On relating type theories and set theories*

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Introduction

The original motivation[1] for the work described in this paper was to determine the proof theoretic strength of the type theories implemented in the proof development systems Lego and Coq, [Luo and Pollack 92, Barras et al 96]. These type theories combine the impredicative type of propositions[2], from the calculus of constructions, [Coquand 90], with the inductive types and hierarchy of type universes of Martin-Lof's constructive type theory, [Martin-Lof 84]. Intuitively there is an easy way to determine an upper bound on the proof theoretic strength. This is to use the 'obvious' **types-as-sets** interpretation of these type theories in a strong enough classical axiomatic set theory. The elementary forms of type of Martin-Lof's type theory have their familiar set theoretic interpretation, the impredicative type of propositions can be interpreted as a two element set and the hierarchy of type universes can be interpreted using a corresponding hierarchy of strongly inaccessible cardinal numbers. The assumption of the existence of these cardinal numbers goes beyond the proof theoretic strength of ZFC. But Martin-Lof's type theory, even with its W types and its hierarchy of universes, is not fully impredicative and has proof theoretic strength way below that of second order arithmetic. So it is not clear that the strongly inaccessible cardinals used in our upper bound are really needed. Of course the impredicative type of propositions does give a fully impredicative type theory, which certainly pushes up the proof theoretic strength to a set theory[3], Z^-, whose strength is well above that of second order arithmetic. The hierarchy of type universes will clearly

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[1]The same motivation may be found in [Werner 97]. More or less the same tools are used there as here; i.e. the TS and ST interpretations. But that paper focuses on slightly different results to the ones obtained here.

[2]Here we will ignore the use of any rules for putting types other than Π types into the impredicative type of propositions

[3]The theory Z^- is obtained from Zermelo set theory, Z, by only using formulae with restricted quantifiers in the separation axiom scheme

lead to some further strengthening. But is it necessary to go beyond ZFC to get an upper bound?

Surprisingly perhaps, the 'obvious' types-as-sets interpretation[4] has hardly been studied systematically[5]. So it is the main aim of this paper to start such a systematic study. In section 2 we first present some of the details of the TS interpretation of a type theory $\mathsf{MLW}^{\mathsf{ext}}$ that is a reformulation of Martin-Lof's extensional type theory with W types but no type universes. This interpretation is carried out in the standard axiomatic set theory ZFC and so gives a proof theoretic reduction of $\mathsf{MLW}^{\mathsf{ext}}$ to ZFC. Of course this result is much too crude and we go on in section 2 to describe two approaches to getting a better result.

The first approach is to make the type theory classical by adding the natural formulation of the law of excluded middle. It turns out that to carry through the interpretation we need to strengthen the set theory by adding a global form of the axiom of choice and we get a proof theoretic reduction of $\mathsf{MLW}^{\mathsf{ext}} + \mathsf{EM}$ to ZFGC. Fortunately it is known that the strengthened set theory is not proof theoretically stronger, so that we do get a reduction of $\mathsf{MLW}^{\mathsf{ext}} + \mathsf{EM}$ to ZFC.

Section 2 ends with the second approach, which is to replace the classical set theory by a constructive set theory, CZF^+, that is based on intuitionistic logic rather than classical logic. So we get a reduction of $\mathsf{MLW}^{\mathsf{ext}}$ to CZF^+.

In section 3 we extend the results of section 2 by adding first a type universe reflecting the forms of type of $\mathsf{MLW}^{\mathsf{ext}}$ and then an infinite cumulative hierarchy of such type universes. To extend the TS interpretation to the resulting type theories we use, in classical set theory, strongly inaccessible cardinal numbers for the type theories with EM, and in constructive set theory, *inaccessible sets* as introduced in [Griffor and Rathjen 96]. Finally in section 3, we formulate type theories having rules for the impredicative type of propositions of the calculus of constructions and formulate corresponding axioms of constructive set theory and again describe how each of these type theories has a TS interpretation into a corresponding set theory.

In section 4 we briefly describe how the **sets-as-trees** interpretation of CZF into the type theory MLWU, first presented in [Aczel 78] and then developed further in [Aczel 82, Aczel 86, Griffor and Rathjen 94, Griffor and Rathjen 96], extends to the other set theories, giving reductions to the corresponding type theories with an extra type universe. Fortunately each type theory with an infinite hierarchy of type universes is proof theoretically as strong as the type theory with a type universe added on top, so that we end up with results stating that to each of the type theories we consider that have an infinite hierarchy of type universes there is a corresponding set theory of the same proof theoretic strength. In particular the type theory $\mathsf{MLWPU}_{<\omega}$, that is our aproximation to the type theories implemented in Lego and Coq, has the same proof theoretic strength as the set theory $\mathsf{CZF}^+\mathsf{pu}_{<\omega}$. This last result does not solve the original problem motivating our work as the set theory is unfamiliar. Nevertheless I think that it does give a new handle on the problem. The new set theory is an interesting one and I plan to present some results about it on a future occasion.

In section 1 we set up our particular approach to the syntax of our type theories and the ST interpretation of them. We have tried to make this a simple as possible. We have prefered to focus on extensional Martin-Lof type theories having extensional

[4]Here abbreviated TS interpretation.
[5]But see [Werner 97].

equality types $Eq(A, a_1, a_2)$ for the ST interpretation, as the rules for these types are easily justified. We have also added equality types $EQ(A_1, A_2)$ for the same reason. For the reverse **sets-as-trees** interpretation these equality types are not needed, but nor are any intensional equality types needed, so we can simply drop the extensionality rules.

1 The general form of the syntax and set theoretical semantics of our type theories

1.1 Syntax

We give the general form of the syntax of the type theories we will consider.

1.1.1 Pseudoterms

The **pseudoterms**, M, are given by the following abstract syntax.

$$M ::= x \mid c_0 \mid c_1(M) \mid c_2(M, M) \mid c_3(M, M, M) \mid (Qx : M)M$$

where $x : VAR$, $c_0 : C_0$, $c_1 : C_1$, $c_2 : C_2$, $c_3 : C_3$ and $Q : QUANT$. Here $VARS$ is an infinite set of variables and the finite sets C_i, for $i = 0, 1, 2, 3$, and $QUANT$ will depend on the type theory.

Each Q operates as a variable binder so that free occurrences of x in M' get bound in $(Qx : M)M'$. The notions of free and bound occurrences of variables and the substitution operation are defined in the standard way. We write $M[M_1, \ldots, M_n/x_1, \ldots x_n]$ for the result of simultaneously substituting M_i for x_i in M, for $i = 1, \ldots n$, relabelling bound variables in the usual way so as to avoid variable clashes. For this we assume that the variables $x_1, \ldots, x_n$ are pairwise distinct. In general we will not distinguish between pseodoterms that only differ in a suitable relabelling of the bound variables.

