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INTEGRATION IN FINITE TERMS

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1. The question arises in elementary calculus: Can the indefinite integral of an explicitly given function of one variable always be expressed "explicitly" (or "in closed form", or "in finite terms")? Liouville gave the answer one would expect, "No", and he proved in particular that such is not the case with $\int e^{x^2}dx$. Since we have all fallen into the habit of quoting this result and giving neither proof nor reference, it may be worthwhile to actually state it as precisely as possible and give a proof that is as elementary as the subject matter might suggest.

We must define our terms carefully. To begin with, we are not interested in arbitrary functions, but in **elementary functions,** which are functions of one variable

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built up by using that variable and constants, together with repeated algebraic operations and the taking of exponentials and logarithms. Since we lose no generality by doing so, we shall take all exponentials and logarithms to the base e. We allow ourselves the convenience of the use of complex numbers, for with these the various trigonometric and inverse trigonometric functions turn out to be elementary, as seems reasonable. Thus the integral of a rational function of one real variable is elementary, since it is a linear combination of logarithms, inverse tangents, and rational functions. But we are still deficient in precision, because of the multivaluedness of algebraic functions and logarithms. The functions we work with must be specific objects, each susceptible of an unambiguous sense. We choose to avoid the difficulties associated with multivaluedness by the simplest method, that of restricting ourselves, in any given discussion, to functions on some specific region (that is, nonempty connected open subset) of the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, and furthermore considering only meromorphic functions on the region in question, a meromorphic function on a region being a function whose values are complex numbers or the symbol ∞, with the property that sufficiently near any point z_0 of the region the function is given by a convergent Laurent series in $z - z_0$, that is, a convergent power series in $z - z_0$, with the possible addition of a finite number of negative powers. Thus the rational functions of one variable, which form the field $\mathbb{C}(z)$ got by adjoining the identity function z to the field of constant functions $\mathbb{C}$, are all meromorphic on all of $\mathbb{R}$ or $\mathbb{C}$. The exponential of a function f meromorphic on a certain region of $\mathbb{R}$ or $\mathbb{C}$ is a function meromorphic on the subregion obtained by deleting those points where the value of f is ∞ (and then taking a connected component, if we are working in $\mathbb{R}$), while $\log f$ can be taken to be meromorphic on any simply connected subregion where f takes on neither of the values 0 or ∞, by arbitrarily choosing one of its many values at any particular point of the subregion. Furthermore, the implicit function theorem shows that if we are given a polynomial equation with coefficients which are functions meromorphic on a certain region, the leading coefficient not being zero, then there exists a meromorphic solution on a suitable subregion. Thus any complicated expression for an elementary function, compounded of algebraic operations, exponentials and logarithms, has a realization as a meromorphic function on some region. Now the totality of all meromorphic functions on a given region form a field under the usual operations of functional addition and multiplication, and the restriction of all these functions to any given subregion gives an embedding of fields. The derivative of a function meromorphic on a given region is again meromorphic, as is an indefinite integral, if one exists, of the function. Note that the rational functions on a region, that is the restriction of $\mathbb{C}(z)$ to this region, are a field of meromorphic functions on the region that are closed under differentiation, and that if we have any field of meromorphic functions on a region that is closed under differentiation and get a larger field of meromorphic functions on the region by adjoining the exponential or a logarithm of a function in our field, or a solution of a polynomial equation with coefficients in the field, we again get a field of meromorphic functions on the region that is closed under differentiation. Thus the proper objects of study are seen to be fields of meromorphic functions on given regions in $\mathbb{R}$ or $\mathbb{C}$ which are closed under differentiation. If a function in such a field has an indefinite integral that is expressible "in finite terms," then by restricting all functions, if necessary, to a suitable subregion, we see that we have a tower of such fields of meromorphic functions, each larger field being obtained by adjunction of an exponential, or a logarithm, or the solution of an algebraic equation, the tower starting with the original field and culminating in a field containing the indefinite integral. Thus the original loosely worded analytic problem, when formulated as a precise analytic problem, becomes algebraic.

