Formal Reasoning 2018
Solutions Test Blocks 1, 2 and 3: Additional Test
(09/01/19)

1. Does the following hold? (20 points)

$$(\exists x \in D\,(P(x) \to Q(x))) \vDash ((\exists x \in D\,P(x)) \to (\exists x \in D\,Q(x)))$$

Explain your answer.

No, it doesn't hold.

It suffices to give a single model M_1 and interpretation I_1 for which it doesn't hold.

Let M_1 be the model

Domain(s)	$\mathbb{N}$
Predicate(s)	is even
	is negative
Relation(s)	–

And let I_1 be the interpretation

D	$\mathbb{N}$
$P(x)$	x is even
$Q(x)$	x is negative

Then $(\exists x \in D\,(P(x) \to Q(x)))$ holds, because we can take $x = 1$, which is not even, so $P(x)$ doesn't hold and automatically $P(x) \to Q(x)$ does hold, independent of $Q(x)$.

However $((\exists x \in D\,P(x)) \to (\exists x \in D\,Q(x)))$ does not hold, since the first part $((\exists x \in D\,P(x))$ does hold because we can take $x = 2$, but the second part $\exists x \in D\,Q(x))$ does not hold, since there are no negative natural numbers.

2. (a) We define: (20 points)

$$L_{a,b} := \{uavbw \mid u, v, w \in \{a,b\}^* \text{ and } |u| + |w| = |v|\}$$

For example $ababaabb \in L_{a,b}$, with $u = ab$, $v = baa$ and $w = b$. Note that all words in $L_{a,b}$ have even length.
Show that the language $L_{a,b}$ is context-free.

$$\begin{aligned} S &\to AB \\ A &\to a \mid aAa \mid aAb \mid bAa \mid bAb \\ B &\to b \mid aBa \mid aBb \mid bBa \mid bBb \end{aligned}$$

Non-terminal A builds u, the obliged a and the first part of v; non-terminal b builds the second part of v, the obliged b and w. Each time u is expanded, v is expanded with the same length. And the same holds for w. Therefore $|u| + |w| = |v|$.

Furthermore, note that A and B together create all words of odd length.

An equivalent grammar is:

$$\begin{aligned} S &\to AB \\ A &\to a \mid CAC \\ B &\to b \mid CBC \\ C &\to a \mid b \end{aligned}$$

(b) We define: (20 points)

$$L_2 := \{uu \mid u \in \{a,b\}^*\}$$

For example $abbabb \in L_2$, with $u = abb$. Note that all words in L_2 have even length.

Show that the language $\overline{L_2}$ is context-free.

Note that all words of odd length are in $\overline{L_2}$.

Now assume that $x = x_1 x_2 \cdots x_n$ is a word of length n in $\overline{L_2}$ and n is even. Note that every x_i represents a single symbol. Then there must be an i in $1, \ldots, n/2$ where $x_i = a$ and $x_{i+n/2} = b$ (or vice versa). The first case implies that we can write $x = uavbw$ with $|u| + |w| = |v|$, hence $x \in L_{a,b}$. The second case implies that we can write $x = ubvaw$ with $|u| + |w| = |v|$, hence $x \in L_{b,a}$.

So this means that $\overline{L_2}$ can be created with the grammar

$$\begin{aligned} S &\to A \mid B \mid AB \mid BA \\ A &\to a \mid aAa \mid aAb \mid bAa \mid bAb \\ B &\to b \mid aBa \mid aBb \mid bBa \mid bBb \end{aligned}$$

3. Given a graph $G_1 = \langle V_1, E_1 \rangle$ and a graph $G_2 = \langle V_2, E_2 \rangle$, we say that G_1 is a *subgraph* of G_2 if and only if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. Count the number of subgraphs of the complete graph on five points K_5. (30 points)

If you give an expression for this number in terms of binomial coefficients and arithmetic operations, but do not have time to compute this to an explicit number, you can still get partial points for this exercise.

We can create a subgraph of G_2 in two steps.

Step 1 Choose 0, 1, 2, 3, 4, or 5 vertices out of the set $\{1, 2, 3, 4, 5\}$.

Step 2 For each of the possible edges between the chosen vertices, choose whether it should be included or not.

So we get this:

- Choose 0 vertices. This can be done in $\binom{5}{0} = 1$ way. And it automatically has 0 edges, so there is one subgraph of this type.
- Choose 1 vertex. This can be done in $\binom{5}{1} = 5$ ways. Obviously, these subgraphs have no edges. So there are 5 subgraphs of this type.

- Choose 2 vertices. This can be done in $\binom{5}{2} = 10$ ways. For each graph there is 1 possible edges, so we get $2^1 = 2$ graphs for each set of 2 vertices. So there are $10 \cdot 2 = 20$ subgraphs of this type.
- Choose 3 vertices. This can be done in $\binom{5}{3} = 10$ ways. For each graph there are 3 possible edges, so we get $2^3 = 8$ graphs for each set of 3 vertices. So there are $10 \cdot 8 = 80$ subgraphs of this type.
- Choose 4 vertices. This can be done in $\binom{5}{4} = 5$ ways. For each graph there are 6 possible edges, so we get $2^6 = 64$ graphs for each set of 4 vertices. So there are $5 \cdot 64 = 320$ subgraphs of this type.
- Choose 5 vertices. This can be done in $\binom{5}{5} = 1$ way. For each graph there are 10 possible edges, so we get $2^{10} = 1024$ graphs for each set of 5 vertices. So there are $5 \cdot 64 = 320$ subgraphs of this type.

If we add all these values we get

$$1 + 5 + 20 + 80 + 320 + 1024 = 1450$$

possible subgraphs.

The numbers above are based upon the fact that we know that a graph of n vertices has $\frac{1}{2} \cdot n \cdot (n-1)$ edges. So we get basically this formula:

$$\sum_{i=0}^{5} \binom{5}{i} \cdot 2^{\frac{1}{2} \cdot i \cdot (i-1)}$$