Finite Topology Concepts and Neighbourhoods

Let X be a general set. If a subset $U(x)$ of X is determined for each element x in X, we call the pair $(X, U(\cdot))$ a finite topological space. The subset $U(x)$ is called a neighbourhood of x. In most cases, we assume $x \in U(x)$ (in this case, we call X *filled*), but it is not necessary in all cases. $U(x)$ is interpreted as a set of points neighbouring to x.

Using the notion of a neighbourhood, we can define many concepts.

Definition (Boundary). A *boundary* A^∂ of a subset A of X is defined as:

$$A^\partial = \{x : U(x) \cap A \neq \emptyset \textit{ and } U(x) \cap A^c \neq \emptyset\},$$

where A^c is the complement of A and $\emptyset$ is the empty set.

In the field of image processing, the process of obtaining A^∂ from A is called an extraction of the outline of A, which is shown in Figure 1.

We can divide A^∂ into the following two parts – an inner boudary A^{∂_i} and an outer boundary A^{∂_o} which are defined as:

$$A^{\partial_i} = A \cap A^\partial \textit{ and } A^{\partial_o} = A^c \cap A^\partial.$$

Of course, we see from these definitions that $A^\partial = A^{\partial_i} \cup A^{\partial_o}$.

Definition (Interior). An *interior* A^i of a subset A of X is defined as:

$$A^i = \{x : x \in A \Rightarrow U(x) \subseteq A\}.$$

In image processing, the process of obtaining A^i is called a contraction of A.

Definition (Closure). A *closure* A^b of a subset A of X is defined as:

$$A^b = \{x : U(x) \cap A \neq \emptyset\}.$$

This concept is referred to as "expansion" in image processing.

Definition (Isolated Points). A set of *isolated points* A^s of a subset A of X is defined as:

$$A^s = \{x : x \in A \textit{ and } (U(x) \setminus x) \cap A = \emptyset\}.$$

A point in A^s is referred to as "noise" in image processing. Conversely, we define a continuous part A^n of A as $A^n = A \setminus A^s$, which can also be written as:

$$A^n = \{x : x \in A \textit{ and } (U(x) \setminus x) \cap A \neq \emptyset\}.$$

A^n is called a "noise elimination" of A in image processing.

Definition (Inflation). An *inflation* A^f of a subset A of X is defined as:

$$A^f = \{x : (\exists y)(y \in A,\, x \in U(y))\} = \bigcup_{y \in A} U(y).$$

This concept seems similar to "closure", but they do not always coincide. We present the following definition.

Definition (Symmetry). A neighbourhood $U(\cdot)$ of a finite topological space $(X, U(\cdot))$ is called *symmetric* if and only if:

$$\textit{For all } x, y \in X,\ y \in U(x) \textit{ implies } x \in U(y).$$

The neighbourhoods (a), (b), and (c) shown in Figure 2 are symmetric, but neighbourhoods (d), (e) and (f) are not.

By using the concept of symmetricity, we can give the following lemma.

Lemma 2.1. If a neighbourhood $U(\cdot)$ is symmetric, then for all subsets A:

$$A^b = A^f.$$

Proof. Let us look at the following formulae.

$$\begin{aligned} x \in A^f &\Leftrightarrow \textit{for some } y \in A,\ x \in U(y) \\ &\Leftrightarrow \textit{for some } y \in A,\ y \in U(x)\ (\textit{by symmetricity}) \\ &\Leftrightarrow U(x) \cap A \neq \emptyset \\ &\Leftrightarrow x \in A^b. \end{aligned}$$

Thus, we get the result.

Q.E.D.

The following two definitions are analogies of very popular concepts of openness and closedness in continuous topologies.

Definition (Open Set). A subset G of X is *open* if and only if:

$$G = G^i.$$

Definition (Closed Set). A subset F of X is *closed* if and only if:

$$F = F^b.$$

We present here a second type of closure and interior.

