Strong Reduction for the Pure λ-calculus by Benjamin Grégoire and Xavier Leroy

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β-reduction

- Weak β-reduction
- Strong β-reduction

The Pure λ-calculus

Terms:

$$a ::= x|\lambda x.a|a_1 a_2$$

Rules:

$$(\lambda x.a)a' \Rightarrow a\{x \leftarrow a'\} \qquad (\beta)$$

$$\Gamma(a) \Rightarrow \Gamma(a') \quad \textit{if } a \Rightarrow a' \qquad (\textit{context})$$

with $\Gamma ::= \lambda x.[] \mid []\ a \mid a\ []$. We assume all λ-terms a are strongly normalizing.

Two computational problems

- To compute the normal form $\mathcal{N}(a)$ of a closed, strongly normalizing term a.
- To decide whether two closed, strongly normalizing term a_1 and a_2 are β-equivalent, written as $a_1 \approx a_2$.

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Strong reduction by iterated symbolic weak reduction and readback

$$\mathcal{N}(a) = \mathcal{N}(\lambda x.a') = \lambda x.\mathcal{N}(a')$$

Problem: a' is not necessarily closed.

The extended version

Terms:

$$b ::= x \mid \lambda x.b \mid b_1 b_2 \mid [\tilde{x} v_1 ... v_n]$$

Values:

$$v ::= \lambda x.b \mid [\tilde{x} v_1 ... v_n]$$

Rules:

$$(\lambda x.b)v \rightarrow_v b\{x \leftarrow v\} \qquad (\beta_v)$$

$$[\tilde{x} v_1 ... v_n]v \rightarrow_v [\tilde{x} v_1 ... v_n v] \qquad (\beta_s)$$

$$\Gamma_v(a) \rightarrow_v \Gamma_v(a') \quad \textit{if } a \rightarrow_v a' \qquad (context_v)$$

with $\Gamma_v ::= [\,]\ v \mid b\ [\,]$.

Strong normalization procedure

1. Normalize weakly
2. Read back

Readback

$$\mathcal{N}(b) = \mathcal{R}(\mathcal{V}(b)) \tag{1}$$

$$\mathcal{R}(\lambda x.b) = \lambda y.\mathcal{N}((\lambda x.b)[\tilde{y}]) \qquad (y \text{ fresh}) \tag{2}$$

$$\mathcal{R}([\tilde{x} v_1 ... v_n]) = x\mathcal{R}(v_1)...\mathcal{R}(v_n) \tag{3}$$

$\mathcal{R}$ transforms values v into normalized source terms a.

Example

Consider the following source term

$$a = (\lambda x.x)(\lambda y.(\lambda z.z)y(\lambda t.t)).$$

Weak symbolic evaluation reduces a to

$$v = \lambda y.(\lambda z.z)y(\lambda t.t).$$

The readback will restart weak symbolic evaluation on

$$b = (\lambda y.(\lambda z.z)y(\lambda t.t))[\tilde{u}].$$

After the weak symbolic evaluation, the value is

$$v' = [\tilde{u}(\lambda t.t)].$$

Eventually, we will get

$$\mathcal{N}(a) = \mathcal{R}(v) = \lambda u.u(\lambda w.w).$$

Questions?