1	$0, 2, 6$
2	$1, 3, 5$
3	$0, 2, 4$
4	$3, 5, 7$
5	$2, 4, 6$
6	$1, 5, 7$
7	$0, 4, 6$

Note that this is actually the hypercube graph Q_3.

(b) Give a *not* connected planar graph G'_{10} with eight vertices, where every vertex has degree three. Explain your answer. (Make sure that you do not forget any properties!) (2 points)

Take for instance G'_{10}:



This graph complies to all requirements:

- It has eight vertices, namely $\{0, 1, 2, 3, 4, 5, 6, 7\}$.

- It is not connected, because there is no path from vertex 2 to vertex 6.
- The graph is planar because there are no crossing edges.
- Each vertex has degree three. Here we present a table of the vertices each vertex is connected to by an edge. And apparently every vertex is connected to three vertices.

vertex	connected to
0	$1, 6, 7$
1	$0, 6, 7$
2	$3, 4, 5$
3	$2, 4, 5$
4	$2, 3, 5$
5	$2, 3, 4$
6	$0, 1, 7$
7	$0, 1, 6$

(c) Indicate for each graph whether it has a Hamilton cycle or not. (2 points)

Graph G_{10} has a Hamilton cycle. The cycle $0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 0$ visits each vertex exactly once (except, of course, the begin and end point).

Graph G'_{10} doesn't have a Hamilton cycle. Because there is no path from vertex 2 to vertex 6, there is certainly no cycle that visits each vertex exactly once, because such a cycle would visit both the vertex 2 and vertex 6 and hence there would have been a path from vertex 2 to vertex 6 in that case.

11. Consider the following recursive definition of the sequence of numbers a_n:

$$a_0 = 3$$
$$a_{n+1} = 3a_n - 3 \qquad \text{voor } n \geq 0$$

(a) Compute a_5 using this recursive definition. (3 points)

$$\begin{array}{lllllll} a_0 & = & 3 & & & & \\ a_1 & = & 3a_0 - 3 & = & 3 \cdot 3 - 3 & = & 6 \\ a_2 & = & 3a_1 - 3 & = & 3 \cdot 6 - 3 & = & 15 \\ a_3 & = & 3a_2 - 3 & = & 3 \cdot 15 - 3 & = & 42 \\ a_4 & = & 3a_3 - 3 & = & 3 \cdot 42 - 3 & = & 123 \\ a_5 & = & 3a_4 - 3 & = & 3 \cdot 123 - 3 & = & 366 \end{array}$$

(b) Prove that (3 points)

$$a_n = \tfrac{3}{2}(3^n + 1)$$

for all $n \geq 0$.

Proposition: 0

$a_n = \frac{3}{2}(3^n + 1)$ for all $n \geq 0$.

Proof by induction on n. 1

We first define our predicate P as:

$$P(n) \;:=\; a_n = \frac{3}{2}(3^n + 1)$$ 2

Base Case. We show that $P(0)$ holds, i.e. we show that 3

$a_0 = \frac{3}{2}(3^0 + 1)$

This indeed holds, because 4

$$a_0 = 3 = \frac{3}{2} \cdot 2 = \frac{3}{2}(1+1) = \frac{3}{2}(3^0 + 1)$$

Induction Step. Let k be any natural number such that $k \geq 1$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6

$a_k = \frac{3}{2}(3^k + 1)$ (Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7

$a_{k+1} = \frac{3}{2}(3^{k+1} + 1)$

This indeed holds, because 8

$$\begin{aligned}
a_{k+1} &= 3a_k - 3 \\
&\overset{\text{IH}}{=} 3 \cdot \frac{3}{2}(3^k + 1) - 3 \\
&= \frac{3}{2}(3^{k+1} + 3) - 3 \\
&= \frac{3}{2} \cdot 3^{k+1} + \frac{3}{2} \cdot 3 - 3 \\
&= \frac{3}{2} \cdot 3^{k+1} + \frac{3}{2} \cdot 3 - \frac{2}{2} \cdot 3 \\
&= \frac{3}{2} \cdot 3^{k+1} + \frac{9}{2} - \frac{6}{2} \\
&= \frac{3}{2} \cdot 3^{k+1} + \frac{3}{2} \\
&= \frac{3}{2}(3^{k+1} + 1)
\end{aligned}$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$. 9

12\. (a) Indicate in Pascal's triangle where you can find the binomial coefficients $\binom{6}{k}$. (3 points)

```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
[1]   [6]  [15]  [20]  [15]  [6]   [1]
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(b) Compute (3 points)

$$\binom{6}{0} + \binom{6}{1}3 + \binom{6}{2}3^2 + \binom{6}{3}3^3 + \binom{6}{4}3^4 + \binom{6}{5}3^5 + \binom{6}{6}3^6$$

According to Newton's Binomium we have:

$$\begin{aligned} & \binom{6}{0} + \binom{6}{1}3 + \binom{6}{2}3^2 + \binom{6}{3}3^3 + \binom{6}{4}3^4 + \binom{6}{5}3^5 + \binom{6}{6}3^6 \\ = \;& (1+3)^6 \\ = \;& 4^6 \\ = \;& 4096 \end{aligned}$$

13. Give a formula in the epistemic logic that formalizes the following sentence as well as possible: (6 points)

Either I know that Klaas is the mole, or I know that Klaas is not the mole, but I don't know which one of these two it is.

Use as dictionary:

K Klaas is the mole

$$((\Box K \vee \Box\neg K) \wedge \neg(\Box K \wedge \Box\neg K)) \wedge (\neg\Box\Box K \wedge \neg\Box\Box\neg K)$$

Explanation:

- $\Box K \vee \Box\neg K$ means 'I know that Klaas is the mole or I know that Klaas is not the mole, or both'.
- $\neg(\Box K \wedge \Box\neg K)$ is needed for the 'either…or…' and excludes the 'both'.
- $\neg\Box\Box K \wedge \neg\Box\Box\neg K$ indicates that I don't know whether the first ($\Box K$) holds, but also that I don't know whether the second ($\Box\neg K$) holds.

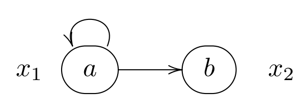
14. Give a Kripke model $\mathcal{M}_{14}$ in which the formula of the modal logic (6 points)

$$\Box(a \vee b) \to \Box a \vee \Box b$$

is not true. Explain your answer.

Take for instance:

$\mathcal{M}_{14}$:



Explanation:

$\Vdash$	a	b	$a \vee b$	$\Box(a \vee b)$	$\Box a$	$\Box b$	$\Box a \vee \Box b$	$\Box(a \vee b) \to \Box a \vee \Box b$
x_1	1	0	1	1	0	0	0	0
x_2	0	1	1	1	1	1	1	1

From the table it follows that $x_1 \nVdash \Box(a \vee b) \to \Box a \vee \Box b$. But from this it follows that $\mathcal{M}_{14} \nvDash \Box(a \vee b) \to \Box a \vee \Box b$.

15. Give an LTL formula that describes the situation in which always if a is true and b is one moment later true, c will be true another moment later. (6 points)

$$\mathcal{G}(a \wedge \mathcal{X}b \rightarrow \mathcal{X}\mathcal{X}c)$$