

Breaking the Loop

Recursive Proofs for Coinductive Predicates in Fibrations

Henning Basold¹

CNRS, ENS de Lyon, henning.basold@ens-lyon.fr

Abstract The purpose of this paper is to develop and study recursive proofs of coinductive predicates. Such recursive proofs allow one to discover proof goals in the construction of a proof of a coinductive predicate, while still allowing the use of up-to techniques. This approach lifts the burden to guess invariants, like bisimulation relations, beforehand. Rather, they allow one to start with the sought-after proof goal and develop the proof from there until a point is reached, at which the proof can be closed through a recursion step. Proofs given in this way are both easier to construct and to understand, similarly to proofs given in cyclic proof systems or by appealing parameterised coinduction. In this paper, we develop a framework for recursive proofs of coinductive predicates that are given through fibrational predicate liftings. This framework is built on the so-called later modality, which has made its appearance in type theoretic settings before. In particular, we show the soundness and completeness of recursive proofs, we prove that compatible up-to techniques can be used as inference rules in recursive proofs, and provide some illustrating examples.

1 Introduction

Recursion is one of the most fundamental notions in Computer Science and Mathematics, be it as the foundation of computability, or to define and reason about structures determined by repeated constructions. In this paper, we will focus on the use of recursion as a proof method for coinductive predicates.

The usual way to prove that some objects are contained in a coinductive predicate or are related by a coinductive relation, is to establish an invariant. More specifically, suppose $\Phi: L \rightarrow L$ is a monotone function on a lattice and Φ that has a greatest fixed point $\nu\Phi$. One proves that the coinductive predicate $\nu\Phi$ holds for $x \in L$ by establishing a $y \in L$ with $x \leq y \leq \Phi(y)$. This approach does, however, not fit common practice, as one usually incrementally constructs the invariant y , rather than guessing it, while following the necessary proof steps. Such an incremental construction leads to a recursive proof methodology.

There are several ways that have been proposed to formalise the idea of recursive proofs for coinductive predicates. In the setting of complete lattices, Hur et al. [18] developed so-called parameterised coinduction. Their techniques were later streamlined using the companion by Pous [23]. Another approach is to use ideas from game theory [22,29] to prove coinductive predicates. There are also

type theoretic approaches that use systems of equations to prove coinductive predicates [1,7,9,15]. Finally, recursion has also entered syntactic proof systems in the form of cyclic proof systems, e.g. [11,13,25,28]. Cyclic proof systems are particularly useful in settings that require proofs by induction or coinduction because cyclic proof systems ease proofs enormously compared to, for example, invariant-based method from above or (co)induction schemes. Nothing comes for free though: In this case checking proofs becomes more difficult, as the correctness conditions are typically global for a proof tree and not compositional. For the same reason, also soundness proofs are often rather complex.

In this paper, we will study an approach to proving coinductive predicates through recursive proofs. Recursion in such proofs is thereby controlled by using the so-called later modality [21], which allows checking of recursive proofs on a per-rule basis. This results in straightforward proof checking, a per-rule soundness proof, and proofs that can be easily debugged. We will thereby develop the recursive proofs abstractly for a general first-order logic, given in form of a fibration. This generality allows us to obtain recursive proofs for coinductive predicates in many different settings. In particular, we will discuss set-based predicates, quantitative predicates, syntactic first-order logic, and (models of) dependent type theory. An instance of this is the syntactic first-order logic given by the author in [4] to reason about program equivalences. This instance was also the original motivation of the present paper, as the results in loc. cit. are mostly obtained by hand.

Towards this, we proceed as follows. In Sec. 2, we show that certain fibrations of functors are fibred Cartesian closed, which is the technical machinery that makes recursive proofs work. Next, we develop in Sec. 3 and Sec. 4 a theory of descending chains of predicates in general categories and fibrations, respectively. In the same sections, we also provide the necessary results for the construction of recursive proofs. Section 5 provides some specific results concerning the descending chain that is induced by a lifting of a behaviour functor. In particular, we show how up-to techniques can be used as proof rules. We instantiate these results in Sec. 6 to obtain recursive proofs for some illustrative examples.