1.1.2 Pseudojudgements and the formal judgements of a type theory

Definition 1.1 *A* **pseudojudgement** *has the form*

$$? \;\Rightarrow \mathsf{B}$$

where $?$ *is a pseudocontext and* B *is a pseudobody.*

- *A* **pseudocontext** *is a finite sequence* $x_1 : M_1, \ldots, x_n : M_n$ *of* **pseudodeclarations**, $x_i : M_i$ *for* $i = 1, \ldots, n$ *where each* M_i *is a pseudoterm and each* $x_i : VAR$ *and, for* $1 \leq j < i$, x_i *is distinct from* x_j *and is not free in* M_j.

- *A* **pseudobody** *has one of the following four forms.*

$$\begin{array}{l} M \ \mathsf{type}, \\ M_1 = M_2, \\ M_0 : M, \\ M_1 = M_2 : M \end{array}$$

When the pseudocontext is the empty sequence then we get a pseudojudgement $\Rightarrow \mathsf{B}$ *which will usually simply be written* B.

If ? is a pseudocontext $x_1 : M_1, \ldots, x_n : M_n$ then a variable y is **new to** ? if y is distinct from each x_i and not free in any M_i.

Note: If ? is a pseudocontext $x_1 : M_1, \ldots, x_n : M_n$, x is a variable distinct from each x_i and M is a pseudoterm that has no free occurrences of any x_i then $x_1 : M_1[M/x], \ldots, x_n : M_n[M/x]$ is also a pseudocontext that we will abbreviate ?$[M/x]$. Also we can define the result $\mathsf{B}[M/x]$ of substituting M for x in a pseudobody B in the obvious way. For example $(M_1 = M_2)[M/x]$ is defined to be $M_1[M/x] = M_2[M/x]$.

The rules of inference of the type theories that we will consider will be given schematically and will have **instances** of the following form.

$$\frac{J_1 \quad \cdots \quad J_k}{J}$$

where $k \geq 0$ and $J_1 \cdots J_k$ are the **premisses** and J is the **conclusion** of the instance, both the premisses and the conclusion being pseudojudgements. When $k = 0$, so that there are no premisses then the line above the conclusion will be omitted in writting the inference.

The schemes presenting the rules will have the abbreviated form

$$\frac{?_1 \Rightarrow \mathsf{B}_1 \quad \cdots \quad ?_k \Rightarrow \mathsf{B}_k}{\Delta \Rightarrow \mathsf{B}},$$

which is unabbreviated by making explicit an implicit pseudocontext metavariable ? of the scheme by adding it to the front of the left hand side of each premiss and the conclusion to get the scheme

$$\frac{?, ?_1 \Rightarrow \mathsf{B}_1 \quad \cdots \quad ?, ?_k \Rightarrow \mathsf{B}_k}{?, \Delta \Rightarrow \mathsf{B}}.$$

Note that an unabbreviated scheme will generally involve metavariables and an instance of the scheme will be obtained by substituting for the metavariables, provided that the side conditions of the scheme hold.

A pseudojudgement is a theorem and so a **formal judgement** of the type theory, if it is in the smallest class of pseudojudgements that includes the conclusion whenever it includes the premisses of any instance of a rule of the type theory. Whenever a pseudocontext ? appears in a formal judgement $? \vdash \mathsf{B}$ then we call ? a **context**.

All our type theories will have a common list of general rules of inference. These come under three headings, assumption rules, equality rules and substitution rules.

1.1.3 General Rules

Assumption Rules In these rules the variable x must be new to the implicit context ?; i.e. not appear in ?.

$$\frac{A \ \mathsf{type}}{x : A \Rightarrow x : A} \qquad \frac{\Delta \Rightarrow \mathsf{B} \qquad A \ \mathsf{type}}{x : A, \Delta \Rightarrow \mathsf{B}}$$

Equality Rules

$$\frac{A \ \mathsf{type}}{A = A} \qquad \frac{A_1 = A_2}{A_2 = A_1} \qquad \frac{A_1 = A_2 \qquad A_2 = A_3}{A_1 = A_3}$$

$$\frac{a : A}{a = a : A} \qquad \frac{a_1 = a_2 : A}{a_2 = a_1 : A} \qquad \frac{a_1 = a_2 : A \qquad a_2 = a_3 : A}{a_1 = a_3 : A}$$

$$\frac{a : A_1 \qquad A_1 = A_2}{a : A_2} \qquad\qquad \frac{a_1 = a_2 : A_1 \qquad A_1 = A_2}{a_1 = a_2 : A_2}$$

Substitution Rule

$$\frac{x : A, \Delta \Rightarrow \mathsf{B} \qquad a : A}{\Delta[a/x] \Rightarrow \mathsf{B}[a/x]}$$

Congruence Rules

$$\frac{x : A, \Delta \Rightarrow C\ \mathsf{type} \qquad a_1 = a_2 : A}{\Delta[a_1/x] \Rightarrow C[a_1/x] = C[a_2/x]} \qquad\qquad \frac{x : A, \Delta \Rightarrow c : C \qquad a_1 = a_2 : A}{\Delta[a_1/x] \Rightarrow c[a_1/x] = c[a_2/x] : C[a_1/x]}$$

1.2 Types-as-Sets

We now assume given a fixed type theory T and a fixed set theory S. We will work informally in the set theory S.

A types-as-sets interpretation (TS interpretation) of T in S is determined by the following set theoretic data.

- For each c_0, a set c_0^o
- For each c_n, where $n = 1, 2, 3$, a definable n-place operation c_n^o assigning a set $c_1^o(A_1, \ldots, A_n)$ to each n-tuple $A_1, \ldots, A_n$ of sets.
- For each Q, a definable operation Q^o that assigns to each set B that is a function a set $Q^o(B)$. In practise, if A is a set and F is a definable unary operation on sets then, using the Replacement Axiom Scheme, that will be available in our set theory, we may form the set $B = \{(a, F(a)) \mid a \in A\}$ which is a function defined on A. The result of applying Q^o to this set B will be written $(Q^o a \in A)F(a)$.

1.2.1 The interpretation functions

By a **variable assignment** we mean a set theoretic function that assigns a set $\xi(x)$ to each variable x.

We can define the function mapping each variable assignment ξ to the interpretation $[[M]]_\xi$ of M, for each pseudoterm M. The definition is by structural induction on the formation of the pseudoterm M, using the variable assignment when M is a variable and using the corresponding operation on sets, as illustrated earlier, for each other form of expression.

In the following $n = 1, 2$ or 3.

$$\begin{aligned}
[[x]]_\xi &= \xi(x) \\
[[c_0]]_\xi &= c_0^o \\
[[c_n(M_1, \ldots, M_n)]]_\xi &= c_n^o([[M_1]]_\xi, \ldots, [[M_n]]_\xi) \\
[[(Qx : M)M']]_\xi &= (Q^o a \in [[M]]_\xi)[[M']]_{\xi(a/x)}
\end{aligned}$$

Here $\xi(a/x)$ is the variable assignment ξ' that is like ξ except that $\xi'(x) = a$.

