Formal Reasoning 2016
Solutions Test Block 2: Predicate Logic
(21/09/16)

The first three exercises use the following 'dictionary'.

s	Socrates
E	living beings
$H(x)$	x is a (human) man
$M(x)$	x is mortal
$P(x)$	x is a Great Basin bristlecone pine tree

1. Give a formula of predicate logic that approximates the meaning of the following sentence as well as possible:

   All men are mortal, Socrates is a man, hence Socrates is mortal.

   $$\forall x \in E\,[H(x) \to M(x)] \wedge H(s) \wedge M(s)$$

   Note that this sentence is basically equivalent to

   Socrates is mortal, because all men are mortal and Socrates is a man.

   And for 'because' we have seen before that it is best translated with the 'and' connective.

2. Give a formula of predicate logic with equality that approximates the meaning of the following sentence (after 1 Timothy 6:16) as well as possible:

   There is only one man that has immortality.

   First we define an abbreviation for an immortal man:

   $$I(x) := H(x) \wedge \neg M(x)$$

   Using this abbreviation we get the following formula:

   $$\exists x \in E\,[I(x) \wedge \forall y \in E\,[I(y) \to x = y]]$$

   Or equivalent but shorter:

   $$\exists x \in E\,[\forall y \in E\,[I(y) \leftrightarrow x = y]]$$

   If we do not use the abbreviation, we get these formulas:

   $$\exists x \in E\,[H(x) \wedge \neg M(x) \wedge \forall y \in E\,[H(y) \wedge \neg M(y) \to x = y]]$$

   $$\exists x \in E\,[\forall y \in E\,[H(y) \wedge \neg M(y) \leftrightarrow x = y]]$$

3. Give an English sentence that corresponds to the following formula of predicate logic:

   $$\forall x \in E\,[P(x) \to \neg M(x)] \wedge [\exists x \in E\,(\neg P(x) \wedge \neg M(x))]$$

All Great Basin bristlecone pine trees are immortal, and there exists a living being which is not a Great Basin bristlecone pine tree but is still immortal.

4. Write the formula from the previous exercise according to the official grammar from the course notes.

$$((\forall x \in E \ \ (P(x) \to \neg M(x))) \wedge (\exists x \in E \ \ (\neg P(x) \wedge \neg M(x))))$$

5. Is your answer to exercise 1 above a logically valid formula? Explain your answer.

   Logically valid means that the formula should be true in all models under all interpretations.

   The formula
   $$\forall x \in E\,[H(x) \to M(x)] \wedge H(s) \wedge M(s)$$
   is not logically valid, because it is not true under the following interpretation I_5 in the model $M_5 := (\mathbb{N}, 0, \text{even}, \text{odd}, \text{prime})$:

s	0
E	$\mathbb{N}$
$H(x)$	x is even
$M(x)$	x is odd
$P(x)$	x is prime

   Under this interpretation the sentence means

   For all natural numbers it holds that each even number is also odd and in particular 0 is even and 0 is odd.

   Which is clearly not true since 0 is not odd.

6. Give an interpretation I_6 in the model $M_6 := (\mathbb{Z}, 0, <)$ under which the formula from exercise 3 is true. Explain your answer.

   Take as interpretation I_6:

s	0
E	$\mathbb{Z}$
$H(x)$	$x < 0$
$M(x)$	$x < 0$
$P(x)$	$0 < x$

   Under this interpretation the formula means:

   For each integer it holds that if it is larger than zero, it is not smaller than zero, and there exists an integer which is not larger than zero and not smaller than zero.

The universal part is obviously true: if a number is positive, it cannot be negative at the same time. The existential part is also true: take special value zero, which is by definition not positive and not negative.

Note that the interpretations for s and $H(x)$ are only given because we know that these symbols exist in the whole test, but if we restrict ourselves to exercise 3, we don't know that they exist and hence they can be omitted. Which value we give them doesn't matter for this exercise anyway.