Related Work To a large part, the present paper develops many results of Birkedal et al. [8] in the setting of general fibrations rather than just the codomain fibration $\mathbf{Set}^{\rightarrow} \rightarrow \mathbf{Set}$ of sets. That [8] was so restrictive is not so surprising, as the intention there was to construct models of programming languages, rather than applying the developed techniques to proofs. Going beyond the category of sets also means that one has to involve much more complicated machinery to obtain exponential objects. Later, Bizjak et al. [9] extended the techniques from [8] to dependent type theory, thereby enabling reasoning by means of recursive proofs in a syntactic type theory. However, also this is again a very specific setting, which rules out most examples that we are interested in here. Similarly, also the parameterised coinduction in [18] is too restrictive, as it applies only to lattices. It might be possible to develop parameterised coinduction in the setting of fibrations by using the companion [23,24,5]. We leave this for another time though.

2 Functor Categories and Fibrations

We fix an index category \mathbf{I} in the following and define

$$\frac{F: \mathbf{C} \rightarrow \mathbf{D}}{\overline{F}: [\mathbf{I}, \mathbf{C}] \rightarrow [\mathbf{I}, \mathbf{D}]}$$

by $\overline{F}(\sigma) = F \circ \sigma$. Note that $\overline{F} = [I, F]$, where $[I, -]: \mathbf{Cat} \rightarrow \mathbf{Cat}$ is the strict 2-functor that assigns to a category \mathbf{C} the functor category $[\mathbf{I}, \mathbf{C}]$. Thus, $\overline{(-)}$ preserves composition of functors and applies to natural transformations as well. We use this to define for a morphism $f: X \rightarrow Y$ in \mathbf{C} , a morphism $\overline{f}: K_X \Rightarrow K_Y$ in $[\mathbf{I}, \mathbf{C}]$ where K_X is the constant functor sending any object in \mathbf{I} to X : Note that there is a natural transformation $K_f: K_X \Rightarrow K_Y$, which is given by $K_{f,I} = f$. Thus, we can put $\overline{f} = [I, K_f]$.

Lemma 1. *If $F: \mathbf{C} \rightarrow \mathbf{D}$ and $G: \mathbf{D} \rightarrow \mathbf{C}$ with $F \dashv G$, then $\overline{F} \dashv \overline{G}$.*

Proof. Given F and G as above, the following unique correspondence follows from the point-wise unique correspondence given by the adjunction $F \dashv G$. That this correspondence is natural also follows from uniqueness of the point-wise correspondence. \square

Lemma 2. *The functor $\overline{(-)}$ extends to a fibred functor on the (large) fibration $\mathbf{Fib} \rightarrow \mathbf{Cat}$.*

Proof. A fibration $p: \mathbf{E} \rightarrow \mathbf{B}$ induces a fibration $\overline{p}: \overline{\mathbf{E}} \rightarrow \overline{\mathbf{B}}$, see [19, Ex. 1.8.8] and [35]. Given a map of fibration (F, G) , easily shows that $(\overline{F}, \overline{G})$ is again a map of fibrations. Finally, that $\overline{(-)}$ is fibred follows from the fact that $\overline{(-)}$ preserves strict 2-pullbacks, since it is an enriched right adjoint functor [20]. \square

Let $S: \mathbf{I}^{\text{op}} \times \mathbf{I} \rightarrow \mathbf{C}$ be a functor. The *end* of S is an object $\int_{i \in \mathbf{I}} S(i, i)$ in \mathbf{C} together with a universal extranatural transformation $\pi: \int_{i \in \mathbf{I}} S(i, i) \rightarrow S$. Concretely, this means that α is a family of morphisms indexed by objects in \mathbf{I} , such that the following diagram commutes for all $u: i \rightarrow j$.

$$\begin{array}{ccc} \int_{i \in \mathbf{I}} S(i, i) & \xrightarrow{\pi_j} & S(j, j) \\ \downarrow \pi_i & & \downarrow S(u, \text{id}) \\ S(i, i) & \xrightarrow{S(\text{id}, u)} & S(i, j) \end{array}$$

Moreover, given any other extranatural transformation $\alpha: X \rightarrow S$ there is a unique $f: X \rightarrow \int_{i \in \mathbf{I}} S(i, i)$ with $\pi_i \circ f = \alpha_i$ for every $i \in \mathbf{I}$.

