Monoidal Company for Accessible Functors[*]

Henning Basold[1], Damien Pous[2], and Jurriaan Rot[3]

1 **Radboud University, Nijmegen, The Netherlands**
`henning@basold.eu`
2 **Univ Lyon, CNRS, ENS de Lyon, UCB Lyon 1, LIP, France**
`Damien.Pous@ens-lyon.fr`
3 **Radboud University, Nijmegen, The Netherlands**
`jrot@cs.ru.nl`

—— Abstract ——

Distributive laws between functors are a fundamental tool in the theory of coalgebras. In the context of coinduction in complete lattices, they correspond to the so-called compatible functions, which enable enhancements of the coinductive proof technique. Amongst these, the greatest compatible function, called the companion, has recently been shown to satisfy many good properties.

Categorically, the companion of a functor corresponds to the final object in a category of distributive laws. We show that every accessible functor on a locally presentable category has a companion. Central to this and other constructions in the paper is the presentation of distributive laws as coalgebras for a certain functor. This functor itself has again, what we call, a second-order companion. We show how this companion interacts with the various monoidal structures on functor categories. In particular, both the first- and second-order companion give rise to monads. We use these results to obtain an abstract GSOS-like extension result for specifications involving the second-order companion.

1 Introduction

Coalgebras are an abstract tool for defining and studying the semantics of state-based systems [7]. Distributive laws of various kinds play a crucial role in the theory of coalgebras and coinduction. For instance, they are used in (structural) operational semantics [23, 3], for automata constructions [20], and for coinductive proof techniques [6].

In the context of complete lattices, distributive laws for a given functor correspond to functions *compatible* with a given function [15]. Those were introduced to obtain a modular theory of enhancements of the coinductive proof method (up-to techniques): most of the useful enhancements can be presented as compatible functions, and their class is closed under union and composition. In particular, the union of all compatible functions is always a compatible function, the greatest one. This greatest compatible function, called the *companion*, subsumes all compatible functions and is a closure operator [16].

The last two authors recently gave a categorical account of the companion of a functor [17]: it can be defined as the final object in the category of distributive laws over that functor; if it exists it is a monad (and the underlying distributive law is that of a monad); and under some conditions, it can be constructed explicitly as the codensity monad of the final sequence of the starting functor. The latter existence result corresponds to a Kleene-like fixpoint theorem, and yields a characterisation of the companion in complete lattices similar to that from Parrow and Weber's work [14]. It is also shown in [17] that the companion of a polynomial functor can be characterised using an abstract notion of causal algebra.

Here we pursue this line of work in another direction, by investigating structural properties and existence of the companion via a Knaster-Tarski-like theorem [10, 22]: accessible functors on a locally presentable category have a final coalgebra [13, 1].

In Section 3 we establish an adjunction between coalgebras for an endofunctor B and distributive laws over B, so that a final coalgebra for B can be obtained from the companion of B. We also recover that the companion is a distributive law of a monad by observing that the (strict) monoidal structure of composition in the category of endofunctors lifts to the category of distributive laws. Monoids for this lifted monoidal structure are precisely distributive laws of monads, and so is the companion (which is a monoid by finality).

Being defined as the largest compatible function, the companion in complete lattices can be obtained as the greatest fixpoint of a carefully chosen functional [16]. We extend this idea categorically in Section 4, by associating to a given endofunctor B a second-order endofunctor $\mathbb{B}$ whose category of coalgebras is isomorphic to the category of distributive laws over B. Slightly more generally, we have the following bijective correspondence:

$$\frac{FB \Rightarrow BG}{F \Rightarrow \mathbb{B}(G)}$$

The second-order functor $\mathbb{B}$ is defined using right Kan extensions. It was used by Street to establish another correspondence, between distributive laws of monads and monad maps [21]. We show that $\mathbb{B}$ is lax monoidal, so that the aforementioned isomorphism actually is an isomorphism of monoidal categories.

To get the existence of the companion, it suffices to show that $\mathbb{B}$ exists and has a final coalgebra; this is where we use accessibility (Section 5). There is a technical subtlety here. Indeed, for size reasons, we restrict $\mathbb{B}$ to the sub-category of κ-accessible functors, for some large enough regular cardinal κ. Doing so, the companion we obtain as a final $\mathbb{B}$-coalgebra depends on κ: it is κ-accessible, and it subsumes only those distributive laws that are κ-accessible. By considering larger and larger cardinals, one can thus obtain a sequence of companions relative to those cardinals. (In small categories, this sequence actually converges.)

Building on those results, we give a new account for some results in structural operational semantics, where one usually considers more permissive notions of distributive laws. For instance, one often works with natural transformations of the shape $\rho\colon FB \Rightarrow BF^\star$, where $F^\star$ is the free monad over F. This is fine because every such natural transformation can be turned into a distributive law $\rho^\sharp\colon F^\star B \Rightarrow BF^\star$.

According to the previous bijection, natural transformations such as the above ρ are in one-to-one correspondence with coalgebras for the composite functor $\mathbb{B}(-^\star)$, which can be considered as coalgebras for $\mathbb{B}$ up to $-^\star$. This observation makes it possible to reuse the theory of up-to techniques to propose a new format (Section 6). Indeed, the second-order functor $\mathbb{B}$ being accessible, it admits a second order companion, $\mathbb{T}$, and one can work with *distributive laws up to* $\mathbb{T}$, that is, natural transformations of type $FB \Rightarrow B\mathbb{T}(F)$.

By lifting another monoidal structure, we prove that $\mathbb{T}$ produces monads: given any functor F, $\mathbb{T}(F)$ is always a monad (but not the free one). We show that starting from a

distributive law up to $\mathbb{T}$ as above, one obtains a distributive law of the monad $\mathbb{T}(F)$ over B. We illustrate the use of these distributive laws up to $\mathbb{T}$ in the stream calculus [19].

When starting with a natural transformation like the above $\rho\colon FB \Rightarrow BF^\star$, for which tools not requiring the second order companion already exist, we show that the two approaches are consistent: they eventually lead to the same F-algebra on the final B-coalgebra. We conjecture that a similar result holds with respect to abstract GSOS specifications [23].

Most omitted proofs can be found in appendix.

2   Preliminaries

Before we dive into the content of the paper, we introduce some notation and concepts that we use throughout. We use capital, calligraphic letters like $\mathcal{C}, \mathcal{D}, \ldots$ to stand for general categories. We write $[\mathcal{C}, \mathcal{D}]$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$ with natural transformation $\alpha, \beta, \ldots$ as morphisms. Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, we denote by $F^*\colon [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$ the functor that pre-composes with F, and by $F_*\colon [\mathcal{E}, \mathcal{C}] \to [\mathcal{E}, \mathcal{D}]$ the functor that post-composes with F. If X is an object in $\mathcal{D}$, then we write $K_X\colon \mathcal{C} \to \mathcal{D}$ for the constant functor that maps every object in $\mathcal{C}$ to X. For the sake of clarity, we denote general functors between functor categories by blackboard letters $\mathbb{F}, \mathbb{G}, \mathbb{H}, \ldots$ and refer to them as *second-order functors*. Consequently, we call the category of functors between functor categories the *second-order functor category*. Given an endofunctor $B\colon \mathcal{C} \to \mathcal{C}$, we denote the category of B-coalgebras by $\mathsf{coalg}(B)$.

2.1   Locally Presentable Categories and Accessible functors

The construction of the companion in Section 5 crucially uses locally presentable categories and accessible functors thereon. We recall these notions and some of their properties, see [2] for an extensive treatment. Let us first describe locally κ-presentable categories for a regular cardinal κ. A diagram $D\colon \mathcal{I} \to \mathcal{C}$ is said to be κ-*filtered* if the category $\mathcal{I}$ is κ-filtered, that is, if every diagram in $\mathcal{I}$ smaller than κ has a cocone. Whenever the colimit of D exists, it is called κ-filtered as well. Next, we say $X \in \mathcal{C}$ is a κ-*presented* object if the hom-functor $\mathcal{C}(X, -)\colon \mathcal{C} \to \mathsf{Set}$ preserves κ-filtered colimits. Finally, a category $\mathcal{C}$ is *locally κ-presentable* if it is locally small, cocomplete, and there is a set S of κ-presented objects in $\mathcal{C}$ that generates $\mathcal{C}$: every object in $\mathcal{C}$ is a κ-filtered colimit of objects from S.

Central to working with locally presentable categories is the notion of accessible functors. A functor $F\colon \mathcal{C} \to \mathcal{C}$ is κ-*accessible* if it preserves κ-filtered colimits. We denote the category of κ-accessible endofunctors on $\mathcal{C}$ by $[\mathcal{C}, \mathcal{C}]^\kappa$. Note that κ-accessible functors can be composed, thus for a κ-accessible functor $F\colon \mathcal{C} \to \mathcal{C}$, the pre- and post-composition functors F^* and F_* restrict to endofunctors on $[\mathcal{C}, \mathcal{C}]^\kappa$.

We shall mention some important results that we need later. For a locally κ-presentable category $\mathcal{C}$, we denote by $\mathcal{C}_\kappa$ the full subcategory of κ-presentable objects and the inclusion functor by $I\colon \mathcal{C}_\kappa \to \mathcal{C}$. By [13, Proposition 2.1.5], $\mathcal{C}_\kappa$ is *essentially small*, that is, $\mathcal{C}_\kappa$ is equivalent to a small category. Lastly, by [13, Proposition 2.4.3] the pre-composition functor I^* has a left adjoint as in $[\mathcal{C}, \mathcal{C}]^\kappa \xrightarrow{\ I^*\ } [\mathcal{C}_\kappa, \mathcal{C}]$, which also is an adjoint equivalence, see [13, Corollary 2.1.9]. This allows us to represent κ-accessible functors by functors on generators.

2.2   Monoidal Categories and Monads

Monoidal categories are categories that come with a notion of tensor product and a unit for that tensor product. For the purpose of this exposition, we are only interested in *strict monoidal categories*. These are triples $(\mathcal{C}, \otimes, I)$, where $\mathcal{C}$ is a category, $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a

functor, the *tensor*, and $I \in \mathcal{C}$ is an object, the *unit*. This data is subject to the following equations: $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$, $X \otimes I = X$, $I \otimes X = X$. Since we will only encounter strict monoidal categories, we drop the adjective "strict". A category $\mathcal{D}$ is said to be a *monoidal subcategory* of $\mathcal{C}$ if $\mathcal{D}$ is a subcategory of $\mathcal{C}$, and $\mathcal{D}$ is closed under the tensor product and contains the unit I. Given a functor $F \colon \mathcal{C} \to \mathcal{D}$ between monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$, we say that F is a *lax monoidal functor* if there is a morphism $\beta \colon I_{\mathcal{D}} \to F(I_{\mathcal{C}})$ and a natural transformation $\alpha \colon \otimes_{\mathcal{D}} \circ (F \times F) \Rightarrow F \circ \otimes_{\mathcal{C}}$. These morphisms must fulfil the following three equations for all objects $X, Y, Z \in \mathcal{C}$: $\alpha_{I_{\mathcal{C}}, X} \circ (\beta \otimes_{\mathcal{D}} \mathsf{id}_{FX}) = \mathsf{id}_{FX}$, $\alpha_{X, I_{\mathcal{C}}} \circ (\mathsf{id}_{FX} \otimes_{\mathcal{D}} \beta) = \mathsf{id}_{FX}$ and $\alpha_{X \otimes_{\mathcal{C}} Y, Z} \circ (\alpha_{X,Y} \otimes_{\mathcal{D}} \mathsf{id}_{FZ}) = \alpha_{X, Y \otimes_{\mathcal{C}} Z} \circ (\mathsf{id}_{FX} \otimes_{\mathcal{D}} \alpha_{Y,Z})$. If α and β are both identities, we say that F is a *strict monoidal functor*. Finally, we consider monoidal categories to be isomorphic if there is a strict monoidal isomorphism between them.

