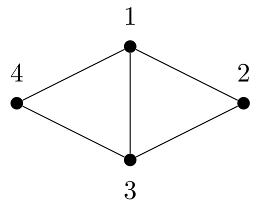
Formal Reasoning 2018
Solutions Test Block 3: Discrete Mathematics and Modal Logic

(19/12/18)

1. (a) Give a connected graph G_1 with a minimal number of vertices such that it has a Hamiltonian circuit, but does not have an Eulerian circuit. (10 points)

G_1

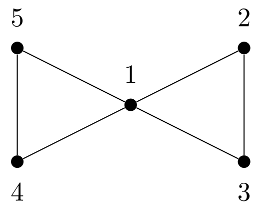


This graph complies to the requirements:

- It is connected, because it has exactly one component.
- It has a Hamiltonian circuit: $1 \to 2 \to 3 \to 4 \to 1$.
- It has no Eulerian circuit because vertices 1 and 3 both have degree 3, which is odd.
- It is minimal with respect to the number of vertices. In order to have a Hamiltonian circuit, we need to have a circuit, so we need at least three vertices. In order to prevent the existence of an Eulerian circuit, we need at least two vertices a and b with an odd degree. However, if $\deg(a) = 1$, then a cannot be part of a Hamiltonian circuit. So $\deg(a) \geq 3$. But if $\deg(a) = 3$, then there must be at least three neighbors of a, which implies that we need at least four vertices. And the diagram shows it can be done with four vertices.

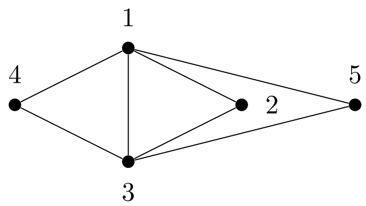
(b) Give a connected graph G_1' with a minimal number of vertices such that it has an Eulerian circuit, but does not have a Hamiltonian circuit. (10 points)

G_1'



or

G_1'



The graph on the left complies to the requirements:

- It is connected, because it has exactly one component.
- It has no Hamiltonian circuit because all four edges $\{1,2\}$, $\{1,3\}$, $\{1,4\}$ and $\{1,5\}$ must be in the circuit, but this implies that vertex 1 is visited twice.

- It has an Eulerian circuit: $1 \to 2 \to 3 \to 1 \to 4 \to 5 \to 1$.
- It is minimal with respect to the number of vertices. In order to have an Eulerian circuit, we need to have a circuit, so we need at least three vertices. But K_3 is the only graph with three vertices and a cycle. However K_3 has a Hamiltonian circuit. Now if we try to create a graph with an Eulerian cycle with four vertices, we see that all vertices have to have degree two, which implies that the graph should contain the cycle graph C_4. But this one has a Hamiltonian circuit. The only edges that we can add to C_4 are the two diagonals, but still we have an Hamiltonian circuit in this case. So it cannot be done with four vertices either. The diagram shows that it can be done with five vertices.

The graph on the right also complies to the requirements:

- It is connected, because it has exactly one component.
- It has no Hamiltonian circuit because all four edges $\{1,2\}$, $\{1,4\}$ and $\{1,5\}$ must be in the circuit, but this implies that vertex 1 is visited twice.
- It has an Eulerian circuit: $1 \to 2 \to 3 \to 1 \to 4 \to 3 \to 5 \to 1$.
- It is minimal with respect to the number of vertices for the same reason as above.

You do not need to explain why your graphs have the required properties.

2. The following equality holds: (20 points)

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! = 23 = 4! - 1$$

We want to show that this pattern holds in general. For this we define:

$$s_n := 1 \cdot 1! + \cdots + (n-1) \cdot (n-1)!$$

For example $s_4 = 23$. This corresponds to the recursion equations:

$$\begin{aligned} s_2 &= 1 \\ s_{n+1} &= s_n + n \cdot n! \qquad \text{for } n \geq 2 \end{aligned}$$

Prove from this with induction that $s_n = n! - 1$.

Proposition: 0

$s_n = n! - 1$ for all $n \geq 2$.