The following lemmas are proved by a routine induction on the structure of the pseudoterm M.

Lemma 1.2 *If the variable x is not free in the pseudoterm M and ξ, ξ' are variable assignments that agree except possibly at x then $[[M]]_\xi = [[M]]_{\xi'}$.*

Lemma 1.3 (Substitution Lemma) *For all pseudoterms M, M', all variables x and all variable assignments ξ*

$$[[M[M'/x]]]_\xi = [[M]]_{\xi([[M']]_\xi/x)}.$$

1.2.2 Soundness

Definition 1.4 *If ? is a pseudocontext $x_1 : M_1, \ldots, x_n : M_n$ then let $\xi \models ?$ if*

$$\xi(x_i) \in [[M_i]]_\xi \textit{ for } i = 1, \ldots n.$$

Lemma 1.5 *If ? is a pseudocontext $x_1 : M_1, \ldots, x_n : M_n$, x is a variable distinct from each x_i and M is a pseudoterm that has no free occurrences of any x_i then for each variable assignment ξ*

$$\xi \models ?\,[M/x] \iff \xi([[M]]_\xi/x) \models ?\,.$$

Definition 1.6 *We define $\xi \models \mathsf{B}$ for each form of pseudobody B.*

- $\xi \models M$ type *for any pseudoterm M,*
- $\xi \models M_1 = M_2$ *if* $[[M_1]]_\xi = [[M_2]]_\xi$,
- $\xi \models M : M'$ *if* $[[M]]_\xi \in [[M']]_\xi$,
- $\xi \models M_1 = M_2 : M'$ *if* $[[M_1]]_\xi = [[M_2]]_\xi \in [[M']]_\xi$,

Lemma 1.7 $\xi \models \mathsf{B}[M/x] \iff \xi([[M]]_\xi/x) \models \mathsf{B}$.

Definition 1.8 *A pseudojudgement $? \Rightarrow \mathsf{B}$ is* **valid**, *written $\models\, ? \Rightarrow \mathsf{B}$ if, for all variable assignments ξ,*

$$\xi \models ? \quad \textit{implies} \quad \xi \models \mathsf{B}.$$

Definition 1.9 *A rule of inference is* **sound** *if, for every instance*

$$\frac{J_1 \;\cdots\; J_k}{J}$$

of the rule, if the premisses are valid then so is the conclusion; i.e.

$$\models J_1 \,\&\cdots\&\, \models J_k \quad \textit{implies} \quad \models J.$$

Proposition 1.10 *Each general rule is sound. Moreover, for each quantifier Q of the type theory the following congruence rule is sound.*

$$\frac{x : M \Rightarrow M_1 = M_2}{(Qx : M)M_1 = (Qx : M)M_2}$$

The proof of this result is straightforward. The assumption and equality rules are trivial. The substitution and congruence rules make use of previously stated lemmas.

The type theory T is **sound** if each of its rules is sound. The following result is by structural induction following the inductive definition of the formal judgements of a type theory.

Lemma 1.11 *If the type theory T is sound then every formal judgement of T is valid.*

When we have a sound TS interpretation of a type theory T in a set theory S we will write $\mathsf{T} \leq_{TS} \mathsf{S}$.

2 The theory $\mathsf{MLW}^{\mathsf{ext}}$

We will start with the theory MLW. The abstract syntax of the theory is determined by the following syntax equations.

$$\begin{array}{rl} c_0 ::= & \mathbf{0} \mid \mathbf{1} \mid \mathbf{2} \mid * \mid 1 \mid 2, \\ c_1 ::= & R_0 \mid \pi_1 \mid \pi_2, \\ c_2 ::= & R_1 \mid pair \mid sup \mid app \mid rec, \\ c_3 ::= & R_2, \\ Q ::= & \Pi \mid \Sigma \mid W \mid \lambda. \end{array}$$

2.1 Some defined forms of pseudoterm

$$\begin{array}{rl} (M_1 \to M_2) & = (\Pi\_ : M_1)M_2 \\ (M_1 \times M_2) & = (\Sigma\_ : M_1)M_2 \\ (M_1 + M_2) & = (\Sigma x : \mathbf{2})R_2(M_1, M_2, x) \\ \mathbb{N} & = (Wx : \mathbf{2})R_2(\mathbf{0}, \mathbf{1}, x) \end{array}$$

Note that the underscore, _, in the first two definitions represents a vacuous variable; i.e. a variable that is being bound by Π and Σ but does not occur in M_2.

2.2 Special Rules for MLW

Type Formation Rules

$$c \ \mathsf{type} \qquad\qquad (c \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\})$$

$$\frac{A_1 \ \mathsf{type} \qquad A_2 \ \mathsf{type} \qquad c : \mathbf{2}}{R_2(A_1, A_2, c) \ \mathsf{type}}$$

$$\frac{x : A \Rightarrow B \ \mathsf{type}}{(Qx : A)B \ \mathsf{type}} \qquad\qquad (Q \in \{\Pi, \Sigma, W\})$$

Using the definitions above we have the following derived type formation rules.

$$\mathbb{N} \ \mathsf{type} \qquad\qquad \frac{A_1 \ \mathsf{type} \qquad A_2 \ \mathsf{type}}{(A_1 \# A_2) \ \mathsf{type}} \qquad (\# \in \{\to, \ \times, \ +\})$$

Introduction Rules

$$* : \mathbf{1} \qquad 1 : \mathbf{2} \qquad 2 : \mathbf{2}$$

$$\frac{x : A \Rightarrow b : B}{(\lambda x : A)b : (\Pi x : A)B}$$

$$\frac{x : A \Rightarrow B \ \mathsf{type} \qquad a : A \qquad b : B[a/x]}{pair(a, b) : (\Sigma x : A)B}$$

$$\frac{x : A \Rightarrow B \ \mathsf{type} \qquad a : A \qquad f : (B[a/x] \to (Wx : A)B)}{sup(a, b) : (Wx : A)B}$$

Special Congruence Rules

$$\frac{x : A \Rightarrow B_1 = B_2}{(Qx : A)B_1 = (Qx : A)B_2} \qquad (Q \in \{\Pi,\ \Sigma,\ W\})$$

$$\frac{x : A \Rightarrow b_1 = b_2 : B}{(\lambda x : A)b_1 = (\lambda x : A)b_2 : (\Pi x : A)B}$$