Definition (f-closure and f-interior). An *f-closure* A^{f_b} and an *f-interior* A^{f_i} of a subset A of X is defined as:

$$A^{f_b} = \bigcap \{F : A \subseteq F,\ F \textit{ is closed}\},$$

$$A^{f_i} = \bigcup \{G : G \subseteq A,\ G \textit{ is open}\}.$$

We can show that an f-closure is gotten by a repetition of an ordinary closure (see Lemma 2.1 in Reference [1]) and an f-interior is gotten by a repetition of an ordinary interior.

Definition (Connected Set). A subset A of X is connected, if and only if, for any B, C in X:

$$A = B \cup C,\ B \neq \emptyset,\ C \neq \emptyset,\ \textit{and } B \cap C = \emptyset \textit{ implies } B^b \cap C \neq \emptyset.$$

This concept coincides with the usual intuitive concept of connectivity. In Figure 3, the image A is connected by the neighbourhood U_9, but not by the neighbourhood U_5.

For connectivity, we present the following theorem.

Theorem 2.1. Let X be filled and finite (i.e., containing only a finite number of points). Then, a subset A of X is connected if and only if for every $x \in A$:

$$(\cdots((\{x\}^b \cap A)^b \cap A)^b \cdots)^b \supseteq A.$$

That is, by a finite process of taking closures of A, A is covered.

Proof. $\Rightarrow$) We assume that for some $x \in A$:

$$(\cdots((\{x\}^b \cap A)^b \cap A)^b \cdots)^b \not\supseteq A.$$

For convenience, we denote an intersection of A and an n-th closure as P_n ($= P^b_{n-1} \cap A$, $P_1 = \{x\}^b \cap A$). As X is finite, we can assume that $P_{n+1} = P_n$. If we let

$$B = P_n,\ C = A \setminus B\ (= A \setminus P_n),$$

then $B \cup C = B \cup (A \setminus B) = A \cup B = A$ and $C \neq \emptyset$ (as $P_n \not\supseteq A$). Since X is filled, $x \in P_n$. Hence, $x \in B = P_n \neq \emptyset$. $B \cap C = \emptyset$ is clear. But,

$$B^b \cap C = P^b_n \cap C = P_n \cap (A \setminus P_n) = \emptyset.$$

Therefore, X is not connected.

$\Leftarrow$) Let B and C be non-void subsets of A such that $B \cap C = \emptyset$ and $B^b \cap C = \emptyset$. Then, there exists an element x in B, and we can construct a set P_n as a procedure described previously and $P_{n+1} = P_n$.

Let us show that $(B^b \cap A)^b \cap C = \emptyset$. If the left hand side is not empty, there exists some element z in it such that:

$$z \in C \ \mathit{and}\ z \in (B^b \cap A)^b.$$

Thus, $U(z) \cap B^b \cap A \neq \emptyset$, where $B^b \cap A = B^b \cap (B \cup C) = (B^b \cap B) \cup (B^b \cap C) = B \cup B^b \cap C$. Hence, $U(z) \cap B^b \cap C \neq \emptyset$ (as $B \cap U(z) = \emptyset$ is clear), which contradicts with $B^b \cap C = \emptyset$. Thus, we can set $B = B^b \cap A$. If $P_i \subseteq B^b \cap A$, then $P_{i+1} \subseteq B^b \cup A$ because:

$$P_{i+1} = P^b_i \cap A \subseteq (B^b \cap A)^b \cap A = B^b \cap A.$$

Therefore, we obtain the result:

$$P_n \subseteq B^b \cap A = B^b \cap (B \cup C) = B,$$

i.e.,

$$P_n \not\supseteq A.$$

Q.E.D.

The following facts are easily derived:

1. $((A^c)^i)^c = A^b$, $((A^c)^b)^c = A^i$.
2. $A^\partial = A^b \cap (A^c)^b$, $(A^c)^\partial = A^\partial$.
3. If $x \in A^s$, then $X \not\in (A \setminus \{x\})^b$.
4. If $A^s \neq \emptyset$, then A is not connected.
5. If X is filled, then $A \subseteq A^b$, $A^i \subseteq A$.
6. If A is open, then A^c is closed. Conversely, if A is closed, then A^c is open.