The two relevant examples of monoidal structures are given by the different ways functors can be composed. For any category $\mathcal{C}$, there is an obvious monoidal structure on the functor category $[\mathcal{C}, \mathcal{C}]$, defined in terms of functor composition. We denote this structure by $([\mathcal{C}, \mathcal{C}], *, \mathsf{Id})$, where the tensor product on functors $F, G \in [\mathcal{C}, \mathcal{C}]$ is defined by $F * G = F \circ G$, and for natural transformations α and β the tensor $\alpha * \beta$ is given by horizontal composition. If $(\mathcal{D}, \otimes, I)$ is itself a monoidal category, then there is a second way of turning $[\mathcal{D}, \mathcal{D}]$ into a monoidal category, by point-wise tensoring of functors. That is, one defines a tensor product $\otimes'$ by $(F \otimes' G)(D) = F(D) \otimes G(D)$ and on natural transformations by $(\alpha \otimes' \beta)_D = \alpha_D \otimes \beta_D$. The identity is given by the constant functor K_I with $K_I(D) = I$. In this paper we will use the particular instance with $(\mathcal{D}, \otimes, I) = ([\mathcal{C}, \mathcal{C}], *, \mathsf{Id})$ on the second-order functor category; we denote this instance by $([[\mathcal{C}, \mathcal{C}], [\mathcal{C}, \mathcal{C}]], \circledast, K_{\mathsf{Id}})$.

2.3   Monads and Distributive Laws

We will make good use of the folklore phrase "a monad is just a monoid in the category of endofunctors". A monoid in a monoidal category $(\mathcal{C}, \otimes, I)$ is a triple (X, m, e), where $X \in \mathcal{C}$, $m \colon X \otimes X \to X$ and $e \colon I \to X$, such that $m \circ (\mathsf{id} \otimes e) = \mathsf{id}$, $m \circ (e \otimes \mathsf{id}) = \mathsf{id}$ and $m \circ (m \otimes \mathsf{id}) = m \circ (\mathsf{id} \otimes m)$. Accordingly, a monad is a triple (F, μ, η),[1] where F is an endofunctor on $\mathcal{C}$, and $\mu \colon F * F \Rightarrow F$ and $\eta \colon \mathsf{Id} \Rightarrow F$ are natural transformations; the monoid laws are then just the usual monad laws. A *monad map* from (F, μ^F, η^G) to (G, μ^G, η^G) is a natural transformation $F \Rightarrow G$ that makes the evident coherence diagrams commute. Finally note that lax monoidal functors preserve monoids.

The central objects of study in this paper are distributive laws. Given endofunctors $B, F \colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$, a *distributive law* is a natural transformation $FB \Rightarrow BF$. Such a distributive law induces a lifting of F to $\mathsf{coalg}(B)$. If (F, μ^F, η^F) is a monad, we say that $\rho \colon FB \Rightarrow BF$ is a *distributive law of a monad (over B)*, provided that $\rho \circ \eta^F B = B \eta^F$ and $\rho \circ \mu^F B = B \mu^F \circ \rho F \circ F \rho$ hold. On rare occasions, we need to generalise distributive laws to *asymmetric distributive laws*, which are natural transformations $FB \Rightarrow BG$ for another endofunctor G on $\mathcal{C}$. Similarly, if G carries a monad structure (G, μ^G, η^G), we say that $\rho \colon FB \Rightarrow BG$ is an *asymmetric distributive law of monads (over B)*, whenever $\rho \circ \eta^F B = B \eta^G$ and $\rho \circ \mu^F B = B \mu^G \circ \rho G \circ F \rho$ hold.[2]

---

[1] We divert from the usual order (F, η, μ) for denoting monads, since for monoids and monoidal categories the multiplication comes first.

[2] Note that Street [21] refers in this situation to (B, ρ) as a monad functor.

3   The Companion of a Functor

We define the notion of companion, and show several of its properties. Throughout this section, let $\mathcal{F}$ be a full, monoidal subcategory of $([\mathcal{C}, \mathcal{C}], *, \mathsf{Id})$, and $B \colon \mathcal{C} \to \mathcal{C}$ a functor in $\mathcal{F}$. The reader can safely assume $\mathcal{F} = [\mathcal{C}, \mathcal{C}]$. The slight generalisation to subcategories is required for the material in Section 5, where we restrict to categories of accessible functors.

▶ **Definition 1.** The category $\mathsf{DL}(B)$ of distributive laws (w.r.t. the subcategory $\mathcal{F}$) is defined as follows. An object is a pair (F, λ) where $F \colon \mathcal{C} \to \mathcal{C}$ is a functor in $\mathcal{F}$ and $\lambda \colon FB \Rightarrow BF$ is a natural transformation. A *morphism of distributive laws* from (F, λ) to (G, ρ) is a natural transformation $\delta \colon F \Rightarrow G$ such that $\rho \circ \delta B = B\delta \circ \lambda$, see [18, 24, 11, 9]. The *companion* of B is the final object of $\mathsf{DL}(B)$, if it exists. We typically denote the companion of B by (T^B, τ^B), or (T, τ) if B is clear from the context. Given an object (F, λ) in $\mathsf{DL}(B)$, we write $\lambda^\dagger \colon F \Rightarrow T$ for the unique morphism obtained by finality.

3.1   Final Coalgebra from the Companion

Suppose the underlying category $\mathcal{C}$ has an initial object 0. Consider the functor $\mathsf{ev}_0 \colon \mathsf{DL}(B) \to \mathsf{coalg}(B)$, defined on objects as $\mathsf{ev}_0(F, \lambda) = \lambda_0 \circ F!_{B0} \colon F0 \to BF0$. In [17], we showed that applying ev_0 to the companion yields a final B-coalgebra. Here we generalise the situation to subcategories of $[\mathcal{C}, \mathcal{C}]$ and show a stronger result: the functor ev_0 has a left adjoint.

A coalgebra $f \colon X \to BX$ is a distributive law of the constant functor K_X over B. If K_X is an object of $\mathcal{F}$, for each X, then this gives rise to a functor, which is left adjoint to ev_0.

▶ **Theorem 2.** *Suppose that $\mathcal{F}$ contains all constant functors K_X for X in $\mathcal{C}$. Then K extends to a functor $K \colon \mathsf{coalg}(B) \to \mathsf{DL}(B)$ which is a left adjoint of ev_0.*

$$\mathsf{coalg}(B) \underset{\mathsf{ev}_0}{\overset{K}{\rightleftarrows}} \perp \mathsf{DL}(B)$$

Hence, if (T, τ) is the companion, then $\mathsf{ev}_0(T, \tau)$ is a final B-coalgebra.

It follows from the above theorem that not every functor B has a companion.

3.2   Monoidal Structure of Distributive Laws

In [17], we showed that the companion, if it exists, is always a monad. It turns out that this result can be phrased slightly more generally based on the monoidal structure of $\mathcal{F}$ given by composition. The main observation is that the monoidal structure of $[\mathcal{C}, \mathcal{C}]$ lifts to $\mathsf{DL}(B)$, and that a monoid in $\mathsf{DL}(B)$ corresponds to a monad with a distributive law over B.

▶ **Theorem 3.** *The category $\mathsf{DL}(B)$ is strict monoidal, with tensor product $*$ given by*

$$(F, \lambda) * (G, \rho) = (FG, \lambda G \circ F\rho)$$

and the unit by trivial distributive law $(\mathsf{Id}, \mathsf{id} \colon B \Rightarrow B)$. *An object (F, λ) is a monoid in* $(\mathsf{DL}(B), *, \mathsf{id})$ *if and only if F is a monad and λ a distributive law of that monad over B.*

Proof. For an object (F, λ) of $\mathsf{DL}(B)$, (F, μ, η) is a monoid iff
1. η is a morphism from $(\mathsf{Id}, \mathsf{id})$ to (F, λ), i.e., $B\eta = \lambda \circ \eta B$;
2. μ is a morphism from $(F, \lambda) * (F, \lambda)$ to (F, λ), i.e., $\lambda \circ \mu B = B\mu \circ \lambda F \circ F\lambda$;

3. (F, μ, η) is a monoid in $\mathcal{F}$.

The first two items are the axioms of distributive laws of monad over functor, the third is equivalent to (F, μ, η) being a monad. ◀

It is straightforward that the final object of any monoidal category, if it exists, is a monoid. Instantiating this to $\mathsf{DL}(B)$ and applying Theorem 3, we obtain the following result. The first item appeared in [17], for $\mathcal{F} = [\mathcal{C}, \mathcal{C}]$.

▶ **Corollary 4.** *Suppose (T, τ) is the companion of B.*
1. *There are unique $\eta\colon \mathsf{Id} \Rightarrow T$ and $\mu\colon TT \Rightarrow T$ such that (T, μ, η) is a monad and $\tau\colon TB \Rightarrow BT$ is a distributive law of this monad over B.*
2. *For any (F, λ) in $\mathsf{DL}(B)$, if F is a monad and λ a distributive law of that monad over B, then $\lambda^\dagger\colon F \Rightarrow T$ is a monad map.*

4   Distributive Laws as Coalgebras

In this section we work again with an endofunctor B on a monoidal subcategory $\mathcal{F}$ of $[\mathcal{C}, \mathcal{C}]$. An important idea underlying the current paper is that distributive laws over B can be characterised as coalgebras for a certain functor $\mathbb{B}\colon \mathcal{F} \to \mathcal{F}$. To obtain $\mathbb{B}$, suppose that B has a (global) right Kan extension $\mathsf{Ran}_B(-)\colon \mathcal{F} \to \mathcal{F}$, that is, a right adjoint to the pre-composition functor B^*. This gives us a bijective correspondence

$$\frac{FB \Rightarrow G}{F \Rightarrow \mathsf{Ran}_B G} \textbf{(Kan)}$$

natural in F and G. We obtain the correspondence between coalgebras and distributive laws from **(Kan)** by taking $G = BF$. More precisely, we compose Ran_B and B_* to the functor

$$\mathbb{B} = \mathsf{Ran}_B(B-)\colon \mathcal{F} \to \mathcal{F}. \tag{1}$$

We call this functor the *familiar of B*. From **(Kan)** we get the announced correspondence.

▶ **Lemma 5.** *The category $\mathsf{coalg}(\mathbb{B})$ is isomorphic to $\mathsf{DL}(B)$.*

Proof. The isomorphism is immediate by the natural bijection **(Kan)**. Note that the functor $\mathsf{coalg}(\mathbb{B}) \to \mathsf{DL}(B)$ is given by transposing along the adjunction, that is, it maps $\lambda\colon F \Rightarrow \mathbb{B}(F)$ to $\epsilon_F \circ \lambda B\colon FB \Rightarrow BF$, where ϵ_F is the counit of the Kan extension $\mathbb{B}(F)$. ◀

In particular, the companion of B is the final $\mathbb{B}$-coalgebra.