Elimination rules

$$\frac{x : \mathbf{0} \Rightarrow C \ \mathsf{type} \qquad a : \mathbf{0}}{R_0(a) : C[a/x]}$$

$$\frac{x : \mathbf{1} \Rightarrow C \ \mathsf{type} \qquad a : \mathbf{1} \qquad c : C[*/x]}{R_1(c, a) : C[a/x]}$$

$$\frac{x : \mathbf{2} \Rightarrow C \ \mathsf{type} \qquad a : \mathbf{2} \qquad c_1 : C[1/x] \qquad c_2 : C[/x]}{R_2(c_1, c_2, a) : C[a/x]}$$

$$\frac{x : A \Rightarrow B \ \mathsf{type} \qquad f : (\Pi x : A)B \qquad a : A}{app(f, a) : B[a/x]}$$

$$\frac{x : A \Rightarrow B \ \mathsf{type} \qquad c : (\Sigma x : A)B}{\begin{cases} \pi_1(c) : A \\ \pi_2(c) : B[\pi_1(c)/x] \end{cases}}$$

$$\frac{\begin{cases} x : A \Rightarrow B \ \mathsf{type} \qquad z : W \Rightarrow C \ \mathsf{type} \\ b : (\Pi x : A)(\Pi u : B \to W)D(x, u) \qquad e : W \end{cases}}{rec(b, e) : C[e/z]}$$

In the last rule we used W to abbreviate $(Wx : A)B$ and $D(x, u)$ to abbreviate $(\Pi y : B)C[app(u, y)/z] \to C[sup(x, u)/z]$.

Computation Rules

$$\frac{A_1 \ \mathsf{type} \qquad A_2 \ \mathsf{type}}{\begin{cases} R_2(A_1, A_2, 1) = A_1 \\ R_2(A_1, A_2, 2) = A_2 \end{cases}}$$

$$\frac{x : \mathbf{1} \Rightarrow C \ \mathsf{type} \qquad c : C[*/x]}{R_1(c, *) = c : C[*/x]}$$

$$\frac{x : \mathbf{2} \Rightarrow C \ \mathsf{type} \qquad c_1 : C[1/x] \qquad c_2 : C[2/x]}{\begin{cases} R_2(c_1, c_2, 1) = c_1 : C[1/x] \\ R_2(c_1, c_2, 2) = c_2 : C[2/x] \end{cases}}$$

$$\frac{x : A \Rightarrow b : B \qquad a : A}{app((\lambda x : A)b, a) = b[a/x] : B[a/x]}$$

$$\frac{x : A \Rightarrow B \ \mathsf{type} \qquad a : A \qquad b : B[a/x]}{\begin{cases} \pi_1(pair(a, b)) = a : A \\ \pi_2(pair(a, b)) = b : B[a/x] \end{cases}}$$

$$\frac{\begin{cases} x : A \Rightarrow B \ \mathsf{type} \qquad z : W \Rightarrow C \ \mathsf{type} \\ b : (\Pi x : A)(\Pi u : B \to W)D(x, u) \qquad a : A \qquad f : B[a/x] \to W \end{cases}}{rec(b, sup(a, f)) = app(app(app(b, a), f), g) : C[sup(a, f)/z]}$$

In this last rule we used the following abbreviations.

$$\begin{array}{rl} W & \text{for } (Wx : A)B, \\ D(x, u) & \text{for } (\Pi y : B)C[app(u, y)/z] \to C[sup(x, u)/z], \\ g & \text{for } (\lambda y : B[a/x])rec(b, app(f, y)). \end{array}$$

2.3 Extending to $\mathsf{MLW}^{\mathsf{ext}}$

We first extend the syntax equations as follows.

$$\begin{array}{l} c_2 ::= \cdots \mid EQ \\ c_3 ::= \cdots \mid Eq \end{array}$$

We add the rules of inference given by the following schemes in abbreviated form.

$$\frac{A \ \mathsf{type} \qquad a_1 : A \qquad a_2 : A}{Eq(A, a_1, a_2) \ \mathsf{type}} \qquad\qquad \frac{A_1 \ \mathsf{type} \qquad A_2 \ \mathsf{type}}{EQ(A_1, A_2) \ \mathsf{type}}$$

$$\frac{a_1 = a_2 : A}{* : Eq(A, a_1, a_2)} \qquad\qquad \frac{A_1 = A_2}{* : EQ(A_1, A_2)}$$

$$\frac{c : Eq(A, a_1, a_2)}{\begin{cases} a_1 = a_2 : A \\ c = * : Eq(A, a_1, a_2) \end{cases}} \qquad\qquad \frac{c : EQ(A_1, A_2)}{\begin{cases} A_1 = A_2 \\ c = * : EQ(A_1, A_2) \end{cases}}$$

2.4 The TS interpretation of $\mathsf{MLW}^{\mathsf{ext}}$ in ZFC

We will work informally in the set theory ZFC. We use the usual von Neumann definition of the natural numbers; i.e. $0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}$, etc .... Ordered pairs are defined as usual; i.e. for sets a, b we define $(a, b) = \{\{a\}, \{a, b\}\}$. As usual functions are single valued sets of ordered pairs. For any set b, its **domain** is the set $dom(b) = \{x \mid \exists y\ (x, y) \in b\}$.

If a is a set and B is a definable operation that assigns a set $B(x)$ to each $x \in a$ then we let $\Pi_{x\in a}B(x)$ be the set of all the functions f, with domain a, such that $f(x) \in B(x)$ for all $x \in a$. Also, we let $\Sigma_{x\in a}B(x)$ be the set of all pairs (x, y) such that $x \in a$ and $y \in B(x)$.

A **function coding** in set theory consists of a pair of definable operations APP, LAM on sets, APP being binary and LAM being unary, such that the following condition holds. If f is a function and $a \in dom(f)$ then

$$APP(LAM(f), a) = f(a).$$

The standard example of a function coding is given by the definitions

$$\begin{aligned} APP(a, b) &= \{x \mid \exists y[x \in y \ \&\ (b, y) \in a]\} \\ LAM(a) &= a \end{aligned}$$

for all sets a, b. Later it will be convenient to use a non-standard function coding. In the following we assume given some function coding. Given sets a, b, c, d let

$$\begin{aligned} EXP(a, b) &= \{LAM(f) \mid f : a \to b\} \\ PI_{x\in a}B(x) &= \{LAM(f) \mid f \in \Pi_{x\in a}B(x)\} \text{ if } B(x) \text{ is a set for each } x \in a \\ APP_2(a, b, c) &= APP(APP(a, b), c) \\ APP_3(a, b, c, d) &= APP(APP(APP(a, b), c), d) \end{aligned}$$

We now present the set theoretic interpretations of the syntactic operations of $\mathsf{ML}^{\mathsf{ext}}$, leaving the interpretations for the W rules til later.