2. Define a **differential field** to be a field F, together with a **derivation** on F, that is, a map of F into itself, usually denoted $a \mapsto a'$, such that $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$ for all $a, b \in F$. Immediate consequences are that $(a/b)' = (ab' - a'b)/b^2$ if a, $b \in F$, $b \neq 0$, and $(a^n)' = na^{n-1}a'$ for all integers n. Furthermore, $1' = (1^2)' = 2 \cdot 1 \cdot 1'$, so $1' = 0$. Therefore the **constants** of F, that is, all $c \in F$ such that $c' = 0$, are a subfield of F.

If a, b are elements of the differential field F, a being nonzero, let us agree to call a **an exponential of** b, or b **a logarithm of** a, if $b' = a'/a$; this terminology is not unreasonable for our present purposes since the only properties of exponentials and logarithms in which we are interested are their differential properties. We immediately get the "logarithmic derivative identity,"

$$\frac{(a_1^{\nu_1} \cdots a_n^{\nu_n})'}{a_1^{\nu_1} \cdots a_n^{\nu_n}} = \nu_1 \frac{a_1'}{a_1} + \cdots + \nu_n \frac{a_n'}{a_n},$$

for $a_1, \cdots, a_n$ nonzero elements of F and $\nu_1, \cdots, \nu_n$ integers.

3. There is a standard result on algebraic extensions of differential fields which we shall need later. For completeness we prove it here. The result is that if F is a differential field of characteristic zero and K an algebraic extension field of F, then the derivation on F can be extended to a derivation on K, and this extension is unique. (Thus K has a unique differential field structure extending that of F. We remark that the restriction to characteristic zero is not essential; it suffices to assume that K is separable over F, and the following proof will hold in this more general case.) For the reader who is interested only in the classical function-theoretic case, where the fields in question are fields of meromorphic functions on a region of $\mathbb{R}$ or $\mathbb{C}$, the proof is immediate, the existence proof being a direct consequence of the implicit function theorem, uniqueness following from the ordinary method of computing derivatives of functions given implicitly. To prove the result generally, let X be an indeterminate and define the maps D_0, D_1 of the polynomial ring $F[X]$ into itself by

$$D_0\left(\sum_{i=0}^{n} a_iX^i\right) = \sum_{i=0}^{n} a_i'X^i, \quad D_1\left(\sum_{i=0}^{n} a_iX^i\right) = \sum_{i=0}^{n} ia_iX^{i-1}$$

for $a_0, a_1, \cdots, a_n \in F$. If K has a differential field structure extending that of F, then for any $x \in K$ and any $A(X) \in F[X]$ we have

$$(A(x))' = (D_0A)(x) + (D_1A)(x) \cdot x'.$$

If we replace $A(X)$ by the minimal polynomial $f(X)$ of x over F, (that is, the monic irreducible polynomial of which x is a root, indeed a simple root, so that $(D_1f)(x) \neq 0$), we get $x' = -(D_0f)(x)/(D_1f)(x)$. Thus the differential field structure on K that extends that on F is unique, if it exists. We now show that such a structure on K exists. Using the usual field-theoretic arguments, we may assume that K is a finite extension of F, so that we can write $K = F(x)$, for a certain $x \in K$. For some $g(X) \in F[X]$, to be determined later, let the map $D: F[X] \rightarrow F[X]$ be defined by

$$DA = D_0A + g(X)D_1A,$$

for any $A \in F[X]$. It follows immediately that $D(A+B) = DA + DB$ and $D(AB) = (DA)B + A(DB)$ for all $A, B \in F[X]$, since the analogous identities hold for both D_0 and D_1. Note that $Da = a'$ for all $a \in F$. Now look at the natural surjective ring homomorphism $F[X] \rightarrow F[x]$, which is the identity on F and sends X into x. Since $F[x] = F(x) = K$, the map D on $F[X]$ will induce a derivation on K extending that on F if it so happens that D maps the kernel of our ring homomorphism into itself. But the kernel of the homomorphism is the ideal $F[X]f(X)$, where $f(X)$ is the minimal polynomial of x over F. Hence we shall have proved our result once we have shown that D maps $F[X]f(X)$ into itself. The condition for this is simply that D map $f(X)$ into a multiple of itself, that is that Df be any element of $F[X]$ of which x is a root, or that $(Df)(x) = 0$. But this last condition reduces to $(D_0f)(x) + g(x)(D_1f)(x) = 0$. Since $(D_1f)(x) \neq 0$ and $F(x) = F[x]$, a polynomial $g(X) \in F[X]$ can actually be found such that $(Df)(x) = 0$, and this completes the proof of our statement.