▶ **Example 6.** Let $b\colon L \to L$ be a monotone function on a complete lattice L. The associated $\mathbb{B}\colon [L, L] \to [L, L]$ was given in [16] by $\mathbb{B}(f) = \bigvee_{gb \le bf} g$. The correspondence in Lemma 5 means that f is b-compatible (i.e., $fb \le bf$) iff it is a post-fixed point of $\mathbb{B}$. The standard pointwise computation of right Kan extensions by limits gives another characterisation:

$$\mathbb{B}(f)(x) = \bigwedge_{x \le b(y)} bf(y).$$

Street [21] considers the functor $\mathbb{B}$ in the context of monads, to obtain the following correspondence between monad maps and distributive laws. We use this result in Section 6.

▶ **Lemma 7.** *If G is a monad, then $\mathbb{B}(G)$ is a monad as well. Moreover, given monads F, G there is a one-to-one correspondence:*

$$\frac{FB \Rightarrow BG \quad \text{asymm. d.l. of monads over } B}{F \Rightarrow \mathbb{B}(G) \quad \text{monad map}} \textbf{(Street)}$$

We complete the picture by showing that the familiar $\mathbb{B}$ is lax monoidal.

▶ **Theorem 8.** *The functor $\mathbb{B}$ is a lax monoidal endofunctor on $(\mathcal{F}, *, \mathsf{Id})$.*

Proof. To show that $\mathbb{B}$ is lax monoidal, we use that for each $H \colon \mathcal{C} \to \mathcal{C}$, $(\mathbb{B}(H), \epsilon_H)$ is a right Kan extension of BH along B. This allows us to define for functors $F, G \colon \mathcal{C} \to \mathcal{C}$ the mediators $\alpha_{F,G} \colon \mathbb{B}(F)\mathbb{B}(G) \Rightarrow \mathbb{B}(FG)$ and $\beta \colon \mathsf{Id} \Rightarrow \mathbb{B}(\mathsf{Id})$ as the unique natural transformations such that the following two diagrams commute.

$$
\begin{array}{ccc}
\mathbb{B}(F)\mathbb{B}(G)B & \xrightarrow{\alpha_{F,G}B} & \mathbb{B}(FG)B \\
{\scriptstyle \mathbb{B}(F)\epsilon_G}\Big\| & & \Big\|{\scriptstyle \epsilon_{FG}} \\
\mathbb{B}(F)BG & \xrightarrow[\epsilon_F G]{} & BFG
\end{array}
\qquad\qquad
\begin{array}{ccc}
B & \xrightarrow{\beta B} & \mathbb{B}(\mathsf{Id})B \\
 & \searrow & \Big\|{\scriptstyle \epsilon_{\mathsf{Id}}} \\
 & & B
\end{array}
$$

Naturality of α (in F and G) follows via the adjunction of the right Kan extension from naturality of $\epsilon_F G \circ \mathbb{B}(F)\epsilon_G$ (in F and G), we skip the details. It remains to prove the coherence axioms for α and β. Given $F, G, H \colon \mathcal{C} \to \mathcal{C}$, we thus need to prove:

$$
\begin{array}{ccc}
\mathbb{B}(F)\mathbb{B}(G)\mathbb{B}(H) & \xrightarrow{\mathbb{B}(F)\alpha_{G,H}} & \mathbb{B}(F)\mathbb{B}(GH) \\
{\scriptstyle \alpha_{F,G}\mathbb{B}(H)}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_{F,GH}} \\
\mathbb{B}(FG)\mathbb{B}(H) & \xrightarrow[\alpha_{FG,H}]{} & \mathbb{B}(FGH)
\end{array}
\quad
\begin{array}{ccc}
\mathbb{B}(F) & \xrightarrow{\mathbb{B}(F)\beta} & \mathbb{B}(F)\mathbb{B}(\mathsf{Id}) \\
 & \searrow & \Big\downarrow{\scriptstyle \alpha_{F,\mathsf{Id}}} \\
 & & \mathbb{B}(F)
\end{array}
\quad
\begin{array}{ccc}
\mathbb{B}(F) & \xrightarrow{\beta\mathbb{B}(F)} & \mathbb{B}(\mathsf{Id})\mathbb{B}(F) \\
 & \searrow & \Big\downarrow{\scriptstyle \alpha_{\mathsf{Id},F}} \\
 & & \mathbb{B}(F)
\end{array}
$$

All these diagrams commute by appealing to the universal property of the Kan extensions. ◀

The above theorem allows us to turn $\mathsf{coalg}(\mathbb{B})$ into a monoidal category, with tensor product defined by $(F, \lambda) * (G, \rho) = (FG, \alpha_{F,G} \circ (\lambda * \rho))$ and with unit $\mathsf{Id} = (\mathsf{Id}, \beta)$. This monoidal structure is the same, modulo an isomorphism, as the monoidal structure of $\mathsf{DL}(B)$ given by composition, as we show in the following lemma.

▶ **Lemma 9.** *The monoidal category $(\mathsf{coalg}(\mathbb{B}), *, \mathsf{Id})$ is isomorphic to $(\mathsf{DL}(B), *, \mathsf{Id})$. Hence, $((F, \lambda), \mu, \eta)$ is a monoid in $\mathsf{DL}(B)$ if and only if $((F, \lambda^*), \mu, \eta)$ is a monoid in $\mathsf{coalg}(B)$, where λ^* is the coalgebra associated to λ by the isomorphism.*

Proof. It suffices to prove that the functor from $\mathsf{coalg}(\mathbb{B})$ to $\mathsf{DL}(B)$ in Lemma 5 is strict monoidal. Hence, we have to prove for $\lambda \colon F \Rightarrow \mathbb{B}(F)$ and $\rho \colon G \Rightarrow \mathbb{B}(G)$ that transposing $(FG, \alpha_{F,G} \circ (\lambda * \rho))$ is the same as the monoidal product of the transpose of (F, λ) and the transpose of (G, ρ) in $\mathsf{DL}(B)$. In turn, this follows by a routine calculation. ◀

5 Constructing the Companion of an Accessible Functor

We show now that the companion of an accessible functor generally exists. More concretely, we assume that $B \colon \mathcal{C} \to \mathcal{C}$ is a κ-accessible functor on a locally κ-presentable category $\mathcal{C}$. We instantiate the subcategory $\mathcal{F}$ of Section 3 to the category $[\mathcal{C}, \mathcal{C}]^\kappa$ of κ-accessible functors. For the sake of clarity, let us denote the associated category of distributive laws by $\mathsf{DL}_\kappa(B)$, and refer to the companion, the final object in $\mathsf{DL}_\kappa(B)$, as the κ-*companion*. Further, we denote the associated familiar of B by $\mathbb{B}_\kappa$, and call it the κ-*familiar*. The presentation of distributive laws over B as coalgebras for $\mathbb{B}_\kappa$ allows us to construct in Theorem 12 the κ-companion as a final $\mathbb{B}_\kappa$-coalgebra. We begin by showing that the κ-familiar $\mathbb{B}_\kappa$ exists.

▶ **Lemma 10.** *The functor $B^* \colon [\mathcal{C}, \mathcal{C}]^\kappa \to [\mathcal{C}, \mathcal{C}]^\kappa$ has a right adjoint, given by $\mathsf{Ran}_{BI}(-I)$.*

Proof. Recall the inclusion functor $I \colon \mathcal{C}_\kappa \to \mathcal{C}$ and consider the right Kan extension Ran_{BI}:

$$
[\mathcal{C}, \mathcal{C}]^\kappa \underset{\mathsf{Ran}_{BI}}{\overset{B^*}{\rightleftarrows}} [\mathcal{C}, \mathcal{C}]^\kappa \underset{}{\overset{I^*}{\rightleftarrows}} [\mathcal{C}_\kappa, \mathcal{C}]
$$

This Kan extension exists and is computed pointwise by the standard limit formula, see, e.g., [12], using that $\mathcal{C}_\kappa$ is essentially small and $\mathcal{C}$ is complete [2, Corollary 1.28]. The desired adjunction is given by the bijective correspondence below, which is natural both in F and G.

$$
\frac{\dfrac{FB \Rightarrow G}{FBI \Rightarrow GI}}{F \Rightarrow \mathsf{Ran}_{BI}GI} \ \textbf{(Kan)}
$$

The upper correspondence (natural in FB and G) comes from the fact that I^* is part of an equivalence, and the lower one (natural in F and GI) from the above right Kan extension. ◀

Using Lemma 10, we can show that the familiar $\mathbb{B}_\kappa$ exists. Recall that we defined in (1) the familiar as the composition of the right adjoint $\mathsf{Ran}_{BI}(-I)$ and B_*, thus we have $\mathbb{B}_\kappa(F) = \mathsf{Ran}_{BI}(BFI)$. Lemma 5 asserts that $\mathsf{coalg}(\mathbb{B}_\kappa) \cong \mathsf{DL}_\kappa(B)$. The problem of finding a κ-companion now reduces to finding a final $\mathbb{B}_\kappa$-coalgebra, for which we can use standard tools: any accessible functor on a locally presentable category has a final coalgebra.

▶ **Lemma 11.** *The functor $\mathbb{B}_\kappa$ is accessible.*

Proof. The functor B_* is accessible since B is, and colimits in functor categories are computed pointwise. The functor $\mathsf{Ran}_{BI}(-)$ is accessible, since it is a right adjoint on locally presentable categories, which in turn follows from the adjoint functor theorem for locally presentable categories [2, Theorem 1.66]. Since the composition of accessible functors is again accessible, we obtain that $\mathbb{B}_\kappa$ is accessible. ◀

Note that the above lemma states that $\mathbb{B}_\kappa$ is accessible, and *not* that it is κ-accessible. The adjoint functor theorem guarantees accessibility of $\mathsf{Ran}_{BI}(-)$ only for a cardinal that is potentially larger than κ.

▶ **Theorem 12.** *The functor B has a κ-companion.*

Proof. The category $[\mathcal{C}, \mathcal{C}]^\kappa$ is locally presentable, since it is equivalent to the category $[\mathcal{C}_\kappa, \mathcal{C}]$, which is locally presentable [2]. Since $\mathbb{B}_\kappa$ is accessible by Lemma 11, $\mathsf{coalg}(\mathbb{B}_\kappa)$ has a final object, see [13] or [1, Theorem 4.2.12]. This final object gives the κ-companion through the isomorphism between $\mathsf{coalg}(\mathbb{B}_\kappa)$ and $\mathsf{DL}_\kappa(B)$. ◀

▶ **Example 13.** **1.** Any complete lattice L is locally presentable, and any monotone function $b \colon L \to L$ is accessible (both for a sufficiently large cardinal). Hence, we obtain the companion for any such b, and thereby recover the corresponding result in [16].

2. The finite powerset functor $\mathcal{P}_\omega$ on Set is ω-accessible, hence it has an ω-companion. Of course, ω can be replaced here by any regular cardinal.

3. More generally, define the class of Kripke-polynomial functors (on Set) as the least class that contains $\mathcal{P}_\omega$, the constant functors K_A and exponent functors $(-)^A$ for every set A, and which is closed under finite products, non-empty coproducts and composition. It is well-known that every Kripke polynomial functor is accessible, see, e.g., [7, Lemma 4.6.8], hence any Kripke-polynomial functor has a κ-companion, for some κ.

