Course: Type Theory and Coq

Exercises on Normalization

1. In the proof of WN for $\lambda\!\rightarrow$, the height of a type $h(\sigma)$ is defined by

   - $h(\alpha) := 0$
   - $h(\sigma_1 \rightarrow \ldots \rightarrow \sigma_n \rightarrow \alpha) := \max(h(\sigma_1), \ldots, h(\sigma_n)) + 1$.

   Prove that this is the same as taking as the second clause

   - $h(\sigma \rightarrow \tau) := \max(h(\sigma) + 1, h(\tau))$.

2. In the proof of WN for $\lambda\!\rightarrow$, it is stated that, if $M \longrightarrow_\beta N$ by contracting a redex of maximum height, $h(M)$, that is not contained in another redex of maximum height, then this does not create a new redex of maximum height.

   Show that this holds for the case

   $$M := (\lambda x : \sigma . x\,(x\,y))(\lambda z . z\,(\mathbf{I}\,\mathbf{I})) \longrightarrow_\beta N := (\lambda z . z\,(\mathbf{I}\,\mathbf{I}))((\lambda z . z\,(\mathbf{I}\,\mathbf{I}))\,y)$$

   And show that $m(M) >_l m(N)$.

3. Prove that *type reduction* is SN for $\lambda 2$ a la Church. (Define a simple measure on terms that decreases with type reduction.)

4. Prove for that for $A, B \in \text{SAT}$, $A \rightarrow B \in \text{SAT}$.
   (Here, $A \rightarrow B := \{M | \forall N \in A (M\,N \in B)\}$. Check the slides or course notes for the definition of SAT, the collection of *saturated sets*.)

5. The Soundness of the saturated sets model for $\lambda 2$ is proved by induction on the derivation. Do the case for the $\forall$-introduction rule.