4. By a **differential extension field** of a differential field F we mean, of course, a differential field which is an extension field of F whose derivation extends the derivation on F. The following result will be the principal tool for proving the theorem of the next section, and will be used for the verification of our subsequent examples.

LEMMA. *Let F be a differential field, $F(t)$ a differential extension field of F having the same subfield of constants, with t transcendental over F, and with either $t' \in F$ or $t'/t \in F$. If $t' \in F$, then for any polynomial $f(t) \in F[t]$ of positive degree, $(f(t))'$ is a polynomial in $F[t]$ of the same degree as $f(t)$, or degree one less, accrdding as the highest coefficient of $f(t)$ is not, or is, a constant. If $t'/t \in F$, then for any nonzero $a \in F$ and any nonzero integer n we have $(at^n)' = ht^n$, for some nonzero*

$h \in F$, and furthermore, for any polynomial $f(t) \in F[t]$ of positive degree, $(f(t))'$ is a polynomial in $F[t]$ of the same degree, and is a multiple of $f(t)$ only if $f(t)$ is a monomial.

We first consider the case $t' = b \in F$. Let the degree of $f(t)$ be $n > 0$, so that $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$, with $a_0, \cdots, a_n \in F$, $a_n \neq 0$. Then

$$(f(t))' = a_n' t^n + (n a_n b + a_{n-1}') t^{n-1} + \cdots .$$

This is clearly a polynomial in $F[t]$, of degree n if a_n is not constant. If a_n is constant and $n a_n b + a_{n-1}' = 0$, then $(n a_n t + a_{n-1})' = n a_n b + a_{n-1}' = 0$, so that $n a_n t + a_{n-1}$ is a constant, therefore an element of F, so that $t \in F$, contrary to the assumption that t is transcendental over F. Thus if a_n is constant, $(f(t))'$ has degree $n - 1$.

Now suppose that we are in the case $t'/t = b \in F$. Let $a \in F$, $a \neq 0$, and let n be a nonzero integer. Then

$$(at^n)' = a' t^n + n a t^{n-1} t' = (a' + nab) t^n.$$

If $a' + nab = 0$, then $(at^n)' = 0$, so that at^n is constant, therefore an element of F, contradicting the transcendence of t over F. Therefore $a' + nab \neq 0$. Finally, let $f(t) \in F[t]$ have positive degree. Clearly $(f(t))'$ has the same degree. If $(f(t))'$ is a multiple of $f(t)$, it must be by a factor in F. Therefore if $f(t)$ is not a monomial, $a_n t^n$ and $a_m t^m$ being two of its different terms, and $(f(t))'$ is a multiple of $f(t)$, we have

$$\frac{a_n' + n a_n b}{a_n} = \frac{a_m' + m a_m b}{a_m},$$

so

$$\frac{a_n'}{a_n} + n \frac{t'}{t} = \frac{a_m'}{a_m} + m \frac{t'}{t},$$

or $(a_n t^n / a_m t^m)' = 0$, so that $a_n t^n / a_m t^m \in F$, again contradicting the transcendence of t over F. This completes the proof.

5. Let F be a differential field. Define an **elementary extension of** F to be a differential extension field of F which is obtained by successive adjunctions of elements that are algebraic, or logarithms, or exponentials, that is, a differential extension field of the form $F(t_1, \cdots, t_N)$, where for each $i = 1, \cdots, N$, the element t_i is either algebraic over the field $F(t_1, \cdots, t_{i-1})$, or the logarithm or exponential of an element of $F(t_1, \cdots, t_{i-1})$. Note that each intermediate field $F(t_1, \cdots, t_{i-1})$ is a differential field and an elementary extension of F.