▶ Remark. Any constant functor K_X, for X an object of $\mathcal{C}$, is ω-accessible. By Theorem 2, the κ-companion of B yields a final B-coalgebra by evaluation on an initial object of $\mathcal{C}$.

6 Second-Order Companion and Distributive Laws Up-To

The construction of the previous section can be iterated to obtain "higher-order" companions: By Lemma 11, $\mathbb{B}_\kappa$ is again an accessible functor on the locally presentable category $[\mathcal{C}, \mathcal{C}]^\kappa$. Hence, by Theorem 12, the functor $\mathbb{B}_\kappa$ itself has a companion $\mathbb{T}_\kappa$. More generally, we assume in this section that both the familiar $\mathbb{B}$ of B and the companion $\mathbb{T}$ of $\mathbb{B}$ exist. We refer in the sequel to $\mathbb{T}$ as the *second-order companion*.

Such a second-order companion turned out to be a useful tool for proving soundness of up-to techniques in the context of complete lattices [16]. Here, we use second-order companions to propose general GSOS-type specifications presented by distributive laws in Section 6.2. We first provide some background on such specifications.

In the theory of coalgebras, distributive laws $\rho\colon FB \Rightarrow BF$ are frequently used as an abstract specification format, where F represents the syntax (typically F is a polynomial functor representing an algebraic signature), B the type of behaviour and ρ the semantics (see [8] for an overview). For specific instances of B (and F), such natural transformations correspond to concrete syntactic rule formats. It is customary in this approach to start from more permissive types of distributive laws, that also correspond to more general rule formats, such as $\rho\colon FB \Rightarrow B(F + \mathsf{Id})$, or $\rho\colon FB \Rightarrow BF^\star$, where $F^\star$ is the free monad over F. An even more general type is given by the celebrated *abstract GSOS specifications*, of the form $\rho\colon F(B \times \mathsf{Id}) \Rightarrow BF^\star$, which are briefly discussed at the end of the paper.

Such natural transformations can typically be extended to distributive laws, possibly with additional structure. In particular, a natural transformation $\rho\colon FB \Rightarrow BF^\star$ corresponds uniquely to a distributive law $\rho^\sharp\colon F^\star B \Rightarrow BF^\star$ of the free monad $F^\star$ over B. This is typically proved by appealing to initiality of algebras, see, e.g., [3]. We can give an elegant proof by using the familiar as follows, where the second step uses that $\mathbb{B}(F^\star)$ is a monad (Lemma 7).

$$
\cfrac{\cfrac{\cfrac{FB \Rightarrow BF^\star}{F \Rightarrow \mathbb{B}(F^\star)} \;(\mathbf{Kan})}{F^\star \Rightarrow \mathbb{B}(F^\star) \quad \text{monad map}} \;(\mathbf{Free})}{F^\star B \Rightarrow BF^\star \quad \text{d.l. of monad over functor}} \;(\mathbf{Street})
$$

In the remainder of this section we consider a more general type of natural transformation: that of the form $\rho\colon FB \Rightarrow B\mathbb{T}(F)$. We refer to it as a *distributive law up to* $\mathbb{T}$. The main result is that $\mathbb{T}(F)$ carries a monad structure, and that every such ρ extends to a distributive law of this monad over B. The approach is at a high level similar to the one for the case $FB \Rightarrow BF^\star$ (c.f. Theorem 15), but it is significantly more involved, as $\mathbb{T}(F)$ is not a free monad F in general (note, for instance, that it depends on B). Later in this section, we show that distributive laws up to $\mathbb{T}$ properly generalise distributive laws of the form $FB \Rightarrow BF^\star$.

Throughout this section, we let $\mathcal{F}$ be a full monoidal subcategory of $([\mathcal{C}, \mathcal{C}], *, \mathsf{Id})$ and $\mathcal{S}$ be a full subcategory of $[\mathcal{F}, \mathcal{F}]$ such that

1. $\mathcal{S}$ is a monoidal subcategory of $([\mathcal{F}, \mathcal{F}], *, \mathsf{Id})$, that is, $\mathcal{S}$ is closed under composition and contains the identity functor;
2. $\mathcal{S}$ is a monoidal subcategory of $([\mathcal{F}, \mathcal{F}], \circledast, K_{\mathsf{Id}})$, that is, $\mathcal{S}$ contains the constant functor K_{Id}, and if $\mathbb{F}, \mathbb{G} \in \mathcal{S}$, then the functor $\mathbb{F} \circledast \mathbb{G}$ given by $F \mapsto \mathbb{F}(F)\mathbb{G}(F)$ is in $\mathcal{S}$;
3. the familiar $\mathbb{B}$ exists and is an element of $\mathcal{S}$;
4. the familiar $\mathbb{B}$ has a companion $\mathbb{T}$.

The category $\mathsf{DL}(B)$ is given by distributive laws over B (with functors in $\mathcal{F}$), and the category $\mathsf{DL}(\mathbb{B})$ is given by distributive laws over $\mathbb{B}$ (with functors in $\mathcal{S}$). The main instance of interest is given by the accessible functors, for which $\mathbb{B}$ and $\mathbb{T}$ exist whenever B is accessible:

▶ **Lemma 14.** *Let* $B\colon \mathcal{C} \to \mathcal{C}$ *be* κ-*accessible. Let* $\mathcal{F} = [\mathcal{C}, \mathcal{C}]^{\kappa}$ *and* $\mathcal{S} = [\mathcal{F}, \mathcal{F}]^{\lambda}$, *where* $\lambda \geq \kappa$ *is a regular cardinal such that* $\mathbb{B}_{\kappa}$ *is* λ-*accessible. These* $\mathcal{F}$ *and* $\mathcal{S}$ *meet assumptions* **1.-4.**

6.1   Second-Order Companion and Monads

An important feature of the companion is that it is a monad, which allows us to collapse multiple uses of the companion. This result lifts to the second-order companion $\mathbb{T}$ in two ways. First, by Theorem 3 the category $\mathsf{DL}(\mathbb{B})$ of distributive laws over $\mathbb{B}$ inherits a monoidal structure from $(\mathcal{S}, *, \mathsf{Id})$. This gives rise to a monad structure $(\mathbb{T}, \mu, \eta)$ on $\mathbb{T}$, see Corollary 4. We denote the associated distributive law of that monad over $\mathbb{B}$ by $\pi\colon \mathbb{T}\mathbb{B} \Rightarrow \mathbb{B}\mathbb{T}$. Second, more interestingly, $\mathbb{T}$ has a monoid structure in $(\mathcal{S}, \circledast, K_{\mathsf{Id}})$. This is proved in Theorem 15 by using that $(\mathcal{S}, \circledast, K_{\mathsf{Id}})$ lifts to $\mathsf{DL}(\mathbb{B})$. The monoid structure neatly encapsulates the fact that $(\mathbb{T}(F), \dot{\mu}_F, \dot{\eta}_F)$ is a monad, for every functor $F\colon \mathcal{C} \to \mathcal{C}$ in $\mathcal{F}$ (Corollary 16).

▶ **Theorem 15.** *The monoidal structure of* $(\mathcal{S}, \circledast, K_{\mathsf{Id}})$ *lifts to* $\mathsf{DL}(\mathbb{B})$. *This yields a monoid* $(\mathbb{T}, \dot{\mu}\colon \mathbb{T} \circledast \mathbb{T} \to \mathbb{T}, \dot{\eta}\colon K_{\mathsf{Id}} \to \mathbb{T})$ *on the second-order companion.*

Proof. We use that $\mathbb{B}$ is lax monoidal (Theorem 8) with mediators $\beta\colon \mathsf{Id} \Rightarrow \mathbb{B}(\mathsf{Id})$ and $\alpha_{F,G}\colon \mathbb{B}(F)\mathbb{B}(G) \Rightarrow \mathbb{B}(FG)$ (natural in $F, G \in \mathcal{F}$). Now, given $(\mathbb{F}, \lambda)$ and $(\mathbb{G}, \rho)$ in $\mathcal{S}$, for the tensor $(\mathbb{F}, \lambda) \circledast (\mathbb{G}, \rho)$ we have to provide a distributive law of type $(\mathbb{F} \circledast \mathbb{G})\mathbb{B} \Rightarrow \mathbb{B}(\mathbb{F} \circledast \mathbb{G})$, which we define on a component F as:

$$(\mathbb{F} \circledast \mathbb{G})\mathbb{B}(F) =\!\!=\!\!= \mathbb{F}\mathbb{B}(F)\mathbb{G}\mathbb{B}(F) \xrightarrow{\lambda_F * \rho_F} \mathbb{B}\mathbb{F}(F)\mathbb{B}\mathbb{G}(F) \xrightarrow{\alpha_{\mathbb{F}(F),\mathbb{G}(F)}} \mathbb{B}(\mathbb{F}(F)\mathbb{G}(F)) \ .$$

The distributive law for the unit K_{Id} is defined by:

$$K_{\mathsf{Id}}\mathbb{B}(F) =\!\!=\!\!= \mathsf{Id} \xrightarrow{\beta} \mathbb{B}(\mathsf{Id}) =\!\!=\!\!= \mathbb{B}K_{\mathsf{Id}}(F)$$

Naturality and the axioms of monoidal categories are routine calculations.   ◀

Given a functor F in $\mathcal{F}$, we get a strict monoidal functor $\mathsf{ev}_F\colon \mathcal{S} \to \mathcal{F}$ by letting $\mathsf{ev}_F(\mathbb{G}) = \mathbb{G}(F)$. Since monoidal functors preserve monoids, $\mathbb{T}(F)$ is a monoid in $\mathcal{F}$, i.e., a monad.

▶ **Corollary 16.** *For every functor* F *in* $\mathcal{F}$, $(\mathbb{T}(F), \dot{\mu}_F, \dot{\eta}_F)$ *is a monad.*

We now prove that any distributive law $\lambda\colon FB \Rightarrow BF$ gives rise to a distributive law of the monad $(\mathbb{T}(F), \dot{\mu}_F, \dot{\eta}_F)$ over B (Corollary 18 below). To do so, we first extend ev_F to a strict monoidal functor $\mathsf{ev}_{(F,\lambda)}\colon \mathsf{DL}(\mathbb{B}) \to \mathsf{DL}(B)$ (Theorem 17). Let $\rho\colon \mathbb{G}\mathbb{B} \Rightarrow \mathbb{B}\mathbb{G}$ be a distributive law. We obtain a lifting $\overline{\mathbb{G}}\colon \mathsf{coalg}(\mathbb{B}) \to \mathsf{coalg}(\mathbb{B})$ by $\overline{\mathbb{G}}_\rho(G, \gamma) = \rho_G \circ \mathbb{T}(\gamma)$. From this, we construct a distributive law over B by applying the following transformation to λ:

$$\frac{\dfrac{\dfrac{FB \Rightarrow BF}{F \Rightarrow \mathbb{B}(F)}\ \textbf{(Kan)}}{\mathbb{G}(F) \Rightarrow \mathbb{B}\mathbb{G}(F)}\ \textbf{(Lifting } \overline{\mathbb{G}}\textbf{)}}{\mathbb{G}(F)B \Rightarrow B\mathbb{G}(F)}\ \textbf{(Kan)}$$

This construction gives rise to a functor $\mathsf{ev}_{(F,\lambda)}\colon \mathsf{DL}(\mathbb{B}) \to \mathsf{DL}(B)$.

▶ **Theorem 17.** *The functor* $\mathsf{ev}_{(F,\lambda)}$ *is strict monoidal, from* $(\mathsf{DL}(\mathbb{B}), \circledast, K_{\mathsf{Id}})$ *to* $(\mathsf{DL}(B), *, \mathsf{Id})$.

Proof. The functor $\mathsf{ev}_{(F,\lambda)}$ decomposes as a functor $\mathsf{DL}(\mathbb{B}) \to \mathsf{coalg}(\mathbb{B})$ followed by the (monoidal) isomorphism $\mathsf{coalg}(\mathbb{B}) \cong \mathsf{DL}(B)$ from Lemma 9. It is easy to check that the functor $\mathsf{DL}(\mathbb{B}) \to \mathsf{coalg}(\mathbb{B})$ is strict monoidal as well. ◄

By applying this to the monoid on the second-order companion from Theorem 15, we obtain:

▶ **Corollary 18.** *For an object* (F, λ) *in* $\mathsf{DL}(B)$*,* $\mathsf{ev}_{(F,\lambda)}(\mathbb{T}, \pi)$ *is a distributive law of the monad* $(\mathbb{T}(F), \dot{\mu}_F, \dot{\eta}_F)$ *over* B.

6.2 Distributive Laws up to $\mathbb{T}$

Now we show how to extend distributive laws of the form $FB \Rightarrow B\mathbb{T}(F)$ to distributive laws of the *monad* $(\mathbb{T}(F), \dot{\mu}_F, \dot{\eta}_F)$ over B. The idea is to first transpose such a distributive law to $\rho \colon F \Rightarrow \mathbb{B}\mathbb{T}(F)$, and then apply, by using the distributive law $\pi \colon \mathbb{T}\mathbb{B} \Rightarrow \mathbb{B}\mathbb{T}$ of the second-order companion, what is sometimes called the generalised powerset construction [20]. Lemma 20 shows how this extension interacts with the monad $(\mathbb{T}(F), \dot{\mu}_F, \dot{\eta}_F)$. For its proof, we need the following lemma, which relates the two monoid structures on $\mathbb{T}$.
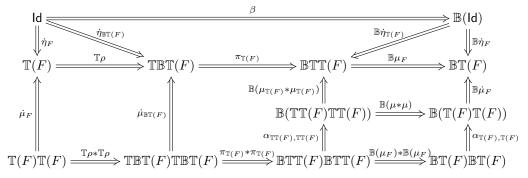
▶ **Lemma 19.** *The following diagrams commute.*

$$
\begin{array}{ccccc}
(\mathbb{T}\mathbb{T}) \circledast (\mathbb{T}\mathbb{T}) & = & (\mathbb{T} \circledast \mathbb{T})\mathbb{T} & \xrightarrow{\dot{\mu}\mathbb{T}} & \mathbb{T}\mathbb{T} \\
\mu \circledast \mu \downarrow & & & & \downarrow \mu \\
\mathbb{T} \circledast \mathbb{T} & & \xrightarrow{\dot{\mu}} & & \mathbb{T}
\end{array}
\qquad
\begin{array}{ccc}
K_{\mathsf{Id}}\mathbb{T} & = & K_{\mathsf{Id}} \\
\dot{\eta}\mathbb{T} \downarrow & & \downarrow \dot{\eta} \\
\mathbb{T}\mathbb{T} & \xrightarrow{\mu} & \mathbb{T}
\end{array}
$$

Proof. Use finality of $\mathbb{T}$ in $\mathsf{DL}(\mathbb{B})$ to prove that the axioms hold in $\mathsf{DL}(\mathbb{B})$. This follows once we show distributivity (on the right) of composition over $\circledast$ in $\mathsf{DL}(\mathbb{B})$, i.e., the equalities in the diagrams should hold in $\mathsf{DL}(\mathbb{B})$ as well. This is a straightforward exercise. ◄

▶ **Lemma 20.** *Extend* $\rho \colon F \Rightarrow \mathbb{B}\mathbb{T}(F)$ *to the natural transformation* $\rho^{\sharp} \colon \mathbb{T}(F) \Rightarrow \mathbb{B}\mathbb{T}(F)$, *which is given by the transformation in the top line in the following diagram.*

$$
\begin{array}{ccccccc}
\mathbb{T}(F) & \xrightarrow{\mathbb{T}\rho} & \mathbb{T}\mathbb{B}\mathbb{T}(F) & \xrightarrow{\pi_{\mathbb{T}(F)}} & \mathbb{B}\mathbb{T}\mathbb{T}(F) & \xrightarrow{\mathbb{B}(\mu_F)} & \mathbb{B}\mathbb{T}(F) \\
\eta_F \Uparrow & & & & & & \\
F & & & \xrightarrow{\rho} & & &
\end{array} \; .
$$

Then $((\mathbb{T}(F), \rho^{\sharp}), \dot{\mu}_F, \dot{\eta}_F)$ *is a monoid in* $\mathsf{coalg}(\mathbb{B})$.

Proof. We only need to prove that $\dot{\eta}_F$ and $\dot{\mu}_F$ are morphisms of the correct type in $\mathsf{coalg}(\mathbb{B})$. This is given by commutativity of the upper and lower parts of the following diagram.

The diagram commutes clockwise, starting from the triangle on the top left, by naturality of $\dot{\eta}$, definition of $\dot{\eta}$, Lemma 19 twice, naturality of α, definition of $\dot{\mu}$ and naturality of $\dot{\mu}$. ◄

The following theorem states the main extension result.

▶ **Theorem 21.** *Let* $\rho\colon FB \Rightarrow B\mathbb{T}(F)$ *with* F *in* $\mathcal{F}$. *There exists a distributive law* $\bar{\rho}\colon \mathbb{T}(F)B \Rightarrow B\mathbb{T}(F)$ *of the monad* $(\mathbb{T}(F), \dot{\mu}_F, \dot{\eta}_F)$ *over* B *such that* $\bar{\rho} \circ \eta_F B = \rho$.

Proof. We have the following inference.

$$
\cfrac{\cfrac{\cfrac{FB \Rightarrow B\mathbb{T}(F)}{F \Rightarrow \mathbb{BT}(F)}\ (\textbf{Kan})}{\mathbb{T}(F) \Rightarrow \mathbb{BT}(F) \quad \text{monoid in } \mathsf{coalg}(\mathbb{B}) \text{ with structure } (\dot{\mu}_F, \dot{\eta}_F)}\ (\textbf{Lemma 20})}{\mathbb{T}(F)B \Rightarrow B\mathbb{T}(F) \quad \text{monoid in } \mathsf{DL}(B) \text{ with structure } (\dot{\mu}_F, \dot{\eta}_F)}\ (\textbf{Lemma 9})
$$

The conclusion corresponds to a distributive law of the monad $(\mathbb{T}(F), \dot{\mu}_F, \dot{\eta}_F)$ over B (Theorem 3). The resulting natural transformation $\bar{\rho}$ satisfies $\bar{\rho} \circ \eta_F B = \rho$. ◀

6.3 A Toolkit for Distributive Laws up to $\mathbb{T}$

As explained before, distributive laws of the shape $FB \Rightarrow BF$ give rise to F-algebras on the final B-coalgebra. A problem with such specifications is that they must define all the involved operations. For instance, the definition of the convolution product on streams requires the simultaneous definition of point-wise addition [19]. Theorem 21 above makes it possible to work instead with distributive laws up to $\mathbb{T}$, of the shape $FB \Rightarrow B\mathbb{T}(F)$. This is convenient in practice as it makes it possible to define distributive laws, and thus operations on the final coalgebra, in a modular way. This is demonstrated in the example on streams below.

Given a functor F, which describes the signature of the operations to be defined, a specification can often be given by finding another functor G and a natural transformation $FB \Rightarrow BG$ that directly describe the behaviour of the operations. To turn such a natural transformation into a distributive law up to $\mathbb{T}$, an explicit or extensional knowledge of $\mathbb{T}$ is usually not required; it suffices to find a natural transformation $G \Rightarrow \mathbb{T}(F)$. Below we provide a small toolkit to construct such natural transformations in a modular way. This toolkit should be seen as a first stepping stone for a typed calculus of specification up to $\mathbb{T}$, which allows the modular construction of coalgebraic specifications.

Recall that T denotes the (first-order) companion of B; we have the following natural transformations:

- $t\colon T \Rightarrow \mathbb{T}(F)$ — If $\mathcal{F}$ has an initial object $\mathbf{0}$ and contains the constant functors K_X for all X. In that case $T = \mathbb{T}(\mathbf{0})$ (Theorem 2), and $t = \mathbb{T}(!_F)\colon \mathbb{T}(\mathbf{0}) \Rightarrow \mathbb{T}(F)$.
- $b\colon B \Rightarrow \mathbb{T}(F)$ — Obtained as the composition of the unique distributive law morphism $B \Rightarrow T$ (from $\mathsf{id}\colon BB \Rightarrow BB$ to the companion T) and t.
- $\eta_F\colon F \Rightarrow \mathbb{T}(F)$ — Unit of the monad on $\mathbb{T}$.
- $\dot{\eta}_F\colon \mathsf{Id} \Rightarrow \mathbb{T}(F)$ — Unit of the monad on $\mathbb{T}(F)$.
- $\dot{\mu}_F\colon \mathbb{T}(F)\mathbb{T}(F) \Rightarrow \mathbb{T}(F)$ — Multiplication of the monad on $\mathbb{T}(F)$.

The monad multiplication $\dot{\mu}_F$ allows us to combine $G \Rightarrow \mathbb{T}(F)$ and $H \Rightarrow \mathbb{T}(F)$ by composition: $GH \Rightarrow \mathbb{T}(F)\mathbb{T}(F) \Rightarrow \mathbb{T}(F)$. Moreover, the monad on $\mathbb{T}(F)$ allows us to extend any $G \Rightarrow \mathbb{T}(F)$ to a monad map $G^\star \Rightarrow \mathbb{T}(F)$, if the free monad G^* of G exists. In particular, we obtain:

- $s\colon F^\star \Rightarrow \mathbb{T}(F)$ — the unique monad map such that $s \circ \iota = \eta_F$, where $\iota\colon F \Rightarrow F^\star$ is the canonical map from F to the associated free monad $F^\star$ and η is the unit of the monad $\mathbb{T}$.

The natural transformation t allows us to reuse distributive laws up to $\mathbb{T}$ in a modular fashion: Any distributive law $\lambda\colon GB \Rightarrow BG$ (or even $\lambda\colon GB \Rightarrow B\mathbb{T}(G)$) gives rise to a natural transformation $\lambda^\dagger\colon G \Rightarrow T$, which can be composed with t to get $t \circ \lambda^\dagger\colon G \Rightarrow \mathbb{T}(F)$.

We can exploit the above toolkit to show that the following types of natural transformations are all (encodable as) instances of distributive laws up to $\mathbb{T}$.

- $FB \Rightarrow BF$ — By using $\eta_F \colon F \Rightarrow \mathbb{T}(F)$.
- $FB \Rightarrow B(F + \mathsf{Id})$ — By using $[\eta_F, \dot{\eta}_F] \colon F + \mathsf{Id} \Rightarrow \mathbb{T}(F)$.
- $FB \Rightarrow BF^\star$ — By using $s \colon F^\star \Rightarrow \mathbb{T}(F)$.
- $FB \Rightarrow B(F + B)^\star$ — By using the unique monad map $(F + B)^\star \Rightarrow \mathbb{T}(F)$ that extends $[\eta_F, b] \colon F + B \Rightarrow \mathbb{T}(F)$.
- $FB \Rightarrow B(F+B+T)^\star$ — Similar to the previous case but with $[\dot{\eta}_F, b, t] \colon F+B+T \Rightarrow \mathbb{T}(F)$.

