Formal Reasoning 2016
Solutions Exam
(17/01/17)

1. Consider the formula f_1 of propositional logic:

$$\neg\neg a \leftrightarrow (a \to \neg b \wedge b) \to \neg b \wedge b$$

   (a) Write this formula according to the official grammar of propositional logic formulas from the course notes. (3 points)

$$\underline{\left(\neg\neg a \leftrightarrow \underline{\left(\underline{\left(a \to \underline{(\neg b \wedge b)}\right)} \to \underline{(\neg b \wedge b)}\right)}\right)}$$

   (b) Give the full truth table of this formula. (3 points)

a	b	$\neg a$	$\neg\neg a$	$\neg b$	$\neg b \wedge b$	$a \to \neg b \wedge b$	$(a \to \neg b \wedge b) \to \neg b \wedge b$	$\neg\neg a \leftrightarrow (a \to \neg b \wedge b) \to \neg b \wedge b$
0	0	1	0	1	0	1	0	1
0	1	1	0	0	0	1	0	1
1	0	0	1	1	0	0	1	1
1	1	0	1	0	0	0	1	1

2. Give a formula f_2 of propositional logic that as best as possible formalizes the meaning of the following English text: (6 points)

   'Orthorhombic ice' is the state of water when the temperature is below 72 K and the pressure is below 100 MPa. It is ferroelectric.

   Use for this the dictionary:

I	water is orthorhombic ice (= 'ice XI')
F	water is ferroelectric
K	the temperature is below 72 K
M	the pressure is below 100 MPa

   (In reality transitions between different types of ice are more complicated than this, but this text roughly corresponds to what is shown in a phase diagram of water on Wikipedia.)

$$(I \leftrightarrow (K \wedge M)) \wedge (I \to F)$$

3. The *principle of explosion* states that for all formulas of propositional logic f and g: (6 points)

$$f \wedge \neg f \vDash g$$

   Does this principle hold? Explain your answer in terms of the definition of $\vDash$.

   Yes, it holds. $f \wedge \neg f \vDash g$ means that in all valuations v such that $v(f \wedge \neg f) = 1$ we also must have that $v(g) = 1$. This holds vacuously, since there is no valuation v such that $v(f \wedge \neg f) = 1$, so there is nothing left to check for $v(g)$. Note that if $v(f) = 1$ then $v(\neg f) = 0$ and hence $v(f \wedge \neg f) = 0$ and if $v(f) = 0$ then $v(\neg f) = 1$ and hence $v(f \wedge \neg f) = 0$.

4. Consider the following formula f_4 of predicate logic:

$$\forall t \in T[J(t) \to W(t)] \wedge \exists t \in T[W(t) \wedge \neg J(t)]$$

(a) Write this formula according to the official grammar of predicate logic formulas from the course notes. (3 points)

$$\Big(\big(\forall t \in T\ (J(t) \to W(t))\big) \wedge \big(\exists t \in T\ (W(t) \wedge \neg J(t))\big)\Big)$$

(b) Give the meaning of this formula as an English sentence. Use for this the dictionary: (3 points)

T	the domain of points of time
$J(x)$	x is in January
$W(x)$	x is in winter

For each moment in time it holds that if this moment is in January, then it is in winter, but there exists a moment in time that is in winter but not in January.

Or less formal:

Each moment in January is also in winter, but not every moment in winter is in January.

Or even less formal:

January is a strict subset of winter.

5. Give a formula f_5 of predicate logic with equality that as best as possible formalizes the meaning of the following English sentence: (6 points)

Every natural number larger than zero has exactly one predecessor.

Use for this the dictionary:

N	the domain of natural numbers
z	the number zero
$L(x, y)$	x is less than y
$P(x, y)$	x is a predecessor of y

$$\Bigg(\forall x \in N\ \Bigg(L(z, x) \to \Bigg(\exists y \in N\ \Bigg(P(y, x) \wedge \Big(\forall u \in N\ \big(P(u, x) \to (u = y)\big)\Big)\Bigg)\Bigg)\Bigg)\Bigg)$$

6. Give an interpretation I_6 in a model M_6 in which the following formula of predicate logic is true: (6 points)

$$[\exists x \in D\, \forall y \in E\, \neg R(x, y)] \wedge [\forall y \in E\, \exists x \in D\, R(x, y)] \wedge$$
$$[\forall x \in D\, \forall y, y' \in E\, (R(x, y) \wedge R(x, y') \to y = y')]$$

Explain your answer.