It is well-known that ends can be computed as certain limits in \mathbf{C} . By analysing carefully the necessary limits, we obtain the following result.

Proposition 3. *Let \mathbf{I} be a small category and \mathbf{C} a category that has finite limits and for every object $i \in \mathbf{I}$ products of the size of the coslice category $i \downarrow \mathbf{I}$. If \mathbf{C} is Cartesian closed, then also $[\mathbf{I}, \mathbf{C}]$ is. The exponential object is then given by*

$$(G^F)(i) = \int_{i \rightarrow j} G(j)^{F(j)}.$$

Proof. More precisely, we define for each $i \in I$ a functor $S_i: (i \downarrow \mathbf{I})^{\text{op}} \times i \downarrow \mathbf{I} \rightarrow \mathbf{C}$ by $S(i \rightarrow j, i \rightarrow k) = G(k)^{F(j)}$ and $S(f: j' \rightarrow j, g: k \rightarrow k') = G(g)^{F(f)}$. The end of S is then given by the equaliser as in the following diagram.

$$\int_{i \rightarrow j} G(i)^{F(i)} \longrightarrow \prod_{\mathbf{I}(i,j)} G(j)^{F(j)} \rightrightarrows \prod_{\mathbf{I}(i,j), \mathbf{I}(i,k)} G(k)^{F(j)}$$

That such an equaliser gives indeed the end of S is standard. Note that both products range only over objects in the coslice category $i \downarrow \mathbf{I}$, hence the products exist in \mathbf{C} . Finally, that the given definition of G^F is an exponential object is folklore, see [30] and cf. [34, Thm. 2.12]. \square

Given that we can construct exponential objects as certain ends, one reasonably might expect that this also works for fibred Cartesian closed categories, which are fibrations $p: \mathbf{E} \rightarrow \mathbf{B}$ in which every fibre is Cartesian closed and reindexing preserves this structure, see [19, Def. 1.8.2]. To prove this, we require a suitable adaption of the co-Yoneda lemma to the setting of fibrations.

Lemma 4 (Fibred co-Yoneda). *Let $p: \mathbf{E} \rightarrow \mathbf{B}$ be a cloven fibration, and suppose $H: \mathbf{I}^{\text{op}} \rightarrow \mathbf{E}$ and $U: \mathbf{I}^{\text{op}} \rightarrow \mathbf{B}$ are functors, such that $p \circ H = U$. Then*

$$H \cong \int^{i \in \mathbf{I}} \sum_{v \in \mathbf{I}(-, i)} \prod_{U(v)} H(i).$$

Theorem 5. *Let \mathbf{I} be a small category and $p: \mathbf{E} \rightarrow \mathbf{B}$ a cloven fibration that has fibred finite limits, fibred exponents and for every object $i \in \mathbf{I}$ fibred products of the size of the coslice category $i \downarrow \mathbf{I}$. Under these conditions, $[\mathbf{I}, p]: [\mathbf{I}, \mathbf{E}] \rightarrow [\mathbf{I}, \mathbf{B}]$ is again a fibred CCC. The exponential object of $F, G \in [\mathbf{I}, \mathbf{E}]_U$ is given by*

$$(G^F)(i) = \int_{v: i \rightarrow j} (U(v)^* G(j))^{U(v)^* F(j)}.$$

Proof. The size of the involved limits to compute the end are given in the same way as in Prop. 3. Note that the end is equivalently given by an end followed by a product:

$$\int_{v: i \rightarrow j} (U(v)^* G(j))^{U(v)^* F(j)} \cong \int_{j \in \mathbf{I}} \prod_{v: i \rightarrow j} (U(v)^* G(j))^{U(v)^* F(j)}$$