The following result is the abstract generalization of Ostrowski's 1946 generalization of Liouville's 1835 theorem on the subject. A proof of the analytic case may be found in Ritt's classic exposition [**4**]. Other algebraic proofs, essentially the same as the one given here, may be seen in [**2**] and [**5**].

THEOREM. *Let F be a differential field of characteristic zero and $\alpha \in F$. If the equation $y' = \alpha$ has a solution in some elementary differential extension field of F having the same subfield of constants, then there are constants $c_1, \cdots, c_n \in F$ and elements $u_1, \cdots, u_n, v \in F$ such that*

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'.$$

A number of comments are in order before we proceed with the proof. First, in the case of greatest interest, in which our fields are fields of meromorphic functions on some subregion of $\mathbb{R}$ or $\mathbb{C}$, the condition that F and its elementary extension field have the same constants will be automatically satisfied as long as $\mathbb{C} \subset F$, since any constant meromorphic function is a complex number. In the general case however, the condition that F and its elementary extension field have the same constants, or some related condition, is essential. This can be seen from the example $F = \mathbb{R}(x)$, the field of real rational functions of a real variable, with $x' = 1$ as usual, and $\alpha = 1/(x^2+1)$. Clearly $\int (1/(x^2+1))dx$ is an element of an elementary extension field of $\mathbb{R}(x)$, and our claim is that the assumption that we can write $1/(x^2+1)$ in the desired form, with $c_1, \cdots, c_n \in \mathbb{R}$ and $u_1, \cdots, u_n$, $v \in \mathbb{R}(x)$, will lead to a contradiction. For if x^2+1 occurs ν_i times in the expression of u_i as a power product of monic irreducible elements of $\mathbb{R}[X]$, then $u_i'/u_i - 2\nu_i x/(x^2+1)$ is an element of $\mathbb{R}(x)$ without x^2+1 in its denominator, while x^2+1, if it occurs in the denominator of v, will occur at least twice in the denominator of v'. Thus x^2+1 divides the denominator of neither v nor v', implying that $1 - \Sigma 2c_i\nu_i x$ is divisible by x^2+1, which is impossible. The final comment is that the theorem has an easy converse: if α can be written as indicated then α has an integral in some elementary extension field of F. This is quite easy to show in the abstract case and is immediate in the classical case where F is a field of meromorphic functions on a subregion of $\mathbb{R}$ or $\mathbb{C}$, as we see by passing to a suitable subregion, where the various $\log u_i$'s can be defined.

Now for the proof of Liouville's theorem. By assumption there is a tower of differential fields

$$F \subset F(t_1) \subset \cdots \subset F(t_1, \cdots, t_N),$$

all with the same subfield of constants, each t_i being algebraic over $F(t_1, \cdots, t_{i-1})$, or the logarithm or exponential of an element of this field, such that there exists an element $y \in F(t_1, \cdots, t_N)$ such that $y' = \alpha$. We shall prove the theorem by induction on N. The case $N = 0$ is trivial, so assume that $N > 0$ and that the theorem holds for $N-1$. Applying the case $N-1$ to the fields $F(t_1) \subset F(t_1, \cdots, t_N)$, we deduce that we can write α in the desired form, but with $u_1, \cdots, u_n, v$ in $F(t_1)$. Setting $t_1 = t$, we have t algebraic over F, or the logarithm or exponential of an element of F, and we know that

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v',$$

with $c_1, \cdots, c_n$ constants of F and $u_1, \cdots, u_n$, $v \in F(t)$, and it remains to find a similar expression for α, possibly with a different n, but with all of $u_1, \cdots, u_n, v$ in F.