(If you do not succeed in satisfying all these requirements, try to satisfy as many as possible, you can still get partial points that way.)

Take as model M_6:

Domain(s)	subsets of natural numbers
Relation(s)	is lower than

And take as interpretation I_6:

D	$\{-1, 1\}$
E	$\{0\}$
$R(x, y)$	$x < y$

The formula is true in this model under this interpretation.

- The part $\exists x \in D\, \forall y \in E\, \neg R(x, y)$ is true because we can take $x = 1 \in D$ and then for all $y \in E$, which happens to be only for $y = 0 \in E$, it is clear that $\neg(1 < 0)$.
- The part $\forall y \in E\, \exists x \in D\, R(x, y)$ is true because we only have to check that it is true if $y = 0 \in E$, which is indeed the case, because we can take $x = -1 \in D$ and then $-1 < 0$ holds.
- The part $\forall x \in D\, \forall y, y' \in E\, (R(x, y) \wedge R(x, y') \to y = y')$ is true, because the conclusion that $y = y'$ automatically holds because E has only one element.

So the conjunction of these three parts is also true.

7. Give a language L_7 with alphabet $\Sigma = \{a, b\}$, such that holds: (6 points)

$$\overline{L_7} = L_7 L_7 \cup \{\lambda\}$$

Explain your answer.

(Hint: one possible solution defines L_7 in terms of the length of the words in the language.)

Define

$$L_7 := \{w \in \{a, b\}^* \mid w \text{ has an odd number of symbols}\}$$

If L_7 is the language with all words over the alphabet $\{a, b\}$ that have an odd length, this means that $\overline{L_7}$ contains all words that have an even length. If we concatenate two words of odd length, we get a word with even length. However, since words with odd length have at least one symbol, a concatenation of these words have at least two symbols, whereas $\overline{L_7}$ also includes words with length zero, which is obviously only the empty word. Hence $\overline{L_7} = L_7 L_7 \cup \{\lambda\}$.

8. Consider the language:

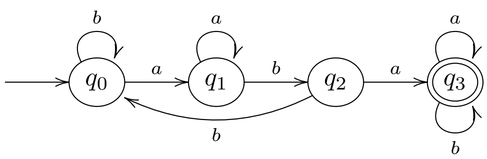
$$L_8 := \{w \in \{a, b\}^* \mid w \text{ contains } aba\}$$

(a) Give a regular expression r_8 that describes this language. (3 points)

Take for instance

$$r_8 = (a \cup b)^* aba (a \cup b)^*$$

(b) Give a deterministic finite automaton M_8 that recognizes this language. (3 points)



9. Consider the context-free grammar G_9:

$$S \to ABA$$
$$A \to aA \mid \lambda$$
$$B \to bB \mid \lambda$$

(a) Is G_9 right linear? Explain your answer. (2 points)

No, it is not. In a right linear grammar all nonterminals must be at the right end of the right hand side of the rules, but this does not hold for the first A and the B in the rule $S \to ABA$.

(b) Is $\mathcal{L}(G_9)$ a regular language? Explain your answer. (2 points)

Yes, it is. The language $\mathcal{L}(G_9) = \mathcal{L}(r_9)$ where

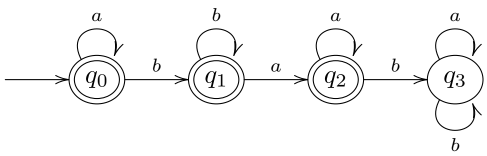
$$r_9 = a^*b^*a^*$$

Furthermore, the language can also be described by a right linear grammar, for instance by this non-minimal one:

$$S \to A \mid B \mid C \mid \lambda$$
$$A \to aA \mid B \mid \lambda$$
$$B \to bB \mid C \mid \lambda$$
$$C \to aC \mid \lambda$$

And we know that languages that are generated by a right linear grammar are automatically regular.

And of course, the language can also be described by a deterministic finite automaton:



The existence of this automaton also implies that the language is regular.