To show that the given exponential is right-adjoint to the product of functors, we consider for $H \in [\mathbf{I}, \mathbf{E}]_U$ the following chain of natural isomorphisms.

$$\begin{aligned} [\mathbf{I}, \mathbf{E}]_U(H, G^F) &\cong \int_{i \in \mathbf{I}} \mathbf{E}_{U(i)}(H(i), (G^F)(i)) \\ &\cong \int_{i \in \mathbf{I}} \mathbf{E}_{U(i)}\left(H(i), \int_{j \in \mathbf{I}} \prod_{v: i \rightarrow j} (U(v)^* G(j))^{U(v)^* F(j)}\right) \\ &\cong \int_{i \in \mathbf{I}} \int_{j \in \mathbf{I}} \prod_{v: i \rightarrow j} \mathbf{E}_{U(i)}\left(H(i), (U(v)^* G(j))^{U(v)^* F(j)}\right) \\ &\cong \int_{i \in \mathbf{I}} \int_{j \in \mathbf{I}} \prod_{v: i \rightarrow j} \mathbf{E}_{U(i)}(H(i) \times (U(v)^* F(j)), U(v)^* G(j)) \end{aligned}$$

$$\begin{aligned}
&\cong \int_{i \in \mathbf{I}} \int_{j \in \mathbf{I}} \prod_{v: i \rightarrow j} \mathbf{E}_{U(i)} \left(\coprod_{U(v)} (H(i) \times (U(v)^* F(j))), G(j) \right) \\
&\cong \int_{i \in \mathbf{I}} \int_{j \in \mathbf{I}} \prod_{v: i \rightarrow j} \mathbf{E}_{U(i)} \left(\left(\coprod_{U(v)} H(i) \right) \times F(j), G(j) \right) \quad (*) \\
&\cong \int_{i \in \mathbf{I}} \int_{j \in \mathbf{I}} \prod_{v: i \rightarrow j} \mathbf{E}_{U(i)} \left(\coprod_{U(v)} H(i), G(j)^{F(j)} \right) \\
&\cong \int_{j \in \mathbf{I}} \mathbf{E}_{U(i)} \left(\int^{i \in \mathbf{I}} \sum_{v: i \rightarrow j} \coprod_{U(v)} H(i), G(j)^{F(j)} \right) \\
&\cong \int_{j \in \mathbf{I}} \mathbf{E}_{U(i)} \left(H(j), G(j)^{F(j)} \right) \quad (**) \\
&\cong \int_{j \in \mathbf{I}} \mathbf{E}_{U(i)} (H(j) \times F(j), G(j)) \\
&\cong [\mathbf{I}, \mathbf{E}]_U (H \times F, G)
\end{aligned}$$

Note that coproducts in p fulfil the Frobenius property in the step (*) because p is a fibred CCC, see [19, Lem. 1.9.11]. Moreover, we do not need to assume the existence of coproducts along morphisms of \mathbf{B} or further colimits explicitly, since $H(j)$ is isomorphic to $\int^{i \in \mathbf{I}} \coprod_{v: i \rightarrow j} \coprod_{U(v)} H(i)$ by the fibred co-Yoneda lemma that we used in the step (**). \square

3 Descending Chains in Categories

In this section, we extend the development in [8] to more general categories. Besides giving us some intuition for the later modality, we also obtain results that we can reuse in later sections of this paper.

Let ω be the poset of finite ordinals, i.e., $\omega = \{0, 1, \dots\}$ with their usual order. Since ω can be seen as a category, we can use its dual category ω^{op} as index category, thereby obtaining a functor $[\omega^{\text{op}}, -] : \mathbf{Cat} \rightarrow \mathbf{Cat}$ as in the last section. We will denote this functor in the following by

$$\overline{(-)} = [\omega^{\text{op}}, -]. \quad (1)$$

The *category of descending chains in \mathbf{C}* is then the presheaf category $\overline{\mathbf{C}}$, the objects of which we denote by σ, τ, \dots . More explicitly, $\sigma \in \overline{\mathbf{C}}$ assigns as a functor $\sigma : \omega^{\text{op}} \rightarrow \mathbf{C}$ to each $n \in \mathbb{N}$ an object $\sigma_n \in \mathbf{C}$ and to each pair of natural numbers with $m \leq n$ a morphism $\sigma(m \leq n) : \sigma_n \rightarrow \sigma_m$ in \mathbf{C} .