Each of the above natural transformations gives rise to a $\rho \colon FB \Rightarrow B\mathbb{T}(F)$ and hence, by Theorem 21, extends to a distributive law $\overline{\rho} \colon \mathbb{T}(F)B \Rightarrow B\mathbb{T}(F)$.

In the beginning of the section, we recalled that any $\rho \colon FB \Rightarrow BF^\star$ extends to a distributive law $\rho^\sharp \colon F^\star B \Rightarrow BF^\star$. We show in the following theorem that this is generalised by the extension of distributive laws up to $\mathbb{T}$ from Theorem 21. To this end, we establish a morphism of distributive laws. Intuitively, this means that the results of the two extensions behave the same, as far as $F^\star$ is concerned.

▶ **Theorem 22.** *Extend* $\rho \colon FB \Rightarrow BF^\star$ *to* $\rho^\sharp \colon F^\star B \Rightarrow BF^\star$ *and* $\overline{(Bs \circ \rho)} \colon \mathbb{T}(F)B \Rightarrow B\mathbb{T}(F)$. *Then* s *is a morphism of distributive laws:* $Bs \circ \rho^\sharp = \overline{(Bs \circ \rho)} \circ sB$.

As a consequence, the F-algebra obtained on the final B-coalgebra from $\rho^\sharp$ coincides with that obtained from $\overline{(Bs \circ \rho)}$.[3]

Example on streams

Let $B \colon \mathsf{Set} \to \mathsf{Set}$ be $BX = \mathbb{R} \times X$, where $\mathbb{R}$ is the set of real numbers. The (carrier of the) final B-coalgebra is given by the set of streams $\mathbb{R}^\omega$. The head and tail of a stream σ are denoted by $\sigma(0)$ and σ' respectively. The following equations define binary operations $+$, $\times$ a unary operation $(-)^*$ and constants $[r]$ for each $r \in \mathbb{R}$ on streams [19]:

$$
\begin{aligned}
[r](0) &= r & [r]' &= [0] \\
(\sigma + \tau)(0) &= \sigma(0) + \tau(0) & (\sigma + \tau)' &= \sigma' + \tau' \\
(\sigma \times \tau)(0) &= \sigma(0) \times \tau(0) & (\sigma \times \tau)' &= \sigma' \times \tau + [\sigma(0)] \times \tau' \\
(\sigma^*)(0) &= 1 & (\sigma^*)' &= \sigma' \times \sigma^*
\end{aligned}
$$

We model these using distributive laws up to $\mathbb{T}$, starting with $(-)^*$. Let $SX = X$ and $MX = X \times X$ be the functors that represent the arity of $(-)^*$ and $\times$. Then we define:

$$
\rho^* \colon SB \Rightarrow BM(\mathsf{Id} + SB)
$$
$$
\rho_X^*(r, x) = (1, (x, (r, x)))
$$

Alternatively, one can use a natural transformation of the form $SB \Rightarrow B(M + S + B)^*$, but the above type reflects more directly the concrete definition. Notice that the functor B on the right hand side signals the occurrence of σ on the right-hand side of the concrete definition of $(\sigma^*)'$. Using the toolkit, which we presented before, it is easy to give a natural transformation $\kappa \colon M(\mathsf{Id} + SB) \Rightarrow M\mathbb{T}(S)$, so we obtain

$$
B\kappa \circ \rho^* \colon SB \Rightarrow BM\mathbb{T}(S) \,.
$$

---

[3] Assuming a final coalgebra (Z, z), every distributive law $\gamma \colon GB \Rightarrow BG$ gives rise to a coalgebra $(GZ, \gamma_Z \circ Gz)$ and thus to a G-algebra by finality of (Z, z); when G is $F^\star$, resp. $\mathbb{T}(F)$, this G-algebra can be turned into an F-algebra by precomposing with ι_Z, resp. $\dot{\eta}_F$.

This natural transformation extends to a distributive law up to $\mathbb{T}$ once we provide semantics of the product, in the form of a natural transformation $M \Rightarrow \mathbb{T}(S)$.

Again, we can give a precise type for the semantics of the product.

$$\rho^\times \colon MB \Rightarrow BP(M(\mathsf{Id} + B) + M(K_\mathbb{R} + \mathsf{Id}))$$
$$\rho_X^\times((r, x), (s, y)) = (r \times s, ((x, (s, y)), (r, y)))$$

Here, $P(X) = X \times X$ models the arity of the sum operator and $K_\mathbb{R}$ that of the $[r]$'s for $r \in \mathbb{R}$. Also this semantics could also be presented as a natural transformation of the form $\rho^\times \colon MB \Rightarrow B(P + M + K_\mathbb{R})^\star$. Either way, we still need natural transformations $K_\mathbb{R} \Rightarrow \mathbb{T}(M)$ and $P \Rightarrow \mathbb{T}(M)$ to complete the specification.

For the sum and constants, we define $\rho^+ \colon PB \Rightarrow BP$ and $\rho^{[-]} \colon K_\mathbb{R} \Rightarrow BK_\mathbb{R}$ by

$$\rho_X^+((r, x), (s, y)) = (r + s, x + y) \qquad \rho^{[-]}(r) = (r, 0)$$

Since these are plain distributive laws, we directly obtain natural transformations $P \Rightarrow T$ and $K_\mathbb{R} \Rightarrow T$ that, by composing with t from the toolkit, extend to natural transformations $P \Rightarrow \mathbb{T}(M)$ and $K_\mathbb{R} \Rightarrow \mathbb{T}(M)$. The toolkit allows us now to combine them into a natural transformation $\delta \colon P(M(\mathsf{Id} + B) + M(K_\mathbb{R} + \mathsf{Id})) \Rightarrow \mathbb{T}(M)$, from which we obtain

$$B\delta \circ \rho^\times \colon MB \Rightarrow B\mathbb{T}(M).$$

Since this is a distributive law up to $\mathbb{T}$ we obtain a natural transformation $M \Rightarrow T$ from Theorem 21 and finality of the companion T. Composing with t gives a natural transformation of the form $M \Rightarrow \mathbb{T}(S)$, which we use to complete ρ^* to a distributive law up to $\mathbb{T}$ of the form $SB \Rightarrow B\mathbb{T}(S)$. Finally, again by Theorem 21, we get a distributive law of $\mathbb{T}(S)$ over B.

Abstract GSOS

An *abstract GSOS specification* is a natural transformation of the form $\rho \colon F(B \times \mathsf{Id}) \Rightarrow BF^\star$. It is not directly clear how to encode abstract GSOS as a distributive law up to $\mathbb{T}$, because of the product with the identity functor in the domain. This problem is reflected in the previous concrete example by the problem of modelling, for instance, the occurrence of σ on the right-hand side of the equation for $(\sigma^\star)'$ as a distributive law. There, we solved the problem by using that distributive laws up to $\mathbb{T}$ permit the use of B on the right-hand side, because of the natural transformation $b \colon B \Rightarrow T$.

The following is an attempt to use this idea to encode an arbitrary abstract GSOS ρ as a distributive law up to $\mathbb{T}$:

$$FB \xrightarrow{F\langle B\eta, b\rangle} F(B \times \mathsf{Id})T \xrightarrow{\rho T} BF^\star T \xrightarrow{BsT} B\mathbb{T}(F)T \xrightarrow{B\mathbb{T}(F)t} B\mathbb{T}(F)\mathbb{T}(F) \xrightarrow{B\mu_F} B\mathbb{T}(F)$$

We conjecture that this distributive law up to $\mathbb{T}$ encodes the behaviour of ρ, in the sense that they define the same F-algebras on the final B-coalgebra. Such a result has been shown for stream systems ($BX = A \times X$ on Set) through an explicit construction of a distributive law in [5], and more abstractly based on the companion for polynomial functors in [17]. The current approach would generalise it to accessible functors. However, we do not currently know if the above construction is indeed correct, and leave this as an open problem.

────── **References** ──────

1 J. Adamek, S. Milius, and L. Moss. Initial algebras and terminal coalgebras: a survey. Preliminary version. Available at `https://www.tu-braunschweig.de/Medien-DB/iti/survey_full.pdf`, 2010.

2 J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories*. Cambridge Tracts in Mathematics. Cambridge University Press, 1994.

3 F. Bartels. *On generalised coinduction and probabilistic specification formats*. PhD thesis, CWI, Amsterdam, April 2004.

4 H. Basold, D. Pous, and J. Rot. Monoidal company for accessible functors (full version, with proofs). Available at `https://hal.archives-ouvertes.fr/hal-01529340/`.

5 F. Bonchi, M. Lee, and J. Rot. Bisimilarity of open terms in stream GSOS. In *FSEN*, 2017. To appear.

6 F. Bonchi, D. Petrisan, D. Pous, and J. Rot. Coinduction up-to in a fibrational setting. In *CSL-LICS*, pages 20:1–20:9. ACM, 2014.

7 B. Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observation*, volume 59 of *Cambridge Tracts in T.C.S.* Cambridge University Press, 2016.

8 B. Klin. Bialgebras for structural operational semantics: An introduction. *Theoretical Computer Science*, 412(38):5043–5069, 2011.

9 B. Klin and B. Nachyla. Presenting morphisms of distributive laws. In *CALCO*, volume 35 of *LIPIcs*, pages 190–204. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015.

10 B. Knaster. Un théorème sur les fonctions d'ensembles. *Annales de la Société Polonaise de Mathématiques*, 6:133–134, 1928.

11 M. Lenisa, J. Power, and H. Watanabe. Distributivity for endofunctors, pointed and co-pointed endofunctors, monads and comonads. *Electronic Notes in Theoretical Computer Science*, 33:230–260, 2000.

12 S. MacLane. *Categories for the working mathematician*. Springer, 1998.

13 M. Makkai and R. Paré. *Accessible Categories: The Foundations of Categorical Model Theory*, volume 104 of *Contemporary mathematics - American Mathematical Society*. American Mathematical Society, 1989.

14 J. Parrow and T. Weber. The largest respectful function. *Logical Methods in Computer Science*, 12(2), 2016.

15 D. Pous. Complete lattices and up-to techniques. In *APLAS*, volume 4807 of *LNCS*, pages 351–366. Springer, 2007.

16 D. Pous. Coinduction all the way up. In *LICS*, pages 307–316. ACM, 2016.

17 D. Pous and J. Rot. Companions, codensity and causality. In *FoSSaCS*, volume 10203 of *LNCS*, pages 106–123, 2017.

18 J. Power and H. Watanabe. Combining a monad and a comonad. *Theoretical Computer Science*, 280(1-2):137–162, 2002.

19 J. J. M. M. Rutten. A coinductive calculus of streams. *Mathematical Structures in Computer Science*, 15(1):93–147, 2005.

20 A. Silva, F. Bonchi, M. M. Bonsangue, and J. J. M. M. Rutten. Generalizing determinization from automata to coalgebras. *Logical Methods in Computer Science*, 9(1), 2013.

21 R. Street. The formal theory of monads. *J. of Pure and Applied Algebra*, 2(2):149–168, 1972.

22 A. Tarski. A Lattice-Theoretical Fixpoint Theorem and its Applications. *Pacific Journal of Mathematics*, 5(2):285–309, June 1955.

