Formal Reasoning 2016
Exam
(17/01/17)

Before you read on, write your name, student number and study on the answer sheet! This exam consists of fifteen exercises (three exercises for each block of the course notes) and each of these exercises is worth six points. The mark for this test is the number of points divided by ten. The first ten points are free. Good luck!

1. Consider the formula f_1 of propositional logic:

$$\neg\neg a \leftrightarrow (a \to \neg b \wedge b) \to \neg b \wedge b$$

   (a) Write this formula according to the official grammar of propositional logic formulas from the course notes. (3 points)

   (b) Give the full truth table of this formula. (3 points)

2. Give a formula f_2 of propositional logic that as best as possible formalizes the meaning of the following English text: (6 points)

   'Orthorhombic ice' is the state of water when the temperature is below 72 K and the pressure is below 100 MPa. It is ferroelectric.

   Use for this the dictionary:

I	water is orthorhombic ice (= 'ice XI')
F	water is ferroelectric
K	the temperature is below 72 K
M	the pressure is below 100 MPa

   (In reality transitions between different types of ice are more complicated than this, but this text roughly corresponds to what is shown in a phase diagram of water on Wikipedia.)

3. The *principle of explosion* states that for all formulas of propositional logic f and g: (6 points)

$$f \wedge \neg f \vDash g$$

   Does this principle hold? Explain your answer in terms of the definition of $\vDash$.

4. Consider the following formula f_4 of predicate logic:

$$\forall t \in T[J(t) \to W(t)] \land \exists t \in T[W(t) \land \neg J(t)]$$

(a) Write this formula according to the official grammar of predicate logic formulas from the course notes. (3 points)

(b) Give the meaning of this formula as an English sentence. Use for this the dictionary: (3 points)

T	the domain of points of time
$J(x)$	x is in January
$W(x)$	x is in winter

5. Give a formula f_5 of predicate logic with equality that as best as possible formalizes the meaning of the following English sentence: (6 points)

Every natural number larger than zero has exactly one predecessor.

Use for this the dictionary:

N	the domain of natural numbers
z	the number zero
$L(x, y)$	x is less than y
$P(x, y)$	x is a predecessor of y

6. Give an interpretation I_6 in a model M_6 in which the following formula of predicate logic is true: (6 points)

$$[\exists x \in D\, \forall y \in E\, \neg R(x, y)] \land [\forall y \in E\, \exists x \in D\, R(x, y)] \land$$
$$[\forall x \in D\, \forall y, y' \in E\, (R(x, y) \land R(x, y') \to y = y')]$$

Explain your answer.

(If you do not succeed in satisfying all these requirements, try to satisfy as many as possible, you can still get partial points that way.)

7. Give a language L_7 with alphabet $\Sigma = \{a, b\}$, such that holds: (6 points)

$$\overline{L_7} = L_7 L_7 \cup \{\lambda\}$$

Explain your answer.

(Hint: one possible solution defines L_7 in terms of the length of the words in the language.)

8. Consider the language:

$$L_8 := \{w \in \{a, b\}^* \mid w \text{ contains } aba\}$$

   (a) Give a regular expression r_8 that describes this language. (3 points)

   (b) Give a deterministic finite automaton M_8 that recognizes this language. (3 points)

9. Consider the context-free grammar G_9:

$$\begin{aligned} S &\to ABA \\ A &\to aA \mid \lambda \\ B &\to bB \mid \lambda \end{aligned}$$

   (a) Is G_9 right linear? Explain your answer. (2 points)

   (b) Is $\mathcal{L}(G_9)$ a regular language? Explain your answer. (2 points)

   (c) Someone claims that (2 points)

$$P(w) := w \text{ does not contain } bab$$

   is an invariant of G_9. Explain why this is not the case.

10. Give a planar bipartite connected graph G_{10} in which each vertex has degree two or more, and in which no Eulerian or Hamiltonian paths exists. Explain your answer. (6 points)

    (If you do not succeed in satisfying all these requirements, try to satisfy as many as possible, you can still get partial points that way.)

11. We define a sequence a_n using the recursive equations:

$$\begin{aligned} a_0 &= 1 \\ a_{n+1} &= 2n + a_n + 7 \qquad \text{for all } n \geq 0 \end{aligned}$$

    (This is sequence A028884 of the On-Line Encyclopedia of Integer Sequences. The value of a_{42} is 2017, happy new year!)

    (a) Compute a_2 using the recursive equations above, and show how you got your answer that way. (3 points)

    (b) Prove using induction that: (3 points)

$$a_n = (n + 3)^2 - 8 \qquad \text{for all } n \geq 0$$

12. Compute the coefficient of x^8 in the expansion of $(\frac{1}{2}x^2 - 2)^6$, and explain which binomial coefficient is relevant for this computation. (6 points)

13. Give a formula f_{13} of modal logic that as best as possible formalizes the meaning of the following English sentence: (6 points)

    If it snows it is winter, but it can be winter without snow.

    Use for this the dictionary:

S	it snows
W	it is winter

14. A formula f is called *logically true in the logic D* if it is true in all *serial* Kripke models.

    The axiom scheme D is;

$$\Box f \to \Diamond f$$

    The axiom scheme T is:

$$\Box f \to f$$

    (a) Show that all instances of the axiom scheme D are logically true in the logic D. (3 points)

    (b) Show that *not* all instances of the axiom scheme T are logically true in the logic D. (3 points)

15. Give an LTL model $\mathcal{M}_{15}$ in which the formula (6 points)

$$\mathcal{G}\big((a \to \mathcal{X}\mathcal{F}\neg a) \wedge (\neg a \to \mathcal{X}a) \wedge \neg(a \wedge \mathcal{X}a)\big)$$

    is true. Explain your answer.

    (If you do not succeed in satisf ying all these requirements, try to satisfy as many as possible, you can still get partial points that way.)