Assumption 6. *Throughout this section, we assume that \mathbf{C} is a category with a terminal object $\mathbf{1}$, finite limits and is Cartesian closed.*

In particular, we get by Prop. 3 that $\overline{\mathbf{C}}$ is also Cartesian closed as follows. Let \underline{n} be the poset of all numbers less or equal to n . Observe now for $n \in \mathbb{N}$ that

$n \downarrow \omega^{\text{op}} = (\omega/n)^{\text{op}} = \underline{n}^{\text{op}}$. Hence, $n \downarrow \omega^{\text{op}}$ is finite and, as assumed, we only need finite limits in \mathbf{C} to obtain Cartesian-closure of $\overline{\mathbf{C}}$ from Prop. 3.

Let us now introduce the later modality, which is the central construction that underlies the recursive proofs that we develop in this paper.

Definition 7. *The later modality on $\overline{\mathbf{C}}$ is the functor $\blacktriangleright: \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ given on objects by*

$$\begin{aligned} (\blacktriangleright\sigma)_0 &= \mathbf{1} \\ (\blacktriangleright\sigma)_{n+1} &= \sigma_n \end{aligned} \quad (\blacktriangleright\sigma)(m \leq n) = \begin{cases} !: \sigma_n \rightarrow \mathbf{1}, & m = 0 \text{ or } n = 0 \\ \sigma(m' \leq n'), & m = m' + 1, n = n' + 1 \end{cases}$$

Theorem 8. *The map \blacktriangleright given in Def. 7 on objects is a functor $\overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$. Moreover, \blacktriangleright has a left adjoint and thereby preserves limits. Finally, there is a natural transformation*

$$\text{next}: \text{Id} \Rightarrow \blacktriangleright,$$

given by $\text{next}_{\sigma,0} = !_{\sigma_0}: \sigma_0 \rightarrow \mathbf{1}$ and $\text{next}_{\sigma,n+1} = \sigma(n \leq n+1)$.

Proof. Functoriality is given by uniqueness of maps into the final object $\mathbf{1}$. The left adjoint to \blacktriangleright is given by \blacktriangleleft with $\blacktriangleleft(\sigma)_n = \sigma_{n+1}$.

$$\frac{\frac{\frac{\blacktriangleleft\sigma \rightarrow \tau}{\forall n. \sigma_{n+1} \rightarrow \tau_n}}{\sigma_0 \rightarrow \mathbf{1} \text{ and } \forall n. \sigma_{n+1} \rightarrow \tau_n}}{\sigma \rightarrow \blacktriangleright\tau}$$

Finally, naturality of next is given again by uniqueness of maps into final objects and by functoriality of chains. \square

Since \blacktriangleright preserves in particular binary products, we obtain the following.

Lemma 9. *For all $\sigma, \tau \in \overline{\mathbf{C}}$ there is a morphism $\blacktriangleright(\sigma^\tau) \rightarrow \blacktriangleright\sigma^{\blacktriangleright\tau}$.*

One the central properties of the later modality is that it allows us to construct fixed points of certain maps in $\overline{\mathbf{C}}$, which are called contractive.

Definition 10. *A map $f: \tau \times \sigma \rightarrow \sigma$ in $\overline{\mathbf{C}}$ is called g -contractive if g is a map $g: \tau \times \blacktriangleright\sigma \rightarrow \sigma$ with $f = g \circ (\text{id} \times \text{next}_\sigma)$. We call $s: \tau \rightarrow \sigma$ a fixed point or solution for f , if the following diagram commutes.*

$$\begin{array}{ccc} \tau & \xrightarrow{s} & \sigma \\ \langle \text{id}, s \rangle \downarrow & \nearrow f & \\ \tau \times \sigma & & \end{array}$$

We can now show that there is a generic operator in $\overline{\mathbf{C}}$ that allows us to construct fixed points.