$$\mathbf{0}^o = 0,\ \mathbf{1}^o = 1,\ \mathbf{2}^o = 2,\ *^o = 0,\ 1^o = 0,\ 2^o = 1$$

$$R_0^o(a) = a,\ \pi_1^o(a) = \{x \mid \exists y\ (x, y) = a\},\ \pi_2^o(a) = \{y \mid \exists x\ (x, y) = a\}$$

$$R_1^o(a, b) = a,\ pair^o(a, b) = (a, b),\ app^o(a, b) = APP(a, b)$$

$$R_2^o(a, b, c) = \{x \mid (c = 1^o \ \&\ x \in a) \vee (c = 2^o \ \&\ x \in b)\}$$

$$EQ^o(a, b) = \{x \mid x = 0 \ \&\ a = b\},\ Eq^o(a, b, c) = \{x \mid x = 0 \ \&\ b = c \ \&\ b \in a\}$$

If b is a function with domain a let

$$\begin{aligned} \lambda^o(b) &= LAM(b) \\ \Pi^o(b) &= PI_{x\in a}b(x) \\ \Sigma^o(b) &= \Sigma_{x\in a}b(x) \end{aligned}$$

To deal with the W rules we will need the following result.

Theorem 2.1

1. *For each set b there is a smallest set W such that*

$$\text{if } x \in dom(b) \text{ and } f \in EXP(b(x), W) \text{ then } (x, f) \in W.$$

We write $\mathcal{W}(b)$ for this set W.

2. *Given a set g let*

$$Y(g) = \Sigma_{x \in dom(g)} \Sigma_{u \in dom(APP(g,x))} dom(APP_2(g, x, u)).$$

There is a smallest set f such that if $(x, (u, v)) \in Y(g)$ and $X_{u,v} \subseteq f$, where $X_{u,v} = \{(APP(u, y), APP(v, y)) \mid y \in dom(u)\}$, then

$$((x, u), APP_3(g, x, u, v)) \in f.$$

We write $\mathcal{R}(g)$ for this set f.

3. *Given sets a, b, c, let*

$$g \in PI_{x \in a} PI_{u \in EXP(b(x),W)} d((x, u)),$$

where $W = \mathcal{W}(b)$ and, for $w = (x, u) \in W$,

$$d(w) = EXP(PI_{y \in b(x)} c(APP(u, y)), c(w)).$$

Then $\mathcal{R}(g)$ is the unique function $f \in \Pi_{w \in W} c(w)$ such that if $w = (x, u) \in W$ then

$$f(w) = APP_3(g, x, u, LAM(H(f, u)).$$

Here $H(f, u)$ is the function $h \in \Pi_{y \in b(x)} c(APP(u, y))$ such that

$$h(y) = f(APP(u, y)) \text{ for } y \in b(x).$$

2.4.1 Proof of the theorem in ZFC

The first two parts of this theorem are applications of the following result.

Lemma 2.2 *Let Θ be a definable operation on sets such that, for some set B, whenever X is a set such that $\Theta(X)$ has an element then there is a surjective function $f : b \to X$ for some $b \in B$. Then there is a smallest class I such that*

$$X \subseteq I \implies \Theta(X) \subseteq I.$$

Moreover I is a set.

To prove part 1 of the theorem ,using this lemma, it suffices to let

$$\Theta(X) = \{(x, LAM(f)) \mid x \in dom(f) \;\&\; f : b(x) \to X \text{ is onto } X\},$$

and choose $B = \{b(x) \mid x \in dom(b)\}$.

For part 2 we let

$$\Theta(X) = \{((x,u), APP_3(g,x,u,v)) \mid (x,(u,v)) \in Y(g) \ \& \ X = X_{u,v}\},$$

and choose $B = \{X_{u,v} \mid (x,(u,v)) \in Y(g)\}$. For part 3 of the theorem, first observe that, by an easy induction following the inductive definition of $\mathcal{R}(g)$, $dom(\mathcal{R}(g)) \subseteq W$. Now, by another easy induction, this time on the inductive definition of W, observe that, for each $w = (x,u) \in W$,

$$APP_3(g,x,u,LAM(H(f,u)))$$

is the unique z such that $(w,z) \in \mathcal{R}(g)$ and moreover $z \in c(w)$. All this shows that $\mathcal{R}(g)$ is an f satisfying the desired conditions. Finally, another proof by induction on W will show that $\mathcal{R}(g)$ is the unique f satisfying these conditions.

We now turn to the proof of the lemma. Let ? be the operation on sets given by

$$?\,(Y) = \bigcup_{X \in Pow(Y)} \Theta(X),$$

for each set Y. The operation ? is monotone and we must show that it has a least fixed point. By transfinite recursion on ordinals we can define sets I^α, for ordinals α, so that

$$I^\alpha = ?\,(I^{<\alpha}),$$

where $I^{<\alpha} = \bigcup_{\beta<\alpha} I^\beta$. Let κ be an infinite regular ordinal such that $card(b) < \kappa$ for all $b \in B$.

Claim 1: $I^\kappa \subseteq I^{<\kappa}$

To see this, let $a \in I^\kappa$. Then $a \in \Theta(X)$ for some set $X \subseteq I^{<\kappa}$. For each $x \in X$ let $h(x)$ be the least ordinal $\gamma < \kappa$ such that $x \in I^\gamma$. By the assumption on Θ there is $b \in B$ and a function $f : b \to X$ that is onto X. If $\alpha = card(b)$ then $\alpha < \kappa$ and there is a function $g : \alpha \to b$ that is onto b. It follows that $h \circ f \circ g : \alpha \to \kappa$. As κ is regular there is $\beta < \kappa$ such that $h \circ f \circ g : \alpha \to \beta$. As $f \circ g$ is onto X it follows that $h : X \to \beta$ so that $X \subseteq I^{<\beta}$ and hence $a \in I^\beta \subseteq I^{<\kappa}$.

It is a standard consequence of this claim that I^κ is the least fixed point of ? and so is the desired set I of the lemma.[6]

To interprete the extra syntax needed for the W rules we use

$$sup^o(a,b) = (a,b),$$

$$rec^o(a,b) = \mathcal{R}(a)(b)$$

and if b is a function

$$W^o(b) = \mathcal{W}(b).$$

[6]This proof of the lemma uses the classical theory of cardinal numbers and uses AC. I do not think that AC can be avoided. Instead of AC it may be possible to use the axiom that there are unboundedly many regular ordinals.

Theorem 2.3 (ZFC) *The type theory* $\mathsf{MLW}^{\mathsf{ext}}$ *is sound.*

This result gives a proof theoretic reduction of the type theory $\mathsf{MLW}^{\mathsf{ext}}$ to the set theory ZFC. We write

$$\mathsf{MLW}^{\mathsf{ext}} \leq_{TS} \mathsf{ZFC}$$

to express this reduction. The type theory is constructive in the sense that when the propositions-as-types idea is used to represent logic then intuitionistic logic is represented and the law of excluded middle is not justified. On the other hand the set theory is classical. In the following two subsections we improve on the result by first making the type theory classical and second by making the set theory constructive.

