Answers to the test Type Theory and Coq, 2011

1. (a)

$$
\dfrac{\dfrac{\dfrac{[(((a \to b) \to a) \to a) \to b^{\,x}] \qquad \dfrac{\dfrac{\dfrac{[(a \to b) \to a^{\,y}] \qquad \dfrac{\dfrac{[(((a \to b) \to a) \to a) \to b^{\,x}] \qquad \dfrac{[a^{\,z}]}{((a \to b) \to a) \to a}\,I[w]{\to}}{b}\,E{\to}}{a \to b}\,I[z]{\to}}{a}\,E{\to}}{((a \to b) \to a) \to a}\,I[y]{\to}}{b}\,E{\to}}{((((a \to b) \to a) \to a) \to b) \to b}\,I[x]{\to}}{}
$$

(b)

$$\lambda x : (((a \to b) \to a) \to a) \to b.\, x\, \lambda y : (a \to b) \to a.\, y\, \lambda z : a.\, x\, \lambda w : (a \to b) \to a.\, z$$

2. (a)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall a : *.\, ((a \to b) \to b) \to a^{\,x}]}{((a \to b) \to b) \to a}\,E\forall \qquad \dfrac{[b^{\,y}]}{(a \to b) \to b}\,I[z]{\to}}{a}\,E{\to}}{b \to a}\,I[y]{\to}}{\forall a : *.\, b \to a}\,I\forall}{(\forall a : *.\, ((a \to b) \to b) \to a) \to \forall a : *.\, b \to a}\,I[x]{\to}}{\forall b : *.\, (\forall a : *.\, ((a \to b) \to b) \to a) \to \forall a : *.\, b \to a}\,I\forall
$$

(b)

$$\lambda b : *.\, \lambda x : (\Pi a : *.\, ((a \to b) \to b) \to a).\, \lambda a : *.\, \lambda y : b.\, x\, a\, \lambda z : a \to b.\, y$$

(c)

$$\Pi b : *.\, (\Pi a : *.\, ((a \to b) \to b) \to a) \to \Pi a : *.\, b \to a$$

3. (a)

$$\Pi c : *.\, (A \to B \to c) \to c$$

(b)

$$\frac{\Gamma \vdash A \wedge B \qquad \Gamma, A, B \vdash C}{\Gamma \vdash C}\ \text{‘}E\wedge\text{’}$$

4. (a)

$$\cfrac{\cfrac{\cfrac{}{\vdash * : \Box} \qquad \cfrac{}{\vdash * : \Box}}{d : * \vdash * : \Box}}{d : *,\ a : * \vdash a : *}$$

(b)

$$\cfrac{\cfrac{\cfrac{}{\vdash * : \Box}}{d : * \vdash d : *} \qquad \cfrac{\cfrac{}{\vdash * : \Box} \qquad \cfrac{}{\vdash * : \Box}}{d : * \vdash * : \Box}}{d : *,\ a : * \vdash d : *}$$

(c)

$$\cfrac{\cfrac{\overset{\text{(a)}}{\vdots}}{d : *,\ a : * \vdash a : *} \qquad \cfrac{\cfrac{\cfrac{\overset{\text{(b)}}{\vdots}}{d : *,\ a : * \vdash d : *} \qquad \cfrac{\cfrac{\overset{\text{(a)}}{\vdots}}{d : *,\ a : * \vdash a : *} \qquad \cfrac{\overset{\text{(b)}}{\vdots}}{d : *,\ a : * \vdash d : *}}{d : *,\ a : *,\ x : d \vdash a : *}}{d : *,\ a : * \vdash (\Pi x : d.\, a) : *} \qquad \cfrac{\overset{\text{(a)}}{\vdots}}{d : *,\ a : * \vdash a : *}}{d : *,\ a : *,\ H : a \vdash (\Pi x : d.\, a) : *}}{d : *,\ a : * \vdash (a \to \Pi x : d.\, a) : *}$$

(d)

$$\lambda H : a.\, \lambda x : d.\, H$$

(e)

$$a \to \forall x.\, a$$

5. (a) `fun x : A => M`
   This notation shows that the term denotes a function.
   (b) `forall x : A, B`
   This notation shows that the product type corresponds to a universally quantified formula under the Curry-Howard isomorphism.
   (c) `Set`, `Prop` and $\texttt{Type}_i$ for all type levels $i \in \mathbb{N}$.
   (d) $*$ has type $\Box$, and $\Box$ does not have a type.
   (e) `Set` and `Prop` both have type $\texttt{Type}_0$, and $\texttt{Type}_i$ has type $\texttt{Type}_{i+1}$.

