# Formal Reasoning 2017
## Solutions Exam
**(22/01/18)**

1. Give a formula of propositional logic that gives the meaning of the English sentence:

   *A rainbow only occurs if it rains and the sun shines at the same time.*

   Use the dictionary:

| | |
|---|---|
| $R$ | it rains |
| $S$ | the sun shines |
| $RB$ | there is a rainbow |

$$RB \to (R \land S)$$

2. Write the formula of propositional logic

$$(a \to b \to c) \to (a \to b) \to a \to c$$

   according to the official grammar in the course notes, and give the truth table.

   The official form is:

$$\Big((a \to (b \to c)) \to ((a \to b) \to (a \to c))\Big)$$

   And the truth table is:

| $a$ | $b$ | $c$ | $b \to c$ | $a \to (b \to c)$ | $a \to b$ | $a \to c$ | $(a \to b) \to (a \to c)$ | $\Big((a \to (b \to c)) \to ((a \to b) \to (a \to c))\Big)$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

3. Give a formula that is logically equivalent to the formula of propositional logic

$$a \to (b \to a)$$

   but that does not contain the atoms $a$ and $b$. This implies that you may use other atoms. Explain your answer.

   The truth table of this formula is:

| $a$ | $b$ | $b \to a$ | $a \to (b \to a)$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 |

   Hence the formula is a tautology, so it is equivalent to any other tautology, in particular it is logically equivalent to the tautology $c \lor \neg c$.

4. Give a formula of predicate logic with equality that gives the meaning of the English sentence:

   *Two is the unique even prime number.*

   Use the dictionary:

   | | | |
   |---|---|---|
   | domains | $N$ | the domain of natural numbers $\mathbb{N}$ |
   | constants | $d$ | the number two |
   | predicates | $E(x)$ | $x$ is even |
   | | $P(x)$ | $x$ is prime |

   Several correct possibilities:

   $$E(d) \wedge P(d) \wedge \forall x \in N\,[E(x) \wedge P(x) \rightarrow (x = d)]$$
   $$E(d) \wedge P(d) \wedge \forall x \in N\,[\neg(x = d) \rightarrow \neg E(x) \vee \neg P(x)]$$
   $$E(d) \wedge P(d) \wedge \neg\exists x \in N\,[\neg(x = d) \wedge E(x) \wedge P(x)]$$
   $$\forall x \in N\,[E(x) \wedge P(x) \leftrightarrow (x = d)]$$

5. Give a model $M_5$, and an interpretation $I_5$ in $M_5$ such that:

   $$(M_5, I_5) \models \forall x \in T\,\neg R(x,x) \wedge \exists x_1, x_2, x_3 \in T\,[R(x_1,x_2) \wedge R(x_2,x_3) \wedge R(x_3,x_1)]$$

   Explain your answer. Make sure that your model is unambiguous and not open for discussion.

   Let $M_5 = (\mathbb{N}, =, \neq)$. Let the interpretation $I_5$ be:

   | | |
   |---|---|
   | $T$ | $\mathbb{N}$ |
   | $R(x,y)$ | $x \neq y$ |

   Then the formula becomes:

   $$\forall x \in \mathbb{N}\,[\neg x \neq x] \wedge \exists x_1, x_2, x_3 \in \mathbb{N}\,[(x_1 \neq x_2) \wedge (x_2 \neq x_3) \wedge (x_3 \neq x_1)]$$

   The first part is true since $x = x$ for all $x \in \mathbb{N}$. The second part is true because we can take $x_1 = 0$, $x_2 = 1$ and $x_3 = 2$. Then $0 \neq 1$, $1 \neq 2$ and $2 \neq 0$.

6. Explain the meaning of the following statement:

   $$\forall x \in D\,[A(x) \rightarrow \neg B(x)] \equiv \neg\exists x \in D\,[A(x) \wedge B(x)]$$

   Does this statement hold? Explain your answer.

   The statement claims that the two formulas are equivalent independent of the chosen model and interpretation. Or in other words, for any model and interpretation the statement 'for each element $x$ in $D$ it holds that $A(x)$ implies that $B(x)$ does not hold' holds if and only if the statement 'there is no element $x$ in $D$ such that $A(x)$ and $B(x)$ both hold' holds.

