Formal Reasoning 2016
Solutions Test Block 1: Propositional Logic
(07/09/16)

The first two exercises use the following 'dictionary':

RB	there is a rainbow
R	it rains
S	the sun shines

1. Give formulas of propositional logic that give the meaning of the following two English sentences:

   (a) *There only is a rainbow if it rains and the sun shines.*

   $$RB \to R \land S$$

   (b) *There is a rainbow because it rains and the sun shines.*
   The longer versions:

   $$(R \land S \to RB) \land R \land S \quad \text{or} \quad RB \land (RB \to R \land S)$$

   The short version:

   $$R \land S \land RB$$

2. Give English sentences that describe the meaning of the following two formulas of propositional logic as well as possible:

   (a)

   $$R \to S \to RB$$

   If it rains then if the sun also shines there is a rainbow.

   (b)

   $$((S \to R) \to S) \to S$$

   The goal of this exercise was to show that formulas are often much more clear than sentences…
   This is a sentence that follows the order of the original formula:
   'If it is true that if it is true that the sun shines implies that it rains, then the sun shines, then the sun shines.'
   And this is a sentence that reverses the order of the original formula:
   'The sun shines if the sun shines if the sun shines implies that it rains.'

   This formula is an instance of Peirce's law and is always true.

3. This exercise is concerned with the formula:

   $$a \leftrightarrow \neg a \leftrightarrow a \leftrightarrow \neg a$$

   (a) Write this formula with parentheses according to the official grammar in the course notes.

   $$(a \leftrightarrow (\neg a \leftrightarrow (a \leftrightarrow \neg a)))$$

(b) Give the truth table of this formula.

a	$\neg a$	$a \leftrightarrow \neg a$	$\neg a \leftrightarrow a \leftrightarrow \neg a$	$a \leftrightarrow \neg a \leftrightarrow a \leftrightarrow \neg a$
0	1	0	0	1
1	0	0	1	1

(c) Does $\models a \leftrightarrow \neg a \leftrightarrow a \leftrightarrow \neg a$ hold? Explain your answer.

Yes, it holds. The formula is logically true. This follows from the observation that the truth table has only ones in its last column.

4. (a) Is there a model v in which $a \to \neg b$ and $\neg b \to a$ are both false? If so, give such a model and explain why it has this property. If not, explain why such a model does not exist.

Such a model cannot exist. We prove this by contradiction. Let v be a model such that $v(a \to \neg b) = 0$ and $v(\neg b \to a) = 0$. From the first claim it follows that $v(a) = 1$. However, if $v(a) = 1$ then $v(\neg b \to a) = 1$, no matter what the value of $v(b)$ is. So if $a \to \neg b$ is false in model v, automatically $\neg b \to a$ is true in that same model.

(b) Are there formulas f and g such that:

$$f \to \neg g \models \neg g \to f$$

If so, give formulas like that and explain why they have this property. If not, explain why formulas like that do not exist.

Yes, such formulas exist. Take $f = a$ and $g = \neg a$. Then the combined truth table for f and g becomes:

a	$\neg a$	$\neg\neg a$	$a \to \neg\neg a$	$\neg\neg a \to a$
0	1	0	1	1
1	0	1	1	1

In the truth table we can see that whenever $a \to \neg\neg a$ has a 1, the formula $\neg\neg a \to a$ also has a 1. So $f \to \neg g \models \neg g \to f$ for this f and g.