Formal Reasoning 2015
Solutions Additional Test
(13/01/16)

1. The operator $|$ named *Sheffer stroke* is defined in such a way that $f \mid g \equiv \neg(f \wedge g)$. Give a formula f_1 in the propositional logic that has the Sheffer stroke as its only connective (so in particular the use of $\neg$, $\wedge$, et cetera is prohibited) such that $f_1 \equiv a \vee b$.

   This *Sheffer stroke* is also known as NAND. In particular it is *functionally complete*, which means that each of the normal operators in the propositional logic can be expressed with formulas using only this Sheffer stroke. For instance like this: [1]

$$\begin{array}{rcl}
\neg f & \equiv & \neg(f \wedge f) \\
& \equiv & f \mid f \\
f \wedge g & \equiv & \neg\neg(f \wedge g) \\
& \equiv & \neg(f \mid g) \\
& \equiv & (f \mid g) \mid (f \mid g) \\
f \vee g & \equiv & \neg\neg f \vee \neg\neg g \\
& \equiv & \neg(\neg f \wedge \neg g) \\
& \equiv & \neg f \mid \neg g \\
& \equiv & (f \mid f) \mid (g \mid g) \\
f \to g & \equiv & \neg f \vee g \\
& \equiv & \neg f \vee \neg\neg g \\
& \equiv & \neg(f \wedge \neg g) \\
& \equiv & f \mid \neg g \\
& \equiv & f \mid (g \mid g) \\
f \leftrightarrow g & \equiv & (f \to g) \wedge (g \to f) \\
& \equiv & (f \mid (g \mid g)) \wedge (g \mid (f \mid f)) \\
& \equiv & ((f \mid (g \mid g)) \mid (g \mid (f \mid f))) \mid ((f \mid (g \mid g)) \mid (g \mid (f \mid f)))
\end{array}$$

   From this list it follows that we can take

$$f_1 = (a \mid a) \mid (b \mid b)$$

2. Give a model in which the following formula of the predicate logic is true.

$$\begin{array}{l}
(\forall x \in D\, \exists y \in D\, \forall y' \in D\, [R(x, y') \leftrightarrow y' = y]) \wedge \\
(\forall x_1, x_2, y \in D\, [R(x_1, y) \wedge R(x_2, y) \to x_1 = x_2]) \wedge \\
(\exists z \in D\, \forall x \in D\, \neg R(x, z))
\end{array}$$

   The formula has three simultaneous requirements:

[1] These are not the shortest ways to express these operators using only the Sheffer stroke.

- For each $x \in D$ there is exactly one $y \in D$ such that $R(x, y)$.
- These y's are unique.
- There exists a $z \in D$ for which no $x \in D$ exists such that $R(x, z)$.

Take as model M_2

Domain(s)	natural numbers
Relation(s)	equality (=)

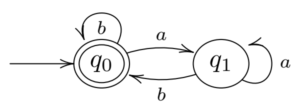
and as interpretation I_2

D	$\mathbb{N}$
$R(x, y)$	$x + 1 = y$

This complies with the first requirement, because given x we can take $y = x + 1$, which is also a natural number. It also complies with the second requirement, because if $x_1 + 1 = y$ and $x_2 + 1 = y$, it follows that $x_1 + 1 = x_2 + 1$ and hence in particular $x_1 = x_2$. And it also complies with the third requirement, because take $z = 0 \in \mathbb{N}$. It is clear that there is no $x \in \mathbb{N}$ such that $x + 1 = 0$.

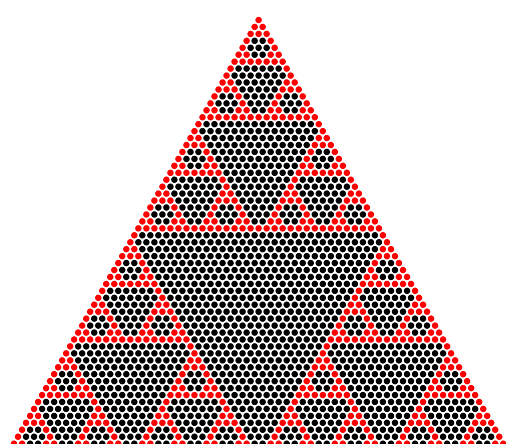
3. Give a finite automaton with a minimal number of states that recognizes the language $\mathcal{L}((a^*b)^*)$.

   The language consists of the words λ and the words that end with a b. An automaton for this language is:



   That the number of states is minimal, follows from the observation that there have to be both final and non-final states. So there must be at least two states. And this automaton has exactly two states.

4.



Above we have 2016 dots, because $(63 \cdot 64)/2 = 2016$. Determine using recursion how many of these dots are red.

The triangle is recursively constructed from smaller triangles. Let r_n be the number of red dots in a triangle of degree n, where this degree indicates the number of levels of triangles that is used to construct the whole triangle. We start with a triangle of degree 1.[2] This is a triangle with six red dots as can be seen in the lower left. So $r_1 = 6$. A triangle of degree $n+1$ is constructed by combining three triangles of degree n using exactly three red dots and fill the remaining space with black dots. So our recursive definition for the number of red dots becomes:

$$r_{n+1} = 3 \cdot r_n + 3$$

The presented triangle is of degree 5. Using the recursive definition we can easily compute r_5:

$$\begin{array}{lllllll}
r_1 & = & 6 & & & & \\
r_2 & = & 3 \cdot r_1 + 3 & = & 3 \cdot 6 + 3 & = & 21 \\
r_3 & = & 3 \cdot r_2 + 3 & = & 3 \cdot 21 + 3 & = & 66 \\
r_4 & = & 3 \cdot r_3 + 3 & = & 3 \cdot 66 + 3 & = & 201 \\
r_5 & = & 3 \cdot r_4 + 3 & = & 3 \cdot 201 + 3 & = & 606
\end{array}$$

5. Give an LTL formula f_5 such that the only Kripke model of f_5 with $V(x_i) \subseteq \{a, b\}$ is the model where

$$V(x_i) = \begin{cases} \{a\} & \text{if } i \text{ is even} \\ \{b\} & \text{if } i \text{ is odd} \end{cases}$$

So we have to find a formula which has as its only model the model such that $x_0 \Vdash (a \wedge \neg b)$, $x_1 \Vdash (b \wedge \neg a)$, $x_2 \Vdash (a \wedge \neg b)$, $x_3 \Vdash (b \wedge \neg a)$, ...So in x_0 a must be true, but b must be false. And after this, the truth values of a and b need to alternate. This can be achieved using the formula

$$f_5 = a \wedge \mathcal{G}(a \leftrightarrow \neg b) \wedge \mathcal{G}(a \rightarrow \mathcal{X}(\neg a \wedge b)) \wedge \mathcal{G}(b \rightarrow \mathcal{X}(\neg b \wedge a))$$

The first part of this formula ensures that $V(x_0) = \{a\}$. The second part of this formula ensures that in each world we have that either a is true or b is true, but never both at the same time. The third part of the formula ensures that if $V(x_i) = \{a\}$, then automatically $V(x_{i+1}) = \{b\}$. And the fourth part of the formula ensures that if $V(x_i) = \{b\}$, then automatically $V(x_{i+1}) = \{a\}$.

[2] It is also possible to start with a triangle of degree 0, which consists of exactly one red dot, but because this doesn't really resemble a triangle, we have chosen to start with degree 1.