Formal Reasoning 2018
Solutions Test Block 1: Propositional and Predicate Logic
(24/09/18)

1.
$$\neg a \wedge b \vee a \to b \leftrightarrow a$$

(a) Write this formula according to the official grammar from the course notes. (10 points)

$$((((\neg a \wedge b) \vee a) \to b) \leftrightarrow a)$$

or

$$\underbrace{\left(\underbrace{\left(\underbrace{\left(\underbrace{(\neg a \wedge b)}_{\wedge} \vee a\right)}_{\vee} \to b\right)}_{\leftarrow} \leftrightarrow a\right)}_{\leftrightarrow}$$

(b) Give the full truth table of this formula. (10 points)

a	b	$\neg a$	$\neg a \wedge b$	$(\neg a \wedge b) \vee a$	$((\neg a \wedge b) \vee a) \to b$	$(((\neg a \wedge b) \vee a) \leftarrow b) \leftrightarrow a$
0	0	1	0	0	1	0
0	1	1	1	1	1	0
1	0	0	0	1	0	0
1	1	0	0	1	1	1

2. *I only am happy if I am well rested, because else I am tired.*

Formalize this English sentence as a formula of propositional logic using the following dictionary as well as possible: (20 points)

H	I am happy
W	I am well rested
T	I am tired

What we had in mind was:

$$(H \to W) \wedge (\neg W \to T)$$

For the translation of 'only if' and 'because' it is evident that they should lead to $(H \to W) \wedge \ldots$, but the 'else' part is less clear so several other solutions like $(\neg H \to T)$ or $(\neg(H \to W) \to T)$ are accepted as well.

3. The following statement holds:

$$\text{If} \models f \vee g \text{ and } f \models h \text{ and } g \models h, \text{ then} \models h.$$

Explain what this statement *says* in terms of truth tables or models. Note that you do not have to *show* that this statement holds. (10 points)

If we translate the symbols into natural language we get:

> If $f \vee g$ is a tautology, and if h is a logical consequence of f, and if h is a logical consequence of g, then h is a tautology.

In terms of models this becomes:

> If the valuation $v_1(f \vee g) = 1$ for all models v_1, and the valuation $v_2(h) = 1$ for all models v_2 in which $v_2(f) = 1$, and the valuation $v_3(h) = 1$ for all models v_3 in which $v_3(g) = 1$, then the valuation $v_4(h) = 1$ for all models v_4.

In terms of truth tables this becomes:

> If the truth table of $f \vee g$ has a 1 on every row, and on every row where the truth table of f has a 1, the truth table of h also as a 1, and on every row where the truth table of g has a 1, the truth table of h also as a 1, then the truth table of h has a 1 on every row.

4. *The prime numbers are the numbers greater than one that are not a product of two numbers greater than one.*

Formalize this English sentence as a formula of predicate logic using the following dictionary as well as possible: (20 points)

N	the domain of numbers
u	the number one
$Pr(x)$	x is a prime number
$Lt(x, y)$	$x < y$
$M(x, y, z)$	$x \times y = z$

$$\forall x \in N\,[Pr(x) \leftrightarrow Lt(u, x) \wedge \neg\exists y_1, y_2 \in N\,[Lt(u, y_1) \wedge Lt(u, y_2) \wedge M(y_1, y_2, x)]]$$

5.

$$\forall x \in D\, P(x) \models \exists x \in D\, P(x)$$

Does this statement hold? Explain your answer. (10 points)

It does not hold. The statement says that for any model and interpretation it holds that if $\forall x \in D\, P(x)$ holds, then $\exists x \in D\, P(x)$ also holds.

This is not true in all models and interpretations. Take model $M_5 = (\emptyset, \text{'is purple'})$ and interpretation I_5 such that

D	$\emptyset$
$P(x)$	x is purple

Because there are no elements in $\emptyset$, the statement 'for each element in the empty set it holds that this element is purple' is vacuously true. But the statement 'there exists an element in the empty set for which it holds that this element is purple' is not true, because there are no elements in the empty set, so in particular no purple ones.

6. Give an interpretation I_6 in the model $M_6 = (\mathbb{N}, +, 1, =, \leq)$ under which the following formula is true: (10 points)

$$\forall x \in D \, \exists y, z \in D \, [R(x, y) \wedge R(y, z) \wedge \neg R(x, z)]$$

Note that you do not need to explain your answer.

For instance, if we define I_6 to be

D	$\mathbb{N}$
$R(x, y)$	$y = x + 1$

Because if $R(x, y)$ and $R(y, z)$ hold, then $y = x + 1$ and $z = y + 1$, and from this naturally follows that $z = (x + 1) + 1 = x + 2 \neq x + 1$.