6. (a) All but the second and fourth can be written with $\to$ notation:

$$\begin{array}{rcl} \Pi x : a.\, b & = & a \to b \\ \Pi a : *.\, b & = & * \to b \\ \Pi x : a.\, * & = & a \to * \\ \Pi a : *.\, * & = & * \to * \end{array}$$

   (b) Only the first is allowed in $\lambda{\to}$.

   (c) The first, second and fifth are allowed in λP.

   (d) The first, third and fourth are allowed in $\lambda 2$.

   (e)

$$\begin{array}{lcl} \Pi x : a.\, b & : & * \\ \Pi x : a.\, p\, x & : & * \\ \Pi a : *.\, b & : & * \\ \Pi a : *.\, a & : & * \\ \Pi x : a.\, * & : & \Box \\ \Pi a : *.\, * & : & \Box \end{array}$$

7. (a)
```
Inductive boollist : Set :=
| nil : boollist
| cons : bool -> boollist -> boollist.
```

   (b)
```
forall P : boollist -> Prop,
  P nil ->
  (forall (b : bool) (l : boollist), P l -> P (cons b l)) ->
  forall l : boollist, P l
```

   (c)
```
Require Import Bool.

Fixpoint andblist (l : boollist) {struct l} : bool :=
  match l with
  | nil => true
  | cons b l' => andb b (andblist l')
  end.
```

8. (a)
```
Require Import Even.

Inductive seq (n : nat) : nat -> nat -> Prop :=
| seq_O : seq n O n
| seq_even : forall (i a b : nat), seq n i a ->
    a = mult (S (S O)) b -> seq n (S i) b
| seq_odd : forall (i a : nat), seq n i a ->
    not (even a) -> seq n (S i) (S (mult (S (S (S O))) a)).
```

   (b)
```
Definition convergent (n : nat) : Prop :=
  n = O \/ exists i : nat, seq n i (S O).
```

   (c)
```
forall n : nat, convergent n
```

9. (a) The version of the derivation without contexts and names for the rules (but with variable names indicated):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg a^{\,x}] \qquad \dfrac{\dfrac{\dfrac{[a^{\,y}]}{\perp\!\!\!\perp}\,[\beta]}{\bot}\,[\gamma]}{\neg a}\,[y]}{\bot}}{(\neg\neg a \to a) \vee \bot}}{\dfrac{\perp\!\!\!\perp}{a}\,[\beta]}\,[\alpha]}{\neg\neg a \to a}\,[x]}{(\neg\neg a \to a) \vee \bot}}{\perp\!\!\!\perp}\,[\alpha]}{(\neg\neg a \to a) \vee \bot}\,[\alpha]
$$

The full derivation was not required, but is given here for completeness. In it, we abbreviate $A := (\neg\neg a \to a) \vee \bot$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\neg a \vdash \neg\neg a;\, a, A}\,\mathit{axiom} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg\neg a,\, a \vdash a;\, \bot,\, a, A}\,\mathit{axiom}}{\neg\neg a,\, a \vdash \perp\!\!\!\perp;\, \bot,\, a, A}\,\mathit{passivate}}{\neg\neg a,\, a \vdash \bot;\, a, A}\,\mathit{activate}}{\neg\neg a \vdash \neg a;\, a, A}\,I{\to}}{\neg\neg a \vdash \bot;\, a, A}\,E{\to}}{\neg\neg a \vdash A;\, a, A}\,I_r\vee}{\neg\neg a \vdash \perp\!\!\!\perp;\, a, A}\,\mathit{passivate}}{\neg\neg a \vdash a;\, A}\,\mathit{activate}}{\vdash \neg\neg a \to a;\, A}\,I{\to}}{\vdash A;\, A}\,I_l\vee}{\vdash \perp\!\!\!\perp;\, A}\,\mathit{passivate}}{\vdash A;}\,\mathit{activate}
$$

(b)

$$
\mu\alpha : ((\neg\neg a \to a) \vee \bot).\, [\alpha]\, \mathsf{inl}\, \big(\lambda x : \neg\neg a.\, \mu\beta : a.\, [\alpha]\, \mathsf{inr}\, \big(x\, \big(\lambda y : a.\, \mu\gamma : \bot.\, [\beta]\, y\big)\big)\big)
$$