Theorem 11. For every $\sigma \in \overline{\mathbf{C}}$ there is a unique morphism, dinatural in σ ,

$$\text{l\"ob}_\sigma : \sigma^{\blacktriangleright\sigma} \rightarrow \sigma,$$

such that for all g -contractive maps f the map $\text{l\"ob}_\sigma \circ \lambda g$ is a solution for f . Dinaturality means thereby that for all $h : \sigma \rightarrow \tau$ the diagram below commutes.

$$\begin{array}{ccccc} & & \tau^{\blacktriangleright\tau} & \xrightarrow{\text{l\"ob}_\tau} & \tau \\ & \nearrow^{h^{\text{id}}} & & & \uparrow h \\ \sigma^{\blacktriangleright\tau} & & & & \sigma \\ & \searrow_{\text{id}^{\blacktriangleright h}} & \sigma^{\blacktriangleright\sigma} & \xrightarrow{\text{l\"ob}_\sigma} & \sigma \end{array}$$

Proof. We define $\text{l\"ob}_n : (\sigma^{\blacktriangleright\sigma})_n \rightarrow \sigma_n$ by iteration on n . For 0, we put

$$\text{l\"ob}_0 := (\sigma^{\blacktriangleright\sigma})_0 \xrightarrow{\langle \text{id}, ! \rangle} (\sigma^{\blacktriangleright\sigma})_0 \times \mathbf{1} = (\sigma^{\blacktriangleright\sigma})_0 \times (\blacktriangleright\sigma)_0 \xrightarrow{\text{ev}_0} \sigma_0,$$

where ev is the counit of $(-) \times \blacktriangleright\sigma \dashv (-)^{\blacktriangleright\sigma}$. In the iteration step, we define

$$\begin{array}{ccc} \text{l\"ob}_{n+1} := (\sigma^{\blacktriangleright\sigma})_{n+1} & \xrightarrow{\langle \text{id}, \sigma^{\blacktriangleright\sigma}(n \leq n+1) \rangle} & (\sigma^{\blacktriangleright\sigma})_{n+1} \times (\sigma^{\blacktriangleright\sigma})_n \\ & \xrightarrow{\text{id} \times \text{l\"ob}_n} & (\sigma^{\blacktriangleright\sigma})_{n+1} \times \sigma_n \\ & \xrightarrow{\text{ev}_{n+1}} & \sigma_{n+1}, \end{array}$$

where $\sigma^{\blacktriangleright\sigma}(n \leq n+1)$ is the functorial action of $\sigma^{\blacktriangleright\sigma}$ (Prop. 3). To show that l\"ob is the unique map making $\text{l\"ob}_\sigma \circ \lambda g$ as solution one first shows that l\"ob is uniquely fulfilling the equation $\text{l\"ob}_n = (\text{ev} \circ (\text{id} \times \text{next}) \circ \langle \text{id}, \text{l\"ob} \rangle)_n$ by induction on n and doing a small diagram chase. Uniqueness of solutions is then given the properties of the adjunction $(-) \times \blacktriangleright\sigma \dashv (-)^{\blacktriangleright\sigma}$. \square

Remark 12. Birkedal et al. [8] give some closure properties of contractive maps. These can be extended to our more general setting, but as we will not need them here, we will not state and prove them. \blacktriangleleft

4 Descending Chains in Fibrations

Now that we have developed some understanding of how descending chains work in general categories, we will essentially lift the results from Sec. 3 to fibrations. This will allow us to construct from a first-order logic, given by a fibration, a new logic of descending chains that admits the same logical structure as the given fibration and admits recursive proofs for coinductive predicates.

Throughout this section, we assume the following.

Assumption 13. Let $p : \mathbf{E} \rightarrow \mathbf{B}$ be a cloven fibration, such that,

- \mathbf{E} has fibred final objects,
- fibred finite limits in \mathbf{E} exist, and
- \mathbf{E} is a fibred CCC.