(c) Someone claims that (2 points)

$$P(w) := w \text{ does not contain } bab$$

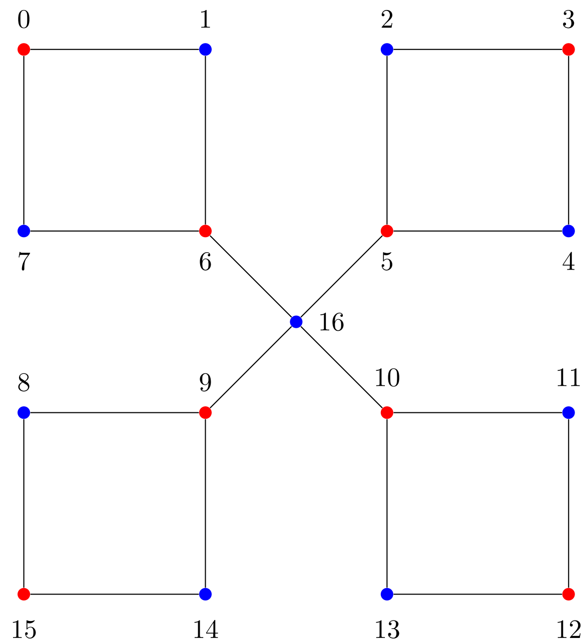
is an invariant of G_9. Explain why this is not the case.

Let $v = baB$ and $v' = babB$. Then it is clear that $P(v)$ holds. In addition, we have that, $v \to v'$ using the rule $B \to bB$. But now it is clear that $P(v')$ does not hold. So the property is not invariant with respect to the rule $B \to bB$.

10. Give a planar bipartite connected graph G_{10} in which each vertex has degree two or more, and in which no Eulerian or Hamiltonian paths exists. Explain your answer. (6 points)

(If you do not succeed in satisfying all these requirements, try to satisfy as many as possible, you can still get partial points that way.)

Take for instance G_{10}:



This graph complies to all requirements:

- It is planar because there are no crossing edges.
- It is bipartite because we can color it with two colors.
- It is connected because from each vertex there exists a path to each other vertex.
- Each vertex has degree two or more because the vertices 0, 1, 2, 3, 4, 7, 8, 11, 12, 13, 14 and 15 have degree two, the vertices 5, 6, 9 and 10 have degree three and the vertex 16 has degree four.
- It has no Eulerian path because there are more than two points with an odd degree, namely four.
- It has no Hamilton path, because if we try to construct such a path, we run into a problem. In order to connect all four squares the edges $(5, 16)$, $(6, 16)$, $(9, 16)$ and $(10, 16)$ all need to be in each Hamilton path. However, this implies that we visit vertex 16 more than once, which is not allowed in a Hamilton path.

Note that if we omit one of the four squares, we still get a graph that satisfies all requirements, but this graph looks nicer.

11. We define a sequence a_n using the recursive equations:

$$a_0 = 1$$
$$a_{n+1} = 2n + a_n + 7 \qquad \text{for all } n \geq 0$$

(This is sequence A028884 of the On-Line Encyclopedia of Integer Sequences. The value of a_{42} is 2017, happy new year!)

(a) Compute a_2 using the recursive equations above, and show how you got your answer that way. (3 points)

$$\begin{array}{lllllllll} a_1 & = & a_{0+1} & = & 2 \cdot 0 + a_0 + 7 & = & 0 + 1 + 7 & = & 8 \\ a_2 & = & a_{1+1} & = & 2 \cdot 1 + a_1 + 7 & = & 2 + 8 + 7 & = & 17 \end{array}$$

(b) Prove using induction that: (3 points)

$$a_n = (n+3)^2 - 8 \qquad \text{for all } n \geq 0$$

Proposition: 0

$a_n = (n+3)^2 - 8$ for all $n \geq 0$.