First suppose that t is algebraic over F. Then there are polynomials $U_1, \cdots, U_n$, $V \in F[X]$ such that $U_1(t) = u_1, \cdots, U_n(t) = u_n$, $V(t) = v$. Let the distinct conjugates of t over F in some suitable algebraic closure of $F(t)$ be $\tau_1 (= t), \tau_2, \cdots, \tau_s$. (In case we are dealing with fields of meromorphic functions on a region in $\mathbb{R}$ or $\mathbb{C}$, the functions $\tau_2, \cdots, \tau_s$ can be taken to be meromorphic functions on a suitable subregion, and it suffices to carry the proof through for functions on the subregion.) Now bear in mind the result of Section 3 on algebraic extensions of differential fields. We have

$$\alpha = \sum_{i=1}^{n} c_i \frac{(U_i(\tau_j))'}{U_i(\tau_j)} + (V(\tau_j))'$$

for $j = 1, \cdots, s$, since this is true for $j = 1$. Application of the operation $(1/s) \sum_{j=1}^{s}$ to both sides of the equation yields

$$\alpha = \sum_{i=1}^{n} \frac{c_i}{s} \frac{(U_i(\tau_1) \cdots U_i(\tau_s))'}{U_i(\tau_1) \cdots U_i(\tau_s)} + \left(\frac{V(\tau_1) + \cdots + V(\tau_s)}{s} \right)' .$$

Since each $U_i(\tau_1) \cdots U_i(\tau_s)$ and $V(\tau_1) + \cdots + V(\tau_s)$ are symmetric polynomials in $\tau_1, \cdots, \tau_s$ with coefficients in F, each of these expressions is actually in F. Hence the last equation is an expression for α of the desired form.

In the remaining cases, where t is the logarithm or exponential of an element of F, we may assume that t is transcendental over F. Then we have

$$\alpha = \sum_{i=1}^{n} c_i \frac{(u_i(t))'}{u_i(t)} + (v(t))',$$

with $u_1(t), \cdots, u_n(t)$, $v(t) \in F(t)$. Each $u_i(t)$ can be written as a power product of a nonzero element of F and various monic irreducible elements of $F[t]$. Hence we may, if necessary, use the logarithmic derivative identity to rewrite $\sum c_i(u_i(t))'/u_i(t)$ in a similar form, but with each $u_i(t)$ either in F or a monic irreducible element of $F[t]$. We therefore assume that $u_1(t), \cdots, u_n(t)$ are distinct, each being an element of F or a monic irreducible element of $F[t]$, and that no c_i is zero. Now look at the partial fraction decomposition of $v(t)$, which expresses $v(t)$ as the sum of an element of $F[t]$ plus various terms of the form $g(t)/(f(t))^r$, where $f(t)$ is a monic irreducible element of $F[t]$, r a positive integer, and $g(t)$ is a nonzero element of $F[t]$ of degree less than that of $f(t)$. Clearly $u_1(t), \cdots, u_n(t), v(t)$ must be of very special form for the right hand side of the last equation to add up to α, which doesn't involve t. To investigate this special form in detail, it now becomes convenient to separate cases. In each case the lemma provides the basic arguments.

First, suppose that t is the logarithm of an element of F, so that $t' = a'/a$, for some $a \in F$. Let $f(t)$ be a monic irreducible element of $F[t]$. Then $(f(t))'$ is also in $F[t]$, and it has degree less than that of $f(t)$, so that $f(t)$ does not divide $(f(t))'$.

Thus if $u_i(t) = f(t)$, then the fraction $(u_i(t))'/u_i(t)$ is already in lowest terms, with denominator $f(t)$. If $g(t)/(f(t))^r$ occurs in the partial fraction expression for $v(t)$, with $g(t) \in F[t]$ of degree less than that of $f(t)$ and $r > 0$ and maximal for given $f(t)$, then $(v(t))'$ will consist of various terms having $f(t)$ in the denominator at most r times plus $(g(t)(1/(f(t))^r)' = -rg(t)(f(t))'/(f(t))^{r+1}$. Since $f(t)$ does not divide $g(t)(f(t))'$, we see that a term with denominator $(f(t))^{r+1}$ actually appears in $(v(t))'$. Thus if $f(t)$ appears as a denominator in the partial fraction expansion of $v(t)$, it will appear in α, which is impossible. Therefore, $f(t)$ does not appear in the denominator of $v(t)$. Therefore $f(t)$ cannot be one of the $u_i(t)$'s either. Since this is true for each monic irreducible $f(t)$, we have each $u_i(t) \in F$ and $v(t) \in F[t]$. Since $(v(t))' \in F$, the lemma implies that $v(t) = ct + d$, with c constant and $d \in F$. Thus

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + c\frac{a'}{a} + d'$$

is an expression for α of the desired form.