Similarly to Sec. 3, we obtain by Lem. 2 that the functor $\bar{p}: \bar{\mathbf{E}} \rightarrow \bar{\mathbf{B}}$ given by post-composition is a fibration. By the above assumptions, we then get by Thm. 5 that \bar{p} is a fibred CCC. We obtain another fibred CCC by change-of-base along the diagonal functor $\delta: \mathbf{B} \rightarrow \bar{\mathbf{B}}$ that sends an object $I \in \mathbf{B}$ to the constant chain $K_I: \omega^{\text{op}} \rightarrow \mathbf{B}$, see [19, Ex. 1.8.8] and [35]:

$$\begin{array}{ccc} \mathbf{B} \times_{\bar{\mathbf{B}}} \bar{\mathbf{E}} & \longrightarrow & \bar{\mathbf{E}} \\ q \downarrow \lrcorner & & \downarrow \bar{p} \\ \mathbf{B} & \xrightarrow{\delta} & \bar{\mathbf{B}} \end{array}$$

Note that for $I \in \bar{\mathbf{B}}$, the fibre of q above I is isomorphic to $\bar{\mathbf{E}}_{K_I}$. Hence, we will simplify notation in the following and just refer to $\bar{\mathbf{E}}_{K_I}$ as $\bar{\mathbf{E}}_I$. Furthermore, we note the following result, which might seem trivial at first, but it allows us to apply, for instance, Lem. 1 to functors between fibres of a given fibration.

Lemma 14. $\bar{\mathbf{E}}_{K_I} = \bar{\mathbf{E}}_I$, which we will denote by $\bar{\mathbf{E}}_I$.

Having worked only abstractly so far, it is about time that we give a few examples. There are four kinds of examples that we shall use here to illustrate different aspects of the theory: predicates over sets, quantitative predicates, syntactic first-order logic, and set families that model dependent types. We begin with the simplest example, namely that of predicates. Despite its simplicity, it is already a quite useful because it allows us to reason about predicates and relations for arbitrary coalgebras in **Set**.

Example 15 (Predicates). A standard fibration is the fibration $\text{Pred} \rightarrow \mathbf{Set}$ of predicates, where an object in Pred is a predicate ($P \subseteq X$) over a set X . Each fibre Pred_X has a final object $\mathbf{1}_X = (X \subseteq X)$ and the fibred binary products are given by intersection. Moreover, exponents also exist in Pred_X by defining

$$Q^P = \{x \in X \mid x \in P \implies x \in Q\}.$$

The fibration $\overline{\text{Pred}}$ consists then of descending chains of predicates. In particular, if $\sigma \in \overline{\text{Pred}}_X$, then σ is a chain with $\sigma_0 \supseteq \sigma_1 \supseteq \dots$. Note now that each fibre Pred_X is a poset, hence equalisers are trivial and (finite) limits are just given as (finite) products. Hence, Thm. 5 applies and we obtain that $\overline{\text{Pred}}$ is a fibred CCC. Since equalisers are trivial, it is easy to see that the exponential for $\sigma, \tau \in \overline{\text{Pred}}_X$ can be defined as follows.

$$(\tau^\sigma)_n = \bigcap_{m \leq n} \tau_n^{\sigma_m} \subseteq X$$

We end this example by noting that fibred constructions, like the above products and exponents, are preserved by a change-of-base, see [19, Lem. 1.8.4]. This induces thus exponents in the fibration of (binary) relations $\text{Rel} \rightarrow \mathbf{Set}$ and the associated fibration $\overline{\text{Rel}} \rightarrow \mathbf{Set}$. Hence, one can also apply the results in this paper to reason, for example, about bisimilarity in coalgebras. \blacktriangleleft

Often, one is not just interested in merely logical predicates, but rather wants to analyse quantitative aspects of system. This is, for instance, particularly relevant for probabilistic or weighted automata. The following example extends the predicate fibration from Ex. 15 to quantitative predