We recall at this point that every topology is a lattice, and indeed a complete Heyting algebra (remember O-3.22!). It is therefore meaningful to search for prime and co-prime elements in $\sigma(L)$ (see 1-3.11, -3.15). To formulate one of our conditions it is necessary to speak of the continuity of an operation (the main topic of the next section). We say that the sup operation is *jointly continuous with respect to the Scott toplogy* provided that the mapping

$$(x, y) \mapsto x \vee y : (L, \sigma(L)) \times (L, \sigma(L)) \to (L, \sigma(L))$$

is continuous in the product toplogy.

1.11. Proposition. *Let L be a complete lattice and $U \subseteq L$.*
(i) *U is a co-prime in $\sigma(L)$ iff $U \in \sigma(L)$ and U is a filter;*
(ii) *U is a prime in $\sigma(L)$ and $U \neq L$ if $U = L \setminus \downarrow u$ for some $u \in L$;*
(iii) *This last condition is also necessary, provided that the* sup *operation is jointly continuous;*
(iv) *This is the case if L is a continuous lattice.*

Proof. (i): Firstly suppose that $U \in \sigma(L)$ is a filter and that U is not a co-prime in $\sigma(L)$. Thus there are $V, W \in \sigma(L)$ such that $U \subseteq V \cup W$ and elements $v \in U \setminus V$ and $w \in U \setminus W$. Since V and W are upper sets we have $vw \notin V \cup W$. But $vw \in U$ since U is a filter. This is a contradiction.

Secondly, suppose that U is a co-prime in $\sigma(L)$. To show that U is a filter, note first that it is an upper set. Suppose $v, w \in U$. Then $U \not\subseteq L \setminus \downarrow v$ and $U \not\subseteq L \setminus \downarrow w$. By 1.4(ii), the sets $L \setminus \downarrow v$ and $L \setminus \downarrow w$ are Scott open. Thus, since U is co-prime,

$$U \not\subseteq ((L \setminus \downarrow v) \cup (L \setminus \downarrow w)) = L \setminus (\downarrow v \cap \downarrow w) = L \setminus \downarrow vw.$$

Thus $U \cap \downarrow vw \neq \emptyset$. Since U is an upper set, this means $vw \in U$.

(ii): Recall that $L \setminus \downarrow u = L \setminus \{u\}^-$ for all u by 1.4(ii). But the complements of singleton closures are prime in any topology.

(iii): Assume that the sup-operation is jointly continuous relative to the Scott topology, and let $U \in$ PRIME $\sigma(L)$, $U \neq L$. Set $A = L \setminus U$ and $u = \sup A$. In order to show that $U = L \setminus \downarrow u$ it suffices to verify $u \in A$. Since A is closed under directed sups by 1.4(i), all we have to verify is that A is directed.

By way of contradiction assume that it is not. Then there are elements $a, b \in A$ with $a \vee b \in U$. By the continuity of the sup-operation we would find Scott-open neighborhoods V and W of a and b, respectively, such that $V \vee W \subseteq U$. But since V and W are upper sets, we have $V \vee W = V \cap W$. Since U is prime, the relation $V \cap W \subseteq U$ implies $V \subseteq U$ or $W \subseteq U$. This would entail $a \in U$ or $b \in U$, which would contradict $a, b \in A = L \setminus U$.

(iv): Finally suppose that L is continuous. In order to show the continuity of the sup-operation at (a, b) we pick some $u \ll a \vee b$. By I-1.6 we have

$$a \vee b = (\sup \twoheaddownarrow a) \vee (\sup \twoheaddownarrow b) = \sup(\twoheaddownarrow a \vee \twoheaddownarrow b)$$

Since $\twoheaddownarrow a \vee \twoheaddownarrow b$ is directed (I-1.2(iii)), we find some $x \ll a$ and $y \ll b$ with $u \ll x \vee y$ (by I-1.19). But then $\twoheaduparrow x$ and $\twoheaduparrow y$ are Scott-open neighborhoods of a and b, respectively, such that

$$\twoheaduparrow x \vee \twoheaduparrow y \subseteq \uparrow(x \vee y) \subseteq \twoheaduparrow u.$$

Since the $\twoheaduparrow u$ with $u \ll a \vee b$ form a basis of $\sigma(L)$-neighborhoods of $a \vee b$ by Proposition 1.10(i), the desired continuity is established. $\square$

