Lemma 7.11 *The six congruences*

$$\begin{array}{rl} 0 & \pmod{2} \\ 0 & \pmod{3} \\ 1 & \pmod{4} \\ 3 & \pmod{8} \\ 7 & \pmod{12} \\ 23 & \pmod{24} \end{array}$$

form a set of covering congruences.

Proof. First, we show that each of the 24 integers $0, 1, \ldots, 23$ satisfies at least one of these six congruences. Every even integer k satisfies $k \equiv 0 \pmod{2}$. For odd integers, we have

$$\begin{array}{rl} 1 \equiv 1 & \pmod{4} \\ 3 \equiv 0 & \pmod{3} \\ 5 \equiv 1 & \pmod{4} \\ 7 \equiv 7 & \pmod{12} \\ 9 \equiv 0 & \pmod{3} \\ 11 \equiv 3 & \pmod{8} \\ 13 \equiv 1 & \pmod{4} \\ 15 \equiv 0 & \pmod{3} \\ 17 \equiv 1 & \pmod{4} \\ 19 \equiv 7 & \pmod{12} \\ 21 \equiv 0 & \pmod{3} \\ 23 \equiv 23 & \pmod{24} \end{array}$$

For every integer k, there is a unique integer $r \in \{0, 1, \ldots, 23\}$ such that

$$k \equiv r \pmod{24}.$$

Choose i so that

$$r \equiv a_i \pmod{m_i},$$

where $a_i \pmod{m_i}$ is one of our six congruences. Each of the six moduli 2, 3, 4, 6, 12, and 24 divides 24, so m_i divides 24 and

$$k \equiv r \pmod{m_i}.$$

Therefore,

$$k \equiv a_i \pmod{m_i}.$$

This completes the proof.