Proof by induction on n. 1

We first define our predicate P as:

$$P(n) \ := \ s_n = n! - 1$$ 2

Base Case. We show that $P(2)$ holds, i.e. we show that 3

$s_2 = 2! - 1$

This indeed holds, because 4

$s_2 = 1 = 2 - 1 = 2! - 1$

Induction Step. Let k be any natural number such that $k \geq 2$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6
$s_k = k! - 1$ (Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7
$s_{k+1} = (k+1)! - 1$

This indeed holds, because 8

$$\begin{aligned} s_{k+1} &= s_k + k \cdot k! && \text{definition of } s_n \\ &= k! - 1 + k \cdot k! && \text{IH} \\ &= k! + k \cdot k! - 1 && \text{elementary algebra} \\ &= (1+k) \cdot k! - 1 && \text{elementary algebra} \\ &= (k+1) \cdot k! - 1 && \text{elementary algebra} \\ &= (k+1)! - 1 && \text{elementary algebra} \end{aligned}$$

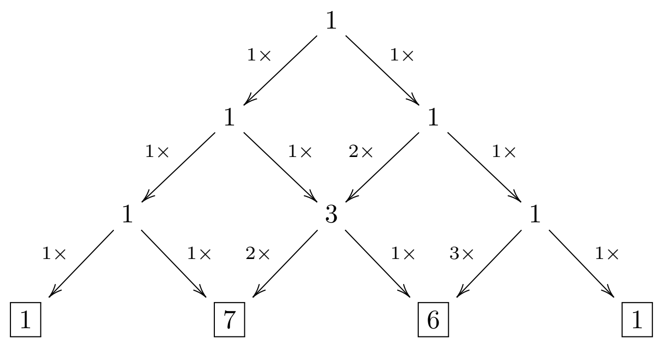
Hence it follows by induction that $P(n)$ holds for all $n \geq 2$. 9

3. There are five rhyme schemes for a poem with three lines: AAA, AAB, ABA, ABB and ABC. Give the number of rhyme schemes for a poem with *four* lines, and show how this number relates to Bell numbers by giving a relevant part of an appropriate triangle of numbers. (20 points)

The Bell number B_n gives the number of ways we can partition a set of n distinguishable elements. So let's try to relate the given rhyme schemes to partitions of $\{1, 2, 3\}$. We can interpret the schemes by matching the line numbers to the positions of the A, B and C in the schemes. So AAA means that line numbers 1, 2 and 3 are all put in the same set, giving the partition $\{\{1, 2, 3\}\}$. And ABB means that line number 1 is in a separate set, and line numbers 2 and 3 are put in the same set, giving the partition $\{\{1\}, \{2, 3\}\}$. Likewise for the other possibilities, giving indeed all partitions of $\{1, 2, 3\}$:

$$\begin{aligned} AAA &\mapsto \{\{1,2,3\}\} \\ AAB &\mapsto \{\{1,2\},\{3\}\} \\ ABA &\mapsto \{\{1,3\},\{2\}\} \\ ABB &\mapsto \{\{1\},\{2,3\}\} \\ ABC &\mapsto \{\{1\},\{2\},\{3\}\} \end{aligned}$$

So we see that the five rhyme schemes correspond to B_3. So the number of rhyme schemes for a four line poem, should correspond to B_4. And $B_4 = 15$, as it can be computed by adding the marked values in the triangle for the Stirling numbers of the second kind:



4. Suppose I lost my keys, but I am not aware of this, so I still think I know my keys are in my pocket. Then the following sentence will be true: (15 points)

> *I do not know that it is not the case that I know my keys are in my pocket.*

Give a formula of epistemic logic *without using negation signs*, that gives the meaning of this sentence. Use for a dictionary:

P	my keys are in my pocket

Literally translated we would get the formula

$$\neg\Box\neg\Box P$$

but this contains negations and is not allowed. However, because $\neg\Box\neg f$ is equivalent to $\Diamond f$ for all formulas f, we can rewrite this formula to

$$\Diamond\Box P$$

5. Give an LTL Kripke model $\mathcal{M}_5$ such that (15 points)

$$\mathcal{M}_5 \models (\mathcal{GF}a) \wedge (\mathcal{GF}\neg a) \wedge \mathcal{G}(a \leftrightarrow \mathcal{XXX}a)$$

The first part of the formula says that a will be true infinitely many times. The second part of the formula says that a will be false infinitely many times. The third part of the formula says that a is true in a world x_i if and only if a is true in a world x_{i+3}. So we need to have a repeating pattern of a being true in a world, followed by two worlds where a is false.

This can be done with the LTL Kripke model $\langle W, R, V\rangle$ where

$$W = \{x_i \mid i \in \mathbb{N}\}$$

and $R(x_i)$ is defined for all $i \in \mathbb{N}$ by

$$R(x_i) = \{x_j \mid j \in \mathbb{N} \text{ and } j \geq i\}$$

and $V(x_i)$ is defined for all $i \in \mathbb{N}$ by

$$V(x_i) = \begin{cases} \{a\} & \text{if } i \bmod 3 = 0 \\ \emptyset & \text{otherwise} \end{cases}$$