   This equivalence statement holds. We can transform the formula on the right to the one on the left using well known equivalences:

   $$\begin{aligned} \neg\exists x \in D\,[A(x) \wedge B(x)] &\equiv \forall x \in D\,[\neg(A(x) \wedge B(x))] \\ &\equiv \forall x \in D\,[\neg A(x) \vee \neg B(x)] \\ &\equiv \forall x \in D\,[A(x) \rightarrow \neg B(x)] \end{aligned}$$

7. Give a non-empty language $L_7 \subseteq \{a,b\}^*$ such that $L_7^R \subseteq \overline{L_7}$, and such that for every word $w \in L_7$ also $aw \in L_7$. Explain your answer.

   Take $L_7 = \mathcal{L}(aa^*b) = \{a^n b \mid n \in \mathbb{N} \text{ and } n \geq 1\}$. So in particular each word in $L_7$ starts with an $a$ and ends with a $b$. Now let $w \in L_7$. Then there exist $n \in \mathbb{N}$ with $n \geq 1$ such that $w = a^n b$. But then it is clear that $aw = aa^n b = a^{n+1} b \in L_7$, because $n+1 \in \mathbb{N}$ and $n+1 \geq 1$. Furthermore $w^R \in \overline{L_7}$ because $w^R$ does not end with a $b$ (or does not start with an $a$). So $L_7^R \subseteq \overline{L_7}$.

8. Give a regular expression for the language:

$$L_8 := \{w \in \{a,b,c\}^* \mid \text{in } w \text{ the symbol } a \text{ is never directly followed by } c\}$$

   Possible solutions (but there are many more):

$$(a^*b \cup b \cup c)^* a^*$$

$$c^*(bc^* \cup a)^*$$

$$((b \cup c)^*(a^*b)^*)^* a^*$$

$$(b \cup c)^*(a(b(b \cup c)^* \cup \lambda))^*$$

9. Consider the context-free grammar $G_9$:

$$S \to aSa \mid B$$
$$B \to bB \mid \lambda$$

   This grammar produces the language:

$$\mathcal{L}(G_9) = \{a^n b^m a^n \mid n, m \in \mathbb{N}\}$$

   Someone wants to show that

$$abaa \notin \mathcal{L}(G_9)$$

   and proposes the property

$$P(w) := [w \text{ is a palindrome, i.e., } w^R = w]$$

   as an invariant to establish this. Does that work? Explain your answer.

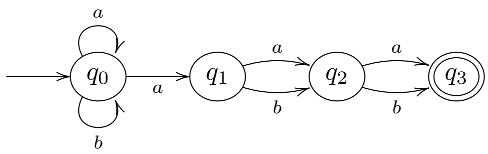
   It doesn't work because $P(w)$ is not an invariant with respect to this grammar. Take $v = B$ and $v' = bB$. Then $P(v)$ holds because $v^R = B^R = B = v$. Furthermore, $B \to bB$. However, $P(v')$ does not hold since $v'^R = (bB)^R = Bb \neq v'$.

10. We define the language:

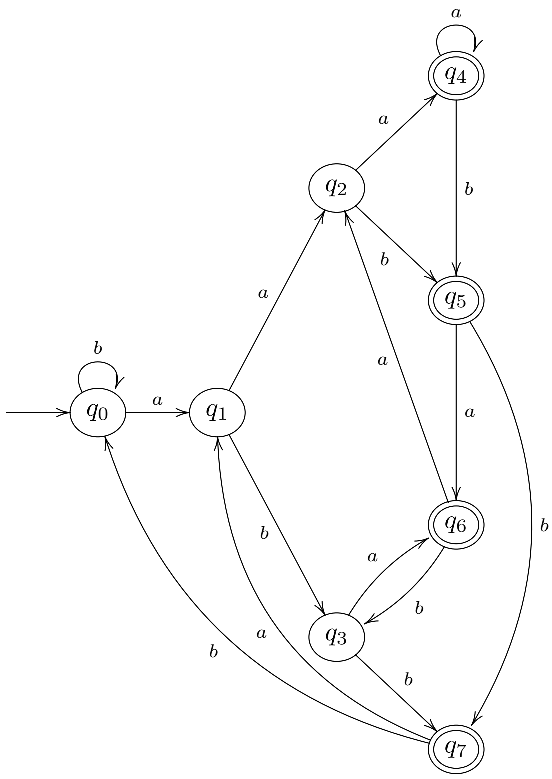
$$\begin{aligned} L_{10} &:= \{uav \mid u, v \in \{a,b\}^* \text{ and } |v| = 2\} \\ &= \{w \in \{a,b\}^* \mid w \text{ has an } a \text{ two places from the end}\} \end{aligned}$$

   For example $\underline{a}bb \in L_{10}$, but $aba\underline{b}ab \notin L_{10}$ and $ab \notin L_{10}$ (too short).