2.5 Adding excluded middle

Recall that the logical notions are represented in MLW by using the propositions-as-types idea. In particular the operation $+$ on types represents disjunction and negation is represented by the operation that maps a type A to the type $A \to \mathbf{0}$. So to add the law EM of excluded middle to the type theory we extend the syntax

$$c_1 ::= \cdots \mid cl$$

and add the following rule.

$$\frac{A \ \ \mathsf{type}}{cl(A) : A + (A \to \mathbf{0})}.$$

We call the resulting theory $\mathsf{MLW} + \mathsf{EM}$.

We need to extend the interpretation by having an equation for the new form of pseudoterm. To do so we strengthen the axiom system ZFC by adding a one-place function symbol CH to the language of ZFC and adding the following global form of the axiom of choice.

$$\forall x[x \neq \emptyset \to CH(x) \in x].$$

The axiom schemes of ZFC should be extended to the extended language. We call the resulting axiom system ZFGC. Working in this axiom system we can define an operation CL where, for each set a,

$$CL(a) = \begin{cases} \langle \emptyset, CH(a) \rangle & \text{if } a \neq \emptyset \\ \langle \{\emptyset\}, \emptyset \rangle & \text{if } a = \emptyset \end{cases}$$

We can now let $cl^o = CL$.

It is easy to check that, for each pseudoterm A and each variable assignment ξ,

$$\xi \models [cl(A) \in A + (A \to \mathbf{0})].$$

So we get the result that

$$\mathsf{MLW}^{\mathsf{ext}} + \mathsf{EM} \leq_{TS} \mathsf{ZFGC}.$$

2.6 Reduction to a constructive set theory

We now follow the other strategy to improve on the result $\mathsf{MLW}^{\mathsf{ext}} \leq_{TS} \mathsf{ZFC}$. This is to weaken ZFC to a constructive set theory. In [Aczel 78] a constructive set theory CZF was introduced that is a subtheory of ZF whose logic is intuitionistic. This set theory was shown to have the property that when excluded middle is added to the logic then a theory $\mathsf{CZF} + EM$ is obtained that has the same theorems as ZF. Here we will consider the extension $\mathsf{CZF}^+ = \mathsf{CZF} + \mathsf{REA}$ of CZF obtained by adding to CZF the following axiom, that was first introduced in [Aczel 86].

Regular Extension Axiom (REA)

> Every set is a subset of a regular set, where a transitive set A is a **regular set** if, for every $a \in A$ and every set $R \subseteq a \times A$ such that $\forall x \in a \exists y \in A[(x, y) \in R]$ there is a set $b \in A$ such that $\forall x \in a \exists y \in b[(x, y) \in R]$ and $\forall y \in b \exists x \in a[(x, y) \in R]$.

The construction, in subsection 2.4, of the TS interpretation of $\mathsf{MLW}^{\mathsf{ext}}$ was carried out in the set theory ZFC. It is straightforward to show that the construction can be carried through in CZF^+. In fact it can all be carried through in CZF, except for the proof of Lemma 2.2 The proof in ZFC that was given here of that lemma used the power set axiom and some of the classical theory of cardinal numbers and needed the axiom of choice. Instead we can apply Theorem 5.2 of [Aczel 86] to see that the lemma is provable in CZF^+.[7]

So we now have the following result.

Theorem 2.4 (CZF^+) *The type theory* $\mathsf{MLW}^{\mathsf{ext}}$ *is sound.*

This can be expressed as

$$\mathsf{MLW}^{\mathsf{ext}} \leq_{TS} \mathsf{CZF}^+.$$

3 Adding type universes

In this section we consider natural ways of extending the type theory MLW with one or more type universes; i.e. types of types. In each case we define a corresponding way of extending set theory so that the TS interpretation extends to include the type universes.

3.1 Adding a single reflecting type universe, U

We extend the type theory MLW to MLWU by adding a type U of types that has rules that reflect the type forming rules of MLW. First we extend the syntax with

$$c_0 ::= \cdots \mid \mathsf{U}.$$

Next we add the rules given by the following schemes in abbreviated form.

$$\mathsf{U}\ \mathsf{type} \qquad \frac{A : \mathsf{U}}{A\ \mathsf{type}} \qquad c : \mathsf{U} \qquad (c \in \{\mathbf{0},\ \mathbf{1},\ \mathbf{2}\})$$

[7] The status of $\mathsf{CZF}^+\mathsf{EM} \equiv \mathsf{ZF} + \mathsf{REA}$ is unclear. Every theorem is a theorem of ZFC. But it is probable that REA is not provable in ZF.

$$\frac{A : \mathsf{U} \qquad x : A \Rightarrow B : \mathsf{U}}{(Qx : A)B : \mathsf{U}} \qquad (Q \in \{\Pi,\ \Sigma,\ W\})$$

When extending $\mathsf{MLW}^{\mathsf{ext}}$ to $\mathsf{MLW}^{\mathsf{ext}}\mathsf{U}$ we also need rules for U to reflect Eq and EQ; i.e.

$$\frac{A : \mathsf{U} \qquad a_1 : A \qquad a_2 : A}{Eq(A, a_1, a_2) : \mathsf{U}} \qquad\qquad \frac{A_1 : \mathsf{U} \qquad A_2 : \mathsf{U}}{EQ(A_1, A_2) : \mathsf{U}}$$

In order to extend the TS interpretation to $\mathsf{MLW}^{\mathsf{ext}}\mathsf{U} + \mathsf{EM}$ it suffices to add to ZFGC the axiom that there is an inaccessible cardinal and interprete U as the set U^o of all sets of set theoretic rank less than the least strongly inaccessible cardinal. If we call the resulting set theory ZFGC_1 then we get the reduction

$$\mathsf{MLW}^{\mathsf{ext}}\mathsf{U} + \mathsf{EM} \leq_{TS} \mathsf{ZFGC}_1.$$

To extend the TS interpretation of $\mathsf{MLW}^{\mathsf{ext}}$ in CZF^+ we add to CZF^+ an individual constant u and axioms expressing that u is an inaccessible set in the sense of Griffor and Rathjen, [Griffor and Rathjen 96] [8] We write $\mathsf{CZF}^+\mathsf{u}$ for the resulting theory. Now it suffices to take $\mathsf{U}^o = \mathsf{u}$ and we get the reduction

$$\mathsf{MLW}^{\mathsf{ext}}\mathsf{U} \leq_{TS} \mathsf{CZF}^+\mathsf{u}.$$

3.2 Adding an infinite hierarchy, $\mathsf{U}_0, \mathsf{U}_1, \ldots$, of reflecting type universes

This time we extend the syntax using

$$c_0 ::= \cdots \mid \mathsf{U}_n \qquad (n = 0, 1, \ldots)$$

and add rules given by the following schemes for $n = 0, 1, \ldots$.