We can immediately rephrase 1.11 in topological terminology, if we recall the concept of a *sober space* (see I-3.36 and the discussion). Remember that a non-empty subset A of a topological space X is called *irreducible* iff it is closed and not the union of two proper closed subsets (that is, the complementary set $X \setminus A \in$ PRIME $\mathcal{O}(X)$; see I-3.11). A space X is called *sober* iff every irreducible set A has a unique dense point (that is, $A = \{a\}^-$ with a unique $a \in A$). Clearly all singleton closures are irreducible. (Notice that a sober space is automatically T_0 since $\{x\}^- = \{y\}^-$ always implies $x = y$.) We now have the following corollaries of 1.11 with a slight sharpening:

1.12. Corollary. *If L is a complete lattice such that the* sup*-operation is jointly Scott-continuous, then $(L, \sigma(L))$ is a sober space.*

Proof. Immediate from 1.11 and the definitions. $\square$

1.13. Corollary. *If L is a continuous lattice, then $(L, \sigma(L))$ is a quasicompact and locally quasicompact sober space. In particular, $(L, \sigma(L))$ is a Baire space.*

Proof. We have to show that a point $x \in L$ has a basis of quasicompact neighborhoods. By 1.10 the sets $\twoheaduparrow y$ with $y \ll x$ form a basis for the neighborhoods of the point. But as we know, if $x \in U \in \sigma(L)$, then actually we have a $y \in U$ with $y \ll x$; hence, $\uparrow y \subseteq U$, and so the sets $\uparrow y$ can be used as neighborhoods. Since $\uparrow y$ (and hence, in particular $L = \uparrow 0$) is trivially quasicompact with respect to any topology whose open sets are upper sets, the assertion is proved. That $(L, \sigma(L))$ is a Baire space follows from I-3.43.10. $\square$

We wish to warn the reader about a subtlety concerning the joint continuity of the sup-operation above. We cannot be satisfied by saying that the sup-operation is a continuous function $(L \times L, \sigma(L \times L)) \to (L, \sigma(L))$; this continuity is weaker, since in general we have a proper containment of topologies: $\sigma(L \times L) \supset \sigma(L) \times \sigma(L)$. We will return to this question at greater length in Section 4 below (see 4.11 ff).

We know enough about the Scott topology now to use it for yet another characterization theorem for continuous lattices.

1.14. Theorem. *For any complete lattice, the following conditions are equivalent:*

(1) *L is continuous;*

(2) *If $U \in \sigma(L)$, then $U = \bigcup\{\twoheaduparrow x : x \in U\}$;*

(3) *Each point has a neighborhood basis (in the Scott topology) of Scott-open filters, and $\sigma(L)$ is a continuous lattice;*

(4) *For each point $x \in L$ we have $x = \sup\{\inf U : x \in U \in \sigma(L)\}$*

(5) *$\sigma(L)$ has enough co-primes and is a continuous lattice;*

(6) *$\sigma(L)$ is completely distributive;*

(7) *Both $\sigma(L)$ and $\sigma(L)^{\mathrm{op}}$ are continuous.*

Proof. (1) implies (2): Use 1.10.

(2) implies (1): Let $x \in L$ and set $y = \sup \twoheaddownarrow x \leq x$. If $y < x$, then $L \setminus \downarrow y$ is a Scott-open neighborhood of x; hence by (2) it contains an open neighborhood $\twoheaduparrow z$ of x with $z \in L \setminus \downarrow y$. But then $z \ll x$, and thus $z \leq \sup \twoheaddownarrow x = y$, a contradiction.

(1) implies (3): By (1) iff (2), x has arbitrarily small neighborhoods of the form $\twoheaduparrow y$ with $y \ll x$. By 1.10 and I-3.3, we know then that x has arbitrarily small Scott-open neighborhoods which are filters. In order to prove the continuity of $\sigma(L)$, we let U be Scott open. For any $x \in U$ we find a $y \in U$ with $y \ll x$ by (2). Then $x \in \twoheaduparrow y \in \sigma(L)$, and we claim that $\twoheaduparrow y \ll U$: Indeed, if D is a directed family of Scott-open sets covering U, then one of its members must contain y, hence $\uparrow y$, since Scott-open sets are upper sets, and thus it contains $\twoheaduparrow y$. We have shown $U = \bigcup\{V : V \ll U\}$.