   Give a *non*-deterministic finite automaton with at most four states and a single final state that recognizes the language $L_{10}$.




11. Give a *deterministic* finite automaton that recognizes the language $L_{10}$ from the previous exercise.




12. Let be given a deterministic finite automaton $M = \langle \Sigma, Q, q_0, F, \delta \rangle$. We want to define a deterministic finite automaton $M' = \langle \Sigma', Q', q'_0, F', \delta' \rangle$ such that

$$L(M') = \overline{L(M)}$$

Give definitions for $\Sigma'$, $Q'$, $q'_0$, $F'$ and $\delta'$ in terms of $\Sigma$, $Q$, $q_0$, $F$ and $\delta$ and explain why $M'$ has the required property.

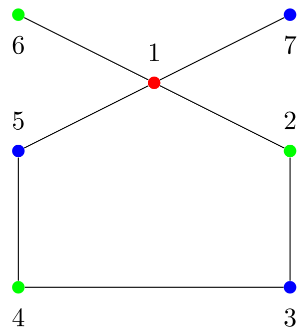
If we want $L(M')$ to be the complement of $L(M)$, all we have to do is swap non-final states and final states:

- Words that were accepted by $M$, ended in a final state in $M$ and will now end in a non-final state in $M'$ and are hence no longer accepted.
- Words that were not accepted by $M$, ended in a non-final state in $M$ and will now end in a final state in $M'$ and are hence they are now accepted.

So we get:

$$\begin{array}{rcl} \Sigma' & := & \Sigma \\ Q' & := & Q \\ q_0' & := & q_0 \\ F' & := & Q \setminus F = Q - F \\ \delta' & := & \delta \end{array}$$

13. Give a connected planar graph $G_{13}$ with chromatic number three that does not contain any triangles (subgraphs isomorphic to $K_3$) and that does not have a Hamilton path. Explain your answer.




This graph complies to all the requirements:

- It is *connected* because from every vertex there exists a path to each other vertex.
- It is *planar* because there are no crossing edges in this representation.
- It has *chromatic number three* because
  - a coloring with three colors exists as can be seen, and
  - a coloring with two colors cannot exist because of the circuit of odd length.
- It has *no triangles* as can be seen.
- It has *no Hamilton path* as each Hamilton path should include both the edges $(1, 6)$ and $(1, 7)$ and at least one of the edges $(1, 2)$ or $(1, 5)$, but this implies that vertex 1 needs to be visited at least twice, which is forbidden in a Hamilton path.

14. We define the sequence $a_3, a_4, a_5, \ldots$ using the recursive equations:

$$a_3 := 2$$
$$a_{n+1} := a_n + 2n + 1 \qquad \text{for } n \geq 3$$

We have for example $a_3 = 2$, $a_4 = 9$, $a_5 = 18$, $\ldots$, $a_{45} = 2018$, $\ldots$

Use induction to prove that

$$a_n = n^2 - 7$$

for all $n \geq 3$.

**Proposition:** 0

$a_n = n^2 - 7$ for all $n \geq 3$.

**Proof** by induction on $n$. 1

We first define our predicate $P$ as:

$$P(n) \quad := \quad a_n = n^2 - 7$$ 2

**Base Case.** We show that $P(3)$ holds, i.e. we show that 3

$a_3 = 3^2 - 7$

This indeed holds, because 4

by definition $a_3 = 2$ and $3^2 - 7 = 9 - 7 = 2$.

**Induction Step.** Let $k$ be any natural number such that $k \geq 3$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6

$a_k = k^2 - 7$ (Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7

$a_{k+1} = (k+1)^2 - 7$

This indeed holds, because 8

$$\begin{aligned} a_{k+1} &= a_k + 2k + 1 \\ &\stackrel{\text{IH}}{=} (k^2 - 7) + 2k + 1 \\ &= (k^2 + 2k + 1) - 7 \\ &= (k+1)^2 - 7 \end{aligned}$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 3$. 9

15. Someone divides nine distinguishable objects by first taking three objects and putting them aside, and then dividing the other six objects in three non-empty indistinguishable piles. Give the number of different combinations of three piles that can be formed this way. If you use binomial coefficients and/or Stirling numbers in your answer, then give a relevant part of the corresponding triangles as well.