23 D. Turi and G. D. Plotkin. Towards a mathematical operational semantics. In *LICS*, pages 280–291. IEEE, 1997.

24 H. Watanabe. Well-behaved translations between structural operational semantics. *Electronic Notes in Theoretical Computer Science*, 65(1):337–357, 2002.

A   Notation

Symbol	Meaning
$[\mathcal{C}, \mathcal{D}]$	Category of functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them
$B, F, G, H, \ldots$	Functors in $[\mathcal{C}, \mathcal{D}]$
F^*	Pre-composition with F
F_*	Post-composition with F
$([\mathcal{C}, \mathcal{C}], *, \mathsf{Id})$	Monoidal structure given by functor composition and horizontal composition of natural transformations. If we need to compose the tensor $*$ with another functor, then we write this as $(\_*\_) \circ F$
$\mathcal{F}$	Full, monoidal subcategory of $([\mathcal{C}, \mathcal{C}], *, \mathsf{Id})$, that is, $\mathcal{F}$ is closed under composition and $\mathsf{Id} \in \mathcal{F}$ – $\mathcal{F}$ stands for first-order
$\mathbb{F}, \mathbb{G}, \mathbb{H}, \ldots$	Second-order functors in $[\mathcal{F}, \mathcal{F}]$
$\mathbb{B}$	The familiar of B
$\mathbb{T}$	The second-order companion of B, which is given as the companion of $\mathbb{B}$
$([\mathcal{F}, \mathcal{F}], \circledast, K_{\mathsf{Id}})$	Monoidal structure given by point-wise composition
$\mathcal{S}$	Full, monoidal subcategory of $([\mathcal{F}, \mathcal{F}], \circledast, K_{\mathsf{Id}})$ – $\mathcal{S}$ stands for second-order
$\mathsf{DL}_{\mathcal{F}}(B)$	Category of distributive laws, in which the possible functors are restricted to $\mathcal{F}$
$\mathsf{DL}(B)$ and $\mathsf{DL}_{\kappa}(B)$	Special cases of $\mathsf{DL}_{\mathcal{F}}(B)$ with $\mathcal{F} = [\mathcal{C}, \mathcal{C}]$ or $\mathcal{F} = [\mathcal{C}, \mathcal{C}]^{\kappa}$
(T^B, τ^B) or (T, τ)	Companion of B
$(\mathbb{T}, \pi)$	Companion of $\mathbb{B}$
$\lambda \colon FB \Rightarrow BG$	(asymmetric) Distributive laws
$\rho \colon FB \Rightarrow B\mathbb{T}(F)$	Distributive laws up-to
$\gamma \colon F \Rightarrow \mathbb{B}(F)$	Coalgebra for $\mathbb{B}$
α, β	Mediators for lax monoidal functors
F^*	Free monad over F
$\mathsf{ev}_F \colon \mathcal{S} \to \mathcal{F}$	Evaluation of second-order functors on F (Section 6.1)
$(\mathbb{T}, \dot{\mu}, \dot{\eta})$ and $(\mathbb{T}(F), \dot{\mu}_F, \dot{\eta}_F)$	Monoid resp. point-wise monad structure on $\mathbb{T}$ (Theorem 15 and Corollary 16)

B   Omitted Proofs

Proof of Theorem 2. Once we prove the adjunction $K \dashv \mathsf{ev}_0$, it immediately follows that $\mathsf{ev}_0(T, \tau)$ is final, since the companion is a final object in $\mathsf{DL}(B)$, and is hence preserved by the right adjoint ev_0.

First of all, notice that ev_0 is indeed a functor: if $\delta \colon (F, \lambda) \Rightarrow (G, \rho)$ is a morphism in $\mathsf{DL}(B)$, then $\mathsf{ev}_0(\delta) = \delta_0$ is a coalgebra morphism:

$$
\begin{array}{ccccc}
F0 & \xrightarrow{F!_{B0}} & FB0 & \xrightarrow{\lambda_0} & BF0 \\
\downarrow{\delta_0} & & \downarrow{\delta_{B0}} & & \downarrow{B\delta_0} \\
G0 & \xrightarrow[G!_{B0}]{} & GB0 & \xrightarrow[\rho_0]{} & BG0
\end{array}
$$

The left square commutes by naturality, the right since δ is a morphism. Next, we prove that for any B-coalgebra (X, f) there is a universal arrow $\eta_f \colon (X, f) \to \mathsf{ev}_0(K(X, f))$, i.e., such that for each (F, λ) and coalgebra morphism $h \colon (X, f) \to \mathsf{ev}_0(F, \lambda)$ there exists a unique

$h^\sharp \colon K(X, f) \to (F, \lambda)$ such that $\mathsf{ev}_0(h^\sharp) \circ \eta_f = h$.

$$
\begin{array}{ccc}
(X, f) \xrightarrow{\ \eta_f\ } \mathsf{ev}_0(K(X, f)) & \quad & K(X, f) \\[2ex]
\quad {}_h\searrow \quad \downarrow {}^{ev_0(h^\sharp)} & \quad & \vline\ {}^{h^\sharp} \\[2ex]
\mathsf{ev}_0(F, \lambda) & \quad & (F, \lambda)
\end{array}
$$

Note that $\mathsf{ev}_0 \circ K = \mathsf{Id}$; we define (the natural transformation) η by $\eta = \mathsf{id}$. We define $h^\sharp$ on a component Y as $h_Y^\sharp = F!_Y \circ h \colon X \to FY$. This is natural, since for any $k \colon Y \to Z$, the following commutes by uniqueness of morphisms from the initial object.

$$
\begin{array}{ccc}
X \xrightarrow{\ h\ } F0 \xrightarrow{\ F!_Y\ } FY \\[2ex]
{}_h\searrow \quad \downarrow {}^{F\mathsf{id}} \quad \downarrow {}^{Fk} \\[2ex]
\quad F0 \xrightarrow[\ F!_Z\ ]{} FZ
\end{array}
$$

Moreover, $h^\sharp$ is a morphism in $\mathsf{DL}(B)$, which follows from commutativity of the following diagram.

$$
\begin{array}{ccc}
X \xrightarrow{\ h\ } B0 \xrightarrow{\ B!_{FY}\ } BFY \\[2ex]
{}_f\downarrow \quad {}^{B!_{F0}}\downarrow \quad \nearrow {}_{BF!_Y} \quad \downarrow {}^{\lambda_Y} \\[2ex]
\quad BF0 \\[2ex]
\quad \downarrow {}^{\lambda_0} \\[2ex]
FX \xrightarrow[\ Fh\ ]{} FB0 \xrightarrow[\ FB!_Y\ ]{} FBY
\end{array}
$$

The left rectangle commutes by the assumption h is a coalgebra morphism, the triangle by uniqueness of arrows from the initial object, and the shape below the triangle by naturality.

We have $\mathsf{ev}_0(h^\sharp) = h_0^\sharp = F!_0 \circ h = h \colon X \to F0$, hence $\mathsf{ev}_0(h^\sharp) \circ \eta_f = h$. Finally, if $k, l \colon K(X, f) \to (F, \lambda)$ are morphisms such that $\mathsf{ev}_0(k) \circ \eta_f = h = \mathsf{ev}_0(l) \circ \eta_f$ then in particular k, l are natural transformations $X \Rightarrow T$ (the domain is the constant-to-X functor), such that $k_0 = l_0$, since $k_0 = \mathsf{ev}_0(k) = \mathsf{ev}_0(k) \circ \eta_f = h = \mathsf{ev}_0(l) \circ \eta_f = \mathsf{ev}_0(l) = l_0$. This implies that k and l agree on any component Y, by commutativity of the following diagram:

$$
\begin{array}{ccc}
X \xrightarrow{\ \ k_Y\ \ } FY \\[2ex]
{}_{l_Y}\downarrow \quad {}^{l_0 = k_0}\searrow \quad \uparrow {}^{F!_Y} \\[2ex]
FY \xleftarrow[\ F!_Y\ ]{} F0
\end{array}
$$

where the two triangles commute by naturality of k and l respectively. ◀
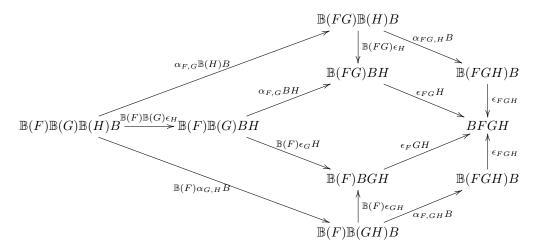
Proof of Lemma 9. It suffices to prove that the functor from $\mathsf{coalg}(\mathbb{B})$ to $\mathsf{DL}(B)$ in Lemma 5 is strict monoidal. Hence, we have to prove for $\lambda \colon F \Rightarrow \mathbb{B}(F)$ and $\rho \colon G \Rightarrow \mathbb{B}(G)$ that transposing $(FG, \alpha_{F,G} \circ (\lambda * \rho))$ is the same as the monoidal product of the transpose of (F, λ) and the transpose of (G, ρ) in $\mathsf{DL}(B)$. Thus, we have to show that the two morphisms on the outside of the following diagram are equal, which follows from commutativity of the diagram. The top-left triangle commutes by definition of $\lambda * \rho$, the upper rectangle by naturality of λ and the lower rectangle by definition of α. ◀

Proof of Lemma 14. By Lemma 11, $\mathbb{B}_\kappa$ exists and is indeed λ-accessible for some $\lambda \geq \kappa$. It therefore has a companion $\mathbb{T}$ by Theorem 12. For $\mathbb{F}, \mathbb{G}$ in $\mathcal{S}$, the functor $\mathbb{F} \circledast \mathbb{G} \colon \mathcal{F} \to \mathcal{F}$ is well-defined since accessible functors are closed under composition, and the constant functors are always accessible. It remains to show that $\mathbb{F} \circledast \mathbb{G}$ is λ-accessible. For a λ-filtered diagram $d \colon I \to \mathcal{F}$ we have:

$$
\begin{aligned}
&(\mathbb{F}(\mathrm{colim}_i d(i)))(\mathbb{G}(\mathrm{colim}_j d(j))) \\
&= (\mathrm{colim}_i \mathbb{F}(d(i)))(\mathrm{colim}_j \mathbb{G}(d(j))) && (\mathbb{F}, \mathbb{G} \text{ accessible}) \\
&= \mathrm{colim}_i (\mathbb{F}(d(i)) \mathrm{colim}_j (\mathbb{G}(d(j)))) && (\text{colimits computed pointwise}) \\
&= \mathrm{colim}_i (\mathrm{colim}_j \mathbb{F}(d(i)) \mathbb{G}(d(j))) && (\mathbb{F}(d(i)) \ (\kappa, \text{ hence } \lambda)\text{-accessible}) \\
&= \mathrm{colim}_i (\mathbb{F}(d(i)) \mathbb{G}(d(i))) && (\text{follows since } I \text{ is filtered})
\end{aligned}
$$

◀

Proof of Theorem 8. We prove the coherence axioms for α and β. By the universal property of the Kan extension, it suffices to prove:

1. $\epsilon_{FGH} \circ \alpha_{FG,H} B \circ \alpha_{F,G} \mathbb{B}(H) B = \epsilon_{FGH} \circ \alpha_{F,GH} B \circ \mathbb{B}(F) \alpha_{G,H} B$:

The top left part commutes by naturality of $\alpha_{F,G}$, and all other parts by definition of α.

