First exercise for the course 'proof assistants'

The exercise is to write a formalization of the following solution to a problem from the International Mathematical Olympiad:

> **Problem** [A1 from IMO 1983]
>
> *Find all functions f defined on the set of positive reals which take positive real values and satisfy:*
>
> $$f(xf(y)) = yf(x) \text{ for all } x,\ y; \text{ and } f(x) \to 0 \text{ as } x \to \infty.$$
>
> **Solution**
>
> If $f(k) = 1$, then $f(x) = f(xf(k)) = kf(x)$, so $k = 1$. Let $y = 1/f(x)$ and set $k = xf(y)$, then $f(k) = f(xf(y)) = yf(x) = 1$. Hence $f(1) = 1$ and $f(1/f(x)) = 1/x$. Also $f(f(y)) = f(1\,f(y)) = y$. Hence $f(1/x) = 1/f(x)$. Finally, let $z = f(y)$, so that $f(z) = y$. Then $f(xy) = f(xf(z)) = zf(x) = f(x)f(y)$.
>
> Now notice that $f(xf(x)) = xf(x)$. Let $k = xf(x)$. We show that $k = 1$. $f(k^2) = f(k)f(k) = k^2$ and by a simple induction $f(k^n) = k^n$, so we cannot have $k > 1$, or $f(x)$ would not tend to 0 as x tends to infinity. But $f(1/k) = 1/k$ and the same argument shows that we cannot have $1/k > 1$. Hence $k = 1$.
>
> So the only such function f is $f(x) = 1/x$. □

The exercise is to write a Mizar article that is error-free, and is in a shape that makes it fit to be submitted to the MML library.

The deadline for this exercise is **May 7**. As a half-way checkpoint, at **March 26** a version of the article has to be handed in that already has the formalization of the statement of the theorem, and that only has `*1` and `*4` errors left.

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Here is a step-by-step guide on how one might go about doing this exercise. It is not at all required to follow this in detail.

1. First, really try to understand the proof that you are formalizing. The proof that is shown above is from the internet, and it is not very clear. Write down a version of the proof, in your own style, that you really understand.
2. Write a version of the statement using Mizar syntax.

   Hint: For the formalization of '$f(x) \to 0$ *as* $x \to \infty$' you might look at the MML article `LIMFUNC1`.

3. Modify the environment such that Mizar accepts the statement with only `*4` errors.

   Hint: Although the exercise only says that f is defined on the set of positive real numbers, it might be easier to have f be a partial function that is defined on the set of *all* real numbers.

You are now at the halfway checkpoint of the exercise.

4. Translate your proof into Mizar syntax, and make it error-free, but do not yet formalize the proofs of statements that clearly are 'lemmas' to the proof. Instead, just add them as lemmas in front of the file.

   Example: One such a lemma might be 'if $k > 1$ then k^n gets arbitrarily large.'

5. Formalize the proofs of the lemmas. Add more lemmas if needed (and prove those too.)
6. Clean up the formalization such that it becomes good enough for submission to the MML.

To give an impression of the complexity of this exercise: my own solution along these lines consisted of 273 lines of Mizar code.

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There are a few things that you can do to make this exercise more interesting. They are not required to get full marks, but they might be fun working on.

– Factor your proof into as many reusable lemmas as possible.
– Look at your lemmas and make them as general as possible. For instance, do not state lemmas about positive real numbers when they hold for all real numbers, do not state them for real numbers when they hold for all complex numbers, etc.
– If your types do not look the way that you like best, add definitions that make it possible to write them in a better way.
  For instance, you might want to define an attribute `positive` that allows you to write

```
positive real number
```

  or maybe even just

```
positive number
```

  instead of

```
positive (real number)
```