Finally, consider the case where t is the exponential of an element of F, say $t'/t = b'$, with $b \in F$. The lemma implies that if $f(t)$ is a monic irreducible element of $F[t]$ other than t itself, then $(f(t))' \in F[t]$ and $f(t)$ does not divide $(f(t))'$. Precisely the same reasoning as above shows that $f(t)$ cannot occur in the denominator of $v(t)$, nor can any $u_i(t)$ equal $f(t)$. Thus $v(t)$ can be written as $v(t) = \sum_j a_j t^j$, where each $a_j \in F$ and j ranges over a finite set of integers, positive, negative, or zero, and each of the quantities $u_1(t), \cdots, u_n(t)$ is in F, with the possible exception that one of these may be t itself. Since each $(u_i(t))'/u_i(t)$ is in F, we have $(v(t))' \in F$, so the lemma implies that $v(t) \in F$. If each $u_i(t)$ is in F, we already have α in the desired form, and are done. If not, only one $u_i(t)$, say $u_1(t)$, is not in F. Then $u_1(t) = t$ and $u_2(t), \cdots, u_n(t) \in F$, so we can write

$$\alpha = c_1 \frac{t'}{t} + \sum_{i=2}^{n} c_i \frac{u_i'}{u_i} + v' = \sum_{i=2}^{n} c_i \frac{u_i'}{u_i} + (c_1 b + v)',$$

with $u_2, \cdots, u_n, c_1 b + v$ all in F. This completes the proof of the theorem.

6. An elementary function is a meromorphic function on some region in $\mathbb{R}$ or $\mathbb{C}$ that is contained in an elementary extension field of the field of rational functions $\mathbb{C}(z)$. We now give some examples of elementary functions with nonelementary indefinite integrals.

As a preliminary comment we note that if $g(z)$ is a non-constant rational function of the complex variable z then e^g is not algebraic over $\mathbb{C}(z)$. This can easily be shown analytically by noting that since $g(z)$ must have at least one pole on the Riemann sphere, e^g will have at least one essential singularity, unlike any algebraic function. Or it can be shown algebraically by looking at the irreducible equation over $\mathbb{C}(z)$ that e^g would otherwise satisfy, say

$$e^{ng} + a_1 e^{(n-1)g} + \cdots + a_n = 0,$$

where $a_1, \cdots, a_n \in \mathbb{C}(z)$, then differentiating this to get

$$ng'e^{ng} + (a_1' + (n-1)a_1 g')e^{(n-1)g} + \cdots + a_n' = 0,$$

which must be proportional to the first equation, so that $ng' = a_n'/a_n$, then noting that a_n'/a_n is either zero or a sum of fractions with constant numerators and linear denominators, whereas ng' can have no linear denominator, so that $g' = 0$, contradicting the assumption that g is nonconstant.

We now want to derive a criterion, due to Liouville, that $\int f(z)e^{g(z)}dz$ be elementary, where $f(z), g(z)$ are given rational functions of z, $f(z)$ being nonzero, and $g(z)$, as above, non-constant. Writing $e^g = t$, we have $t'/t = g'$. Working in the differential field $\mathbb{C}(z,t)$, a pure transcendental extension of $\mathbb{C}(z)$, we see that if $\int fe^g dz$ is elementary, then we can write