$$\mathsf{U}_n\ \mathsf{type} \qquad \frac{A : \mathsf{U}_n}{A\ \mathsf{type}} \qquad c : \mathsf{U}_n \qquad (c \in \{\mathbf{0},\ \mathbf{1},\ \mathbf{2}\})$$

$$\frac{A : \mathsf{U} \qquad x : A \Rightarrow B : \mathsf{U}_n}{(Qx : A)B : \mathsf{U}_n} \qquad (Q \in \{\Pi,\ \Sigma,\ W\})$$

$$U_n : U_{n+1} \qquad \frac{A : \mathsf{U}_n}{A : \mathsf{U}_{n+1}}$$

In the case of $\mathsf{MLW}^{\mathsf{ext}}$ we also need the obvious rules for reflecting Eq and EQ. We get the resulting type theories $\mathsf{MLWU}_{<\omega}$ and $\mathsf{MLW}^{\mathsf{ext}}\mathsf{U}_{<\omega}$. To extend the TS interpretation we need to extend the classical and intuitionistic set theories in the following way. We add an infinite sequence u_n for $n = 0, 1, \ldots$ of individual constants to the set theoretical language and add axioms $\mathsf{u}_n \in \mathsf{u}_{n+1}$ for $n = 0, 1, \ldots$. In the classical case we also add axioms that express that each u_n is the set of sets of rank less than a strongly inaccessible cardinal number and in the constructive case we add axioms that express that each u_n is an inaccessible set. We write $\mathsf{ZFGCu}_{<\omega}$ and $\mathsf{CZF}^+\mathsf{u}_{<\omega}$ for the resulting extensions. We extend the TS interpretation by taking $\mathsf{U}_n^o = \mathsf{u}_n$ for each n and get the following reductions.

$$\mathsf{MLW}^{\mathsf{ext}}\mathsf{U}_{<\omega} + EM \quad \leq_{TS} \mathsf{ZFGCu}_{<\omega}$$

$$\mathsf{MLW}^{\mathsf{ext}}\mathsf{U}_{<\omega} \quad \leq_{TS} \mathsf{CZF}^+\mathsf{u}_{<\omega}$$

[8] i.e. a regular set that is a transitive model of CZF^+.

3.3 Adding an impredicatively Π-closed type universe P

We extend the syntax with

$$c_0 ::= \cdots \mid \mathsf{P}$$

and add rules given by the schemes

$$\mathsf{P}\ \mathsf{type} \qquad \frac{A : \mathsf{P}}{A\ \mathsf{type}} \qquad \frac{A : \mathsf{P} \qquad a_1 : A \qquad a_2 : A}{a_1 = a_2 : A}$$

$$\frac{x : A \Rightarrow B : \mathsf{P}}{(\Pi x : A)B : \mathsf{P}} \qquad \frac{x : A \Rightarrow B_1 = B_2 : \mathsf{P}}{(\Pi x : A)B_1 = (\Pi x : A)B_2 : \mathsf{P}}$$

With these rules the type P behaves like the impredicative type of propositions of the calculus of constructions, with the additional property that all the propositions in P are proof-irrelevant. Adding these rules we get the type theories MLWP and $\mathsf{MLW^{ext}P}$. To get the type theories MLWPU and $\mathsf{MLW^{ext}PU}$ we need to add the previously given rules for U and also the following rules so that U reflects P.

$$\mathsf{P} : \mathsf{U} \qquad \frac{A : \mathsf{P}}{A : \mathsf{U}}$$

Similarly we can define the type theories $\mathsf{MLWPU}_{<\omega}$ and $\mathsf{MLW^{ext}PU}_{<\omega}$.

We show how to extend the TS interpretation so as to interprete the type P and justify its rules. In classical set theory we can interprete P as the set $2 = \{0, 1\}$. But to do so we need to use a non-standard function coding. Recall that our TS interpretation uses an arbitrary function coding and so far the standard one has been good enough. But to justify the rules for P we use the following non-standard function coding.

$$\begin{aligned} APP(a, b) &= \{y \mid (b, y) \in a\} \\ LAM(a) &= \textstyle\bigcup_{(x,z)\in a}(\{x\} \times z) \end{aligned}$$

The advantage of this function coding over the standard one is that we can prove the following result, which we express in a form that still usefully holds in constructive set theory. Recall that $1 = \{0\}$.

Proposition 3.1 *For any set a, if $B(x) \subseteq 1$ for each $x \in a$ then*

$$PI_{x\in a}B(x) = \{y \in 1 \mid \forall x \in a(B(x) = 1)\} \subseteq 1$$

so that

$$PI_{x\in a}B(x) = 1 \iff \forall x \in a(B(x) = 1).$$

Note that in classical set theory the subsets of 1 are just the elements of $2 = \{0, 1\}$. In constructive set theory the subsets of 1 play the role *small extensional propositions* and the above result expresses that the PI operation behaves like universal quantification on such propositions.

Using this result we get the soundness of the rules for P and hence the following reductions.

$$\begin{aligned} \mathsf{MLW^{ext}P} + \mathsf{EM} &\leq_{TS} \mathsf{ZFGC} \\ \mathsf{MLW^{ext}PU} + \mathsf{EM} &\leq_{TS} \mathsf{ZFGC_1} \\ \mathsf{MLW^{ext}PU}_{<\omega} + \mathsf{EM} &\leq_{TS} \mathsf{ZFGCu}_{<\omega} \end{aligned}$$

In constructive set theory we cannot use $Pow(1) = \{x \mid x \subseteq 1\}$ to interprete the type P as the class $Pow(1)$ cannot be shown to be a set in CZF or its constructive extensions. Instead we will here simply extend the theory to give us what we want. So we add a new individual constant p to the language and add the following axioms.

1. $\forall x \in \mathsf{p}\; x \subseteq 1$,
2. If B is a function with domain the set a such that $\forall x \in a\; B(x) \in \mathsf{p}$ then $PI_{x\in a}B(x) \in \mathsf{p}$.

This gives us the extension $\mathsf{CZF^+p}$. For the theories $\mathsf{CZF^+pu}$, $\mathsf{CZF^+pu}_{<\omega}$ we also need the axioms $\mathsf{p} \in \mathsf{u}$, $\mathsf{p} \in \mathsf{u_0}$ respectively.