We give an algorithm to construct such a combination of three piles.

**Task 1** Choose three objects that will be put aside. This can be done in $\binom{9}{3} = 84$ ways.

**Task 2** Distribute the remaining six objects over exactly three indistinguishable piles. This can be done in $\left\{ {6 \atop 3} \right\} = 90$ ways.

So in total there are $84 \cdot 90 = 7560$ ways to do this.

The relevant part of the triangle for binomial coefficients:

1

1 1

1 2 1

1 3 3 1

1 4 6 4

1 5 10 10

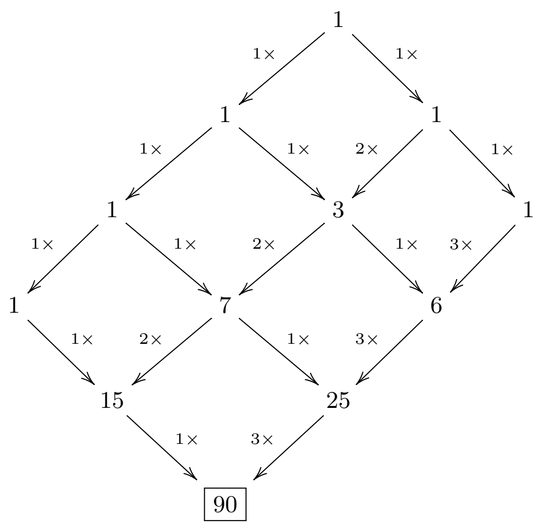
1 6 15 20

7 21 35

28 56

84

The relevant part of the triangle for Stirling numbers of the second kind:



16. Give a formula of modal logic that gives the meaning of the English sentence:

*When it rains, I get wet, but now it rains and I don't get wet!*

Use the dictionary:

| | |
|---|---|
| $R$ | it rains |
| $W$ | I get wet |

$$\Box(R \to W) \wedge R \wedge \neg W$$

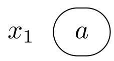
17. Give a Kripke model $\mathcal{M}_{17}$ such that:

$$\mathcal{M}_{17} \vDash \Box(a \to b) \wedge a \wedge \neg b$$

Explain what this statement means, and explain why your Kripke model makes this true.

The statement means that within model $\mathcal{M}_{17}$ it holds that for all worlds the formulas $a$, $\neg b$ and $\Box(a \to b)$ hold, where the last formula means that $a \to b$ should hold in all accessible worlds from all worlds in the model.

Let $\mathcal{M}_{17}$ be:




Since $x_1$ has no accessible worlds, $x_1 \Vdash \Box(a \to b)$ trivially holds. And because $a \in V(x_1)$ also $x_1 \Vdash a$ holds. And because $b \notin V(x_1)$ also $x_1 \Vdash \neg b$ holds. So $x_1 \Vdash \Box(a \to b) \wedge a \wedge \neg b$ holds. And because $x_1$ is the only world in $\mathcal{M}_{17}$ we may deduce that $\mathcal{M}_{17} \vDash \Box(a \to b) \wedge a \wedge \neg b$ holds.

Note that as soon as there is a world in $\mathcal{M}_{17}$ which has an accessible world, the model will be wrong, because then in this accessible world both $a \to b$ and $a \wedge \neg b$ must be true, which cannot be.

18. Give an LTL formula in which the connectives $\mathcal{G}$ and $\to$ do not occur, that says that directly after the proposition $a$ is true, the proposition $c$ is not true.

Note that $\Box$ is alternative syntax for $\mathcal{G}$ and therefore is also disallowed. Also, if you only manage to write such a formula using these forbidden connectives, you can do so for partial points.

We start by giving the most natural LTL formula without checking which operators are allowed or not.

The sentence indicates that in every world where $a$ holds, in the next world $c$ should not hold. This can be expressed by

$$\mathcal{G}(a \to \mathcal{X}\neg c)$$

However, this formula contains the forbidden $\mathcal{G}$ and $\to$. Because $\mathcal{G}f \equiv \neg\mathcal{F}\neg f$ for each formula $f$, our formula is equivalent to

$$\neg\mathcal{F}\neg(a \to \mathcal{X}\neg c)$$

And because $f \to g \equiv \neg f \vee g$ for all formulas $f$ and $g$, our formula is equivalent to

$$\neg\mathcal{F}\neg(\neg a \vee \mathcal{X}\neg c)$$