Proof by induction on n. 1

We first define our predicate P as:

$$P(n) \quad := \quad a_n = (n+3)^2 - 8$$ 2

Base Case. We show that $P(0)$ holds, i.e. we show that 3

$a_0 = (0+3)^2 - 8$

This indeed holds, because by definition $a_0 = 1$ and 4

$$(0+3)^2 - 8 = 9 - 8 = 1$$

Induction Step. Let k be any natural number such that $k \geq 0$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6

$a_k = (k+3)^2 - 8$ (Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7

$a_{k+1} = (k+1+3)^2 - 8$

This indeed holds, because 8

$$\begin{array}{lcll} a_{k+1} & = & 2k + a_k + 7 & \text{(by definition of } a_{k+1}\text{)} \\ & \overset{\text{IH}}{=} & 2k + (k+3)^2 - 8 + 7 & \text{(by applying the IH)} \\ & = & 2k + k^2 + 6k + 9 + 7 - 8 & \text{(elementary algebra)} \\ & = & k^2 + 8k + 16 - 8 & \text{(elementary algebra)} \\ & = & (k+4)^2 - 8 & \text{(elementary algebra)} \\ & = & (k+1+3)^2 - 8 & \text{(elementary algebra)} \end{array}$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$. 9

12. Compute the coefficient of x^8 in the expansion of $(\frac{1}{2}x^2 - 2)^6$, and explain which binomial coefficient is relevant for this computation. (6 points)

We compute this using the generalized form of Newton's Binomial Theory, which states:

$$(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}b^6$$

We take $a = \frac{1}{2}x^2$ and $b = -2$. Then it follows that the coefficient of x^8 can be found by expanding

$$\binom{6}{2}a^4b^2 = 15 \cdot (\tfrac{1}{2}x^2)^4 \cdot (-2)^2 = 15 \cdot 4 \cdot \tfrac{1}{16}x^8$$

So the coefficient of x^8 in the expansion is $\frac{15}{4}$ and the relevant binomial coefficient is $\binom{6}{2}$.

13. Give a formula f_{13} of modal logic that as best as possible formalizes the meaning of the following English sentence: (6 points)

If it snows it is winter, but it can be winter without snow.

Use for this the dictionary:

S	it snows
W	it is winter

$$\Box(S \to W) \wedge \Diamond(W \wedge \neg S)$$

14. A formula f is called *logically true in the logic D* if it is true in all *serial* Kripke models.

The axiom scheme D is;

$$\Box f \to \Diamond f$$

The axiom scheme T is:

$$\Box f \to f$$

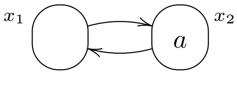
(a) Show that all instances of the axiom scheme D are logically true in the logic D. (3 points)

The formula $\Box f \to \Diamond f$ means that if in a world it is true that for all successors of that world the formula f is true, then there must be a successor in which the formula is true. And because the model is serial, we know that each world has at least one successor. From $\Box f$ it follows that f is true in this successor. So $\Diamond f$ is also true.

(b) Show that *not* all instances of the axiom scheme T are logically true in the logic D. (3 points)

Take for instance $f = a$ and $\mathcal{M} = \mathcal{M}_{14}$ where

$\mathcal{M}_{14}$:



Note that this model is serial, because each world has at least one outgoing arrow. Because $a \notin V(x_1)$ we have $x_1 \nVdash a$. However, because $R(x_1) = \{x_2\}$ and $x_2 \Vdash a$ we have that $x_1 \Vdash \Box a$. But this means that $x_1 \nVdash \Box a \to a$. And hence $\mathcal{M}_{14} \nvDash \Box a \to a$.

15. Give an LTL model $\mathcal{M}_{15}$ in which the formula (6 points)

$$\mathcal{G}\big((a \to \mathcal{X}\mathcal{F}\neg a) \wedge (\neg a \to \mathcal{X}a) \wedge \neg(a \wedge \mathcal{X}a)\big)$$

is true. Explain your answer.

(If you do not succeed in satisf ying all these requirements, try to satisfy as many as possible, you can still get partial points that way.)

We define the LTL-Kripke model $\mathcal{M}_{15} = \langle W, R, V \rangle$, where

$$\begin{array}{rcl} W & = & \{x_i \mid i \in \mathbb{N}\} \\ R(x_i) & = & \{x_j \mid j \in \mathbb{N} \text{ and } j \geq i\} \\ V(x_i) & = & \{a\} \text{ if } i \text{ is even} \\ V(x_i) & = & \emptyset \text{ if } i \text{ is odd} \end{array}$$

The formula says that it should always hold that

- if a is true at a certain moment then there must be some moment later in which a is not true, which is obviously the case because we can simply take the next moment.
- if a is not true in a certain moment then it has to be true in the next moment, which is also the case in this model.
- there are never two moments next to each other, where a is true in both moments, which is also obviously true seeing the alternating nature of the model.

So the total formula is indeed true.