Of course in the TS interpretations in our constructive set theories we let $\mathsf{P}^o = \mathsf{p}$ and get the following reductions.

$$\begin{array}{rl} \mathsf{MLW^{ext}P} & \leq_{TS} \mathsf{CZF^+p} \\ \mathsf{MLW^{ext}PU} & \leq_{TS} \mathsf{CZF^+pu} \\ \mathsf{MLW^{ext}PU}_{<\omega} & \leq_{TS} \mathsf{CZF^+pu}_{<\omega} \end{array}$$

4 Interpreting Set Theories in Type Theories

We now explore to what extent the proof theoretic reductions we have obtained using the TS interpretation can be reversed using what we will here call the ST interpretation. This is the **sets-as-trees** interpretation that was introduced and developed in [Aczel 78, Aczel 82, Aczel 86] and has also been used in [Griffor and Rathjen 94, Griffor and Rathjen 96]. It is used to interprete a set theory in a type theory. The idea for the original interpretation, in [Aczel 78], of CZF in MLWU was to interprete the sets of CZF as the well-founded trees of the type $V = (Wx : \mathsf{U})x$, the membership and equality relations of CZF being interpreted as terms ϵ_V, $=_V$ of type $V \to (V \to \mathsf{U})$. Using the propositions-as-types idea each sentence of CZF was interpreted as a type of MLWU and it was shown that each theorem of CZF is an inhabited type of MLWU; i.e. a type A such that $a : A$ can be derived in MLWU for some term a. In this way a proof theoretic reduction of CZF to MLWU is obtained that will be expressed as [9] $\mathsf{CZF} \leq_{ST} \mathsf{MLWU}$. In fact, as shown in [Aczel 86], we get

$$\mathsf{CZF^+} \leq_{ST} \mathsf{MLWU}.$$

Also, it is easy to see that, using the rule EM of $\mathsf{MLWU} + \mathsf{EM}$ we can justify both the law of excluded middle and global choice for the set theory so as to get the reduction

$$\mathsf{ZFGC} \leq_{ST} \mathsf{MLWU} + \mathsf{EM}.$$

Unfortunately this and the previous reduction do not match up exactly with our earlier TS reductions. The trouble is the need to use a type universe U in our ST interpretation. In order to interprete the type universe in set theory we need to strengthen the set theory with a set theoretic version; i.e. an inaccessible set in the constructive set theory case and a strongly inaccessible cardinal in the classical set theory case. Now, if we wish to extend

[9] Notice that the ST interpretation does not use any kind of equality types, neither intensional nor extensional, so that we have stated the stronger result of a reduction to MLWU rather than to $\mathsf{MLW^{ext}U}$.

the ST interpretation of CZF^+ to an interpretation of $\mathsf{CZF}^+\mathsf{u}$, we need to use two of the type universes U_0, U_1 of $\mathsf{MLWU}_{<\omega}$ and their rules and use the type $V_1 = (Wx : \mathsf{U}_1)x$ to interprete the universe of sets of $\mathsf{CZF}^+\mathsf{u}$. The inaccessible set u of $\mathsf{CZF}^+\mathsf{u}$ can be modelled by $\mathsf{v}_0 = sup(V_0, (\lambda x : V_0)h(x)) : V_1$ where $V_0 = (Wx \in \mathsf{U}_0)x : \mathsf{U}_1$ and $h(x) : V_1$ is defined by transfinite recursion on $x : V_0$ so that

$$h(sup(a, f)) = sup(V_0, (\lambda x : a)h(app(f, x)))$$

for $a : \mathsf{U}_0$ and $f : a \to V_0$; i.e. $h(x)$ is the term $rec(b, x)$ where b is the term $(\lambda x : \mathsf{U}_0)(\lambda y : x \to V_0)(\lambda z : x \to V_1)sup(x, z)$.

We can extend these ideas to more universes, a set theory with n inaccessibles being given an ST interpretation in a type theory with $n+1$ type universes, $\mathsf{U}_0, \ldots, \mathsf{U}_n$, with the universe of sets of the set theory being interpreted as the type $V_n = (Wx : \mathsf{U}_n)x$.

Fortunately we do get a matching of a set theory with a type theory of the same proof theoretic strength when we go to the limit. First consider the type theory $\mathsf{MLWU}_{<\omega}\mathsf{U}$ that is obtained from $\mathsf{MLWU}_{<\omega}$ by adding the type universe U at the top reflecting all the rules of $\mathsf{MLWU}_{<\omega}$ so that in particular we have the rules

$$\mathsf{U}_n : \mathsf{U} \qquad \frac{A : \mathsf{U}_n}{A : \mathsf{U}}$$

for $n = 0, 1, \ldots$. As above we get an ST interpretation of CZF^+_ω into this theory, using $V = (Wx \in \mathsf{U})x$ to interprete the universe of sets of the set theory, giving us

$$\mathsf{CZF}^+_\omega \leq_{ST} \mathsf{MLWU}_{<\omega}\mathsf{U}.$$

Now observe that we have a proof theoretic reduction

$$\mathsf{MLWU}_{<\omega}\mathsf{U} \leq \mathsf{MLWU}_{<\omega}.$$

The idea for this is that any derivation in the left hand type theory can only involve finitely many of the type universes U_i and so can be translated into a derivation in the right hand type theory by replacing the symbol U everywhere by U_n, where n is chosen large enough so that $n > i$ whenever U_i occurs in the derivation. Using a previous TS reduction, we get the next result.

Theorem 4.1 *The following theories are of the same proof theoretic strength.*

- $\mathsf{CZF}^+\mathsf{u}_{<\omega}$
- $\mathsf{MLWU}_{<\omega}\mathsf{U}$
- $\mathsf{MLWU}_{<\omega}$
- $\mathsf{MLW}^{\mathsf{ext}}\mathsf{U}_{<\omega}$

We have the same situation for classical set theory so that, using the fact that global choice does not increase the proof theoretic strength, we get the next result.

Theorem 4.2 *The following theories are of the same proof theoretic strength.*

- $\mathsf{ZFCu}_{<\omega}$

- $\mathsf{ZFGCu}_{<\omega}$
- $\mathsf{MLWU}_{<\omega}\mathsf{U} + \mathsf{EM}$
- $\mathsf{MLWU}_{<\omega} + \mathsf{EM}$
- $\mathsf{MLW}^{\mathsf{ext}}\mathsf{U}_{<\omega} + \mathsf{EM}$

Finally we observe that the ST interpretation carries over to the set theory $\mathsf{CZF}^{+}\mathsf{p}$ to give the reduction

$$\mathsf{CZF}^{+}\mathsf{p} \leq_{ST} \mathsf{MLWUP}$$

and, as above, the reduction

$$\mathsf{CZF}^{+}\mathsf{pu}_{<\omega} \leq_{ST} \mathsf{MLWPU}_{<\omega}.$$

This, with a previous reduction gives us the following result.

Theorem 4.3 *The following theories are of the same proof theoretic strength.*

- $\mathsf{CZF}^{+}\mathsf{pu}_{<\omega}$
- $\mathsf{MLWPU}_{<\omega}$
- $\mathsf{MLW}^{\mathsf{ext}}\mathsf{PU}_{<\omega}$

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