2. $\epsilon_F \circ \alpha_{F,\mathsf{Id}} B \circ \mathbb{B}(F) \beta B = \epsilon_F$:

The triangle commutes by definition of β, the square by definition of α.

3. $\epsilon_F \circ \alpha_{\mathsf{Id},F} B \circ \beta \mathbb{B}(F) B = \epsilon_F$:

The left square commutes by naturality of β, the right square by definition of α, and the crescent by definition of β.

◀

The following Lemma is used in the proof of Theorem 15.

▶ **Lemma 23.** *Let* $(\mathcal{D}, \otimes, I)$ *be a monoidal category and* $\mathcal{S}$ *a full monoidal subcategory of* $([\mathcal{D}, \mathcal{D}], \otimes', K_I)$ *and* $([\mathcal{D}, \mathcal{D}], *, \mathsf{Id})$. *If* $B: (\mathcal{D}, \otimes, I) \to (\mathcal{D}, \otimes, I)$ *is a lax monoidal functor with* $B \in \mathcal{S}$, *then* B_* *is also a lax monoidal functor on* $(\mathcal{S}, \otimes', K_I)$.

Proof. First of all, we note that B_* is indeed a functor on $\mathcal{S}$, because $B \in \mathcal{S}$ and $\mathcal{S}$ closed under composition (that is what it means for $\mathcal{S}$ to be a monoidal subcategory of $([\mathcal{D}, \mathcal{D}], *, \mathsf{Id})$. Suppose B now that is lax monoidal with mediating morphisms $\beta: I \to BI$ and $\alpha: \otimes \circ (B \times B) \Rightarrow B \circ \otimes$. To show that B_* is lax monoidal, we need mediators $\beta': K_I \to B_*(K_I)$ and $\alpha': \otimes' \circ (B_* \times B_*) \Rightarrow B_* \circ \otimes'$. Note that $B_*(K_I)(X) = BI$. Thus, we can define β' to be constantly β, that is, $\beta'_X = \beta$. Next, we note that $F \otimes' G = \otimes \circ \langle F, G \rangle$. Therefore, we can put

$$\alpha'_{F,G} = \alpha * \mathsf{id}_{\langle F,G \rangle}.$$

This definition gives us immediately that $\alpha'_{F,G}$ is natural. More explicitly, we have that $\alpha'_{F,G}$ is given on objects by $\alpha'_{F,G,X} = (\alpha * \mathsf{id}_{\langle F,G \rangle})_X = \alpha_{\langle F,G \rangle X} \circ B(\mathsf{id}_{\langle F,G \rangle, X}) = \alpha_{FX,GX}$. To show that α' is natural in F and G, let $\sigma: F \Rightarrow F'$ and $\tau: G \Rightarrow G'$ be natural transformations. The following diagram expresses component-wise naturality of α' and commutes because α is natural.

Next, we need to check that the mediators β' and α' fulfil the necessary equations. Unsurprisingly, these are checked point-wise for objects $X \in \mathcal{C}$:

$$
\begin{aligned}
(\alpha'_{K_I,F} \circ (\beta' \otimes' \mathsf{id}_{B_*(F)}))_X &= \alpha'_{K_I,F,X} \circ (\beta'_X \otimes \mathsf{id}_{B_*(F),X}) \\
&= \alpha_{K_I X, FX} \circ (\beta \otimes \mathsf{id}_{BFX}) \\
&= \alpha_{I,FX} \circ (\beta \otimes \mathsf{id}_{BFX}) \\
&= \mathsf{id}_{BFX} && B \text{ lax monoidal} \\
&= \mathsf{id}_{B_*(F),X}
\end{aligned}
$$

Similar for $\alpha'_{F,K_I} \circ (\mathsf{id}_{BF} \otimes' \beta') = \mathsf{id}_{BF}$. Finally, we have

$$
\begin{aligned}
(\alpha'_{F \otimes' G, H} \circ (\alpha'_{F,G} \otimes' \mathsf{id}_{B_*(H)}))_X &= \alpha'_{F \otimes' G, H, X} \circ (\alpha'_{F,G,X} \otimes \mathsf{id}_{B_*(H),X}) \\
&= \alpha_{(F \otimes' G)X, HX} \circ (\alpha_{FX,GX} \otimes \mathsf{id}_{B_*(H)(X)}) \\
&= \alpha_{FX \otimes GX, HX} \circ (\alpha_{FX,GX} \otimes \mathsf{id}_{BHX}) \\
&= \alpha_{FX,GX \otimes HX} \circ (\mathsf{id}_{BFX} \otimes \alpha_{GX,HX}) && B \text{ l.m.} \\
&= (\alpha'_{F,G \otimes' H} \circ (\mathsf{id}_{B_*(F)} \otimes' \alpha'_{G,H}))_X.
\end{aligned}
$$

Thus, B_* is lax monoidal with β' and α'. ◀

▶ **Lemma 24.** *For all functors* $\mathbb{F}, \mathbb{G}, \mathbb{H} \colon [\mathcal{F}, \mathcal{F}] \to [\mathcal{F}, \mathcal{F}]$, *we have* $(\mathbb{F} \circledast \mathbb{G})\mathbb{H} = \mathbb{F}\mathbb{H} \circledast \mathbb{G}\mathbb{H}$.

Proof. For all functors $F \in \mathcal{F}$, we have $((\mathbb{F} \circledast \mathbb{G})\mathbb{H})(F) = (\mathbb{F} \circledast \mathbb{G})(\mathbb{H}(F)) = \mathbb{F}\mathbb{H}(F) \circledast \mathbb{G}\mathbb{H}(F)$. Similarly for morphisms. ◀

Proof of Theorem 15. First of all, we note that $\mathbb{B}_*$ is lax monoidal on $(\mathcal{S}, \circledast, K_{\mathsf{Id}})$ by Lemma 23 and because $\mathbb{B}$ is lax monoidal (Theorem 8). We shall refer to the mediators for $\mathbb{B}_*$ by $\beta' \colon K_{\mathsf{Id}} \Rightarrow \mathbb{B}_*(K_{\mathsf{Id}})$ and $\alpha'_{\mathbb{F},\mathbb{G}} \colon \mathbb{B}\mathbb{F} \circledast \mathbb{B}\mathbb{G} \Rightarrow \mathbb{B}(\mathbb{F} \circledast \mathbb{G})$. Now, given $(\mathbb{F}, \lambda)$ and $(\mathbb{G}, \rho)$ in $\mathcal{S}$, for the tensor $(\mathbb{F}, \lambda) \circledast (\mathbb{G}, \rho)$ we have to provide a distributive law of type $(\mathbb{F} \circledast \mathbb{G})\mathbb{B} \Rightarrow \mathbb{B}(\mathbb{F} \circledast \mathbb{G})$. We use Lemma 24 and α' to obtain this distributive law by

$$(\mathbb{F} \circledast \mathbb{G})\mathbb{B} = \mathbb{F}\mathbb{B} \circledast \mathbb{G}\mathbb{B} \xRightarrow{\lambda \circledast \rho} \mathbb{B}\mathbb{F} \circledast \mathbb{B}\mathbb{G} \xRightarrow{\alpha'_{\mathbb{F},\mathbb{G}}} \mathbb{B}(\mathbb{F} \circledast \mathbb{G}).$$

The tensor of morphisms $\delta_1 \colon (\mathbb{F}, \lambda) \to (\mathbb{F}', \lambda')$ and $\delta_2 \colon (\mathbb{G}, \rho) \to (\mathbb{G}', \rho')$, is just given by the tensor product $\delta_1 \circledast \delta_2$ of the underlying natural transformations. Finally, note that $K_{\mathsf{Id}}\mathbb{B} = K_{\mathsf{Id}}$, thus we can use $I = (K_{\mathsf{Id}}, \beta')$ as unit for $\circledast$.

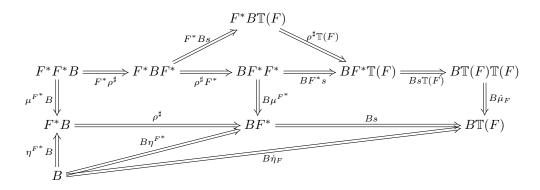Given these definitions, we need to check a couple of things:

1. $\circledast$ is a functor $\mathsf{DL}(\mathbb{B}) \times \mathsf{DL}(\mathbb{B}) \to \mathsf{DL}(\mathbb{B})$. This is immediate by $\circledast$ being a functor $\mathcal{S}^2 \to \mathcal{S}$.

2. $\circledast$ is associative. For $(\mathbb{F}, \lambda), (\mathbb{G}, \rho), (\mathbb{H}, \gamma) \in \mathsf{DL}(\mathbb{B})$, this is given by

$$
\begin{aligned}
((\mathbb{F}, \lambda) \circledast (\mathbb{G}, \rho)) \circledast (\mathbb{H}, \gamma) &= \alpha'_{\mathbb{F}\circledast\mathbb{G},\mathbb{H}} \circ ((\alpha'_{\mathbb{F},\mathbb{G}} \circ (\lambda \circledast \rho)) \circledast \gamma) \\
&= \alpha'_{\mathbb{F}\circledast\mathbb{G},\mathbb{H}} \circ (\alpha'_{\mathbb{F},\mathbb{G}} \circledast \mathsf{id}_{\mathbb{B}\mathbb{H}}) \circ (\lambda \circledast \rho \circledast \gamma) \\
&= \alpha'_{\mathbb{F},\mathbb{G}\circledast\mathbb{H}} \circ (\mathsf{id}_{\mathbb{B}\mathbb{F}} \circledast \alpha'_{\mathbb{G},\mathbb{H}}) \circ (\lambda \circledast \rho \circledast \gamma) \quad \mathbb{B}_* \text{ lax monoidal} \\
&= \alpha'_{\mathbb{F},\mathbb{G}\circledast\mathbb{H}} \circ (\lambda \circledast (\alpha'_{\mathbb{G},\mathbb{H}} \circ (\rho \circledast \gamma))) \\
&= (\mathbb{F}, \lambda) \circledast ((\mathbb{G}, \rho) \circledast (\mathbb{H}, \gamma)).
\end{aligned}
$$

3. I is left and right unit for $\circledast$. This is similar to the proof of associativity, only that we use the two laws involving the mediator β that make $\mathbb{B}_*$ a lax monoidal functor.
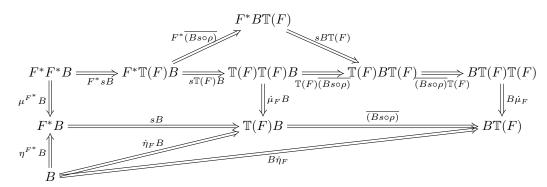
This shows that $\mathsf{DL}(\mathbb{B})$ is a (strict) monoidal category. ◀

Proof of Theorem 22. The second item amounts to commutativity of the diagrams below.

The upper triangle commutes by naturality, the left rectangle and triangle since $\rho^\sharp$ is a

distributive law of monad over functor, and the rest since s as a monad map.

$$
\begin{array}{c}
F^*B\mathbb{T}(F) \\
\end{array}
$$

The upper triangle commutes by naturality, the left rectangle and triangle since s is a monad map, and the rest since $\overline{(Bs \circ \rho)}$ is a distributive law of monad over functor. ◀