Theorem (Knaster-Tarski fixpoint theorem)**.** Let F be a function that maps the subsets of given set U to subsets of U. Suppose further that F is *order-preserving* with respect to set inclusion, i.e., that if

$$X \subseteq Y$$

then also

$$F(X) \subseteq F(Y)$$

Then F has a *fixpoint* μF which is a subset of U such that

$$F(\mu F) = \mu F$$

Proof. Consider the *post-fixpoints* of F, which are the sets X for which it holds that $F(X) \subseteq X$. Call the collection of all post-fixpoints P, so define

$$P := \{X \subseteq U \mid F(X) \subseteq X\}$$

This set is non-empty as U will certainly be in P. Now take μF to be the intersection of all sets in P, so define

$$\mu F := \bigcap_{X \in P} X$$

We claim that this μF will be a fixpoint of F.

First of all $F(\mu F)$ will be a subset of each post-fixpoint. If X is a post-fixpoint, then $\mu F \subseteq X$ by definition, and therefore $F(\mu F) \subseteq F(X) \subseteq X$.

But if $F(\mu F)$ is a subset of all elements of P then it certainly will be a subset of the intersection of those elements, so $F(\mu F) \subseteq \mu F$, i.e., μF will *itself* be a post-fixpoint. Futhermore, since F is order-preserving, $F(\mu F)$ *also* will be a post-fixpoint. (The F-image of any post-fixpoint again is a post-fixpoint, as is easy to see.)

Now by definition μF is smaller than all post-fixpoints, so it also is smaller than $F(\mu F)$! Hence we find that $\mu F \subseteq F(\mu F)$. But we already knew that $F(\mu F) \subseteq \mu F$. Together this gives that $F(\mu F) = \mu F$. $\square$