$$ft = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v',$$

with $c_1, \cdots, c_n \in \mathbb{C}$ and $u_1, \cdots, u_n, v \in \mathbb{C}(z,t)$. Now let $F = \mathbb{C}(z)$, so that $f, g \in F$ and $u_1, \cdots, u_n, v \in F(t)$. By factoring each u_i as a power product of irreducible elements of $F[t]$ and using logarithmic derivatives, if necessary, we can guarantee that the u_i's which are not in F are distinct monic irreducible elements of $F[t]$. Imagine v expanded into partial fractions with respect to $F[t]$. The lemma implies immediately that the only possible monic irreducible factor of a denominator in v is t, which is also the only possible u_i not in F. Thus v is of the form $\sum b_j t^j$, for j ranging over some set of integers and each $b_j \in F$. Since $\sum c_i u_i'/u_i \in F$, we have $ft = (b_1' + b_1 g')t$. Writing $b_1 = a$, we have $f = a' + ag'$, with $a \in \mathbb{C}(z)$. Conversely, if there is an $a \in \mathbb{C}(z)$ such that $f = a' + ag'$ then one elementary integral of fe^g is ae^g. Thus fe^g has an elementary integral if and only if there is an $a \in \mathbb{C}(z)$ such that $f = a' + ag'$.

For given $f, g \in \mathbb{C}(z)$, the possibility of finding $a \in \mathbb{C}(z)$ such that $f = a' + ag'$ can be decided by considering partial fraction expansions for f, g, and a. For $\int e^{z^2}dz$ we have the equation $1 = a' + 2za$, which is easily seen to have no solution $a \in \mathbb{C}(z)$. For $\int (e^z/z)dz$, we have the equation $1/z = a' + a$, which also has no solution in $\mathbb{C}(z)$. Therefore $\int e^{z^2}dz$ and $\int (e^z/z)dz$ are not elementary. By certain changes of variable we can get other nonelementary integrals. For example, if we replace z by e^z in the second integral we get $\int e^{e^z}dz$ nonelementary, and replacing z by $\log z$ we get $\int (1/\log z)dz$ nonelementary. The integral $\int \log\log z dz$ reduces to the previous integral by integration by parts, so it also is nonelementary.

It is slightly more complicated to show that $\int (\sin z/z)dz$ is not elementary. To do this, first change the variable to $\sqrt{-1}\,z$ to slightly simplify the problem to that of showing that $\int ((e^z - e^{-z})/z)dz$ is not elementary. Here again consider the differential field $\mathbb{C}(z,t)$, where $t = e^z$. If our integral is elementary, Liouville's theorem enables us to write

$$\frac{t^2-1}{tz} = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v',$$

with $c_1, \cdots, c_n \in \mathbb{C}$ and $u_1, \cdots, u_n, v \in \mathbb{C}(z,t)$. Again write $F = \mathbb{C}(z)$, so that $u_1, \cdots, u_n$, $v \in F(t)$, again arrange that the u_i's which are not in F are distinct monic irreducible elements of $F[t]$ and that v is expressed in its partial fraction form, and use the lemma. We again get that the only possible u_i not in F is t, so that $\Sigma c_i u_i'/u_i \in F$, and the only possible monic irreducible factor of a denominator in v is t. Writing $v = \Sigma b_j t^j$, as before, with each $b_j \in F$, we deduce as before that $1/z = b_1' + b_1$, which is impossible. Therefore $\int (\sin z/z) dz$ is not elementary.

7. The question arises whether for any explicitly given elementary function of the complex variable z it can be decided whether or not the function has an elementary integral, and if so, finding it. It is not difficult to see, using the method of the previous section, that this can be done for any function in $\mathbb{C}(z, e^g)$, where g is any nonconstant element of $\mathbb{C}(z)$, but the general question is not so easy. Hardy's book [**1**] discusses the systematic integration of the kinds of elementary functions that occur in calculus, the main point being that there really *is* a system (contrary to the sometimes expressed opinion that integration in calculus is as much an art as a science), but the book barely broaches the general decision question, which very quickly leads to once intractable questions about points of finite order on abelian varieties over finitely generated ground fields. A solution to this decision problem has recently been announced by Risch [**3**].

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THE COLLEGE PREPARATION FOR A MATHEMATICIAN IN INDUSTRY

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I should like to express my deep appreciation to this association for inviting me to speak on industrial mathematics, a subject which has been ignored for many

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