(3) implies (4): For $x \in L$ set

$$y = \sup\{\inf U : x \in U \in \sigma(L)\} \leq x.$$

If $y < x$, then $L \setminus \downarrow y$ is a Scott-open neighborhood of x. Let V be a Scott-open neighborhood of x with $V \ll L \setminus \downarrow y$, which exists since $\sigma(L)$ is continuous by (3). Now use (3) to find a Scott-open filter neighborhood U of x within V. By the definition of y we have $\inf U \leq y$. Then

$$L \setminus \downarrow y \subseteq L \setminus \downarrow \inf U = L \setminus \bigcap\{\downarrow u : u \in U\} = \bigcup\{L \setminus \downarrow u : u \in U\}.$$

Since U is a filter, the $L \setminus \downarrow u$ for $u \in U$ form a directed family of Scott-open sets. Since $V \ll L \setminus \downarrow y$, there must be a $u \in U$ such that $V \subseteq L \setminus \downarrow u$ and so $u \notin V$. This is a contradiction to $U \subseteq V$. Thus $x = y$.

(4) implies (1): Clear since for every Scott-open neighborhood U of x one has $\inf U \ll x$.

(3) iff (5): Clear from 1.11.

(5) iff (6) iff (7): Consequence of I-3.15. $\square$

Note that condition (2) is stronger than saying that $\sigma(L)$ has a basis of sets $\Uparrow x$. A parallel result to 1.14 for algebraic lattices reads as follows:

1.15. Corollary. *For any complete lattice L the following conditions are equivalent:*
(1) *L is algebraic.*
(2) *The Scott topology has a basis of sets $\uparrow k$ where $k \in K(L)$.*
(3) *$\sigma(L)$ is algebraic and has enough co-primes.*

Proof. (1) implies (2): Let U be a Scott-open neighborhood of x. We recall that $x = \sup(\downarrow x \cap K(L))$ and $\downarrow x \cap K(L)$ is deirected. Hence by 1.2(ii), we find a $k \in \downarrow x \cap K(L) \cap U$. Then $\uparrow k = \Uparrow k$ is a Scott-open neighborhood of k, hence of x, with $\uparrow k \subseteq U$.

(2) implies (3): If $k \in K(L)$, then $\uparrow k \in \sigma(L)$ since $\Uparrow k = \uparrow k$. Now $\uparrow k$ is a quasicompact set (if $\uparrow k$ is covered by Scott-open sets, then one of them must contain k, hence $\uparrow k$ by 1.2(i)). Thus $\sigma(L)$ is algebraic by I-4.4. Since all $\uparrow k$ are filters, hence co-primes by 1.11, we are done.

(3) implies (1): Since $\sigma(L)$ is algebraic, the Scott topology has a basis of quasicompact sets U. Since there are enough co-primes, U is a union of open filters by 1.11, and thus, by quasicompactness, $U = U_1 \cup \ldots \cup U_n$ with open filters U_k. It is no loss of generality to assume that none of the U_k is contained in the union of the others. Then $V = U_1 \setminus (U_2 \cup \ldots \cup U_n)$ is quasicompact and filtered. We claim that V has a smallest element u_1.

For if not, then $V \subseteq \bigcup\{L \setminus \downarrow v : v \in V\}$; and by quasicompactness and the fact that the $L \setminus \downarrow v$ form a directed family, there would be a $v \in V$ with $V \subseteq L \setminus \downarrow v$, notably $v \notin V$, which is impossible. Since $U_2 \cup \ldots \cup U_n$ is an upper set, it cannot contain $\inf U_1$. Hence $u_1 = \min U_1$. Since U_1 is an upper set, $U_1 = \uparrow u_1$. Since U_1 is Scott open, $u_1 \in K(L)$.

We have shown that $\sigma(L)$ has a basis of sets $\uparrow k$ where $k \in K(L)$. Now let $x \in L$, set $y = \sup(\downarrow x \cap K(L)) \leq x$. If $y < x$, then the Scott-open neighborhood $L \setminus \downarrow y$ of x would contain a basic neighborhood $\uparrow k$ of x. Then there would be a $k \in K(L)$ with $k \leq x$ and $k \not\leq y$ which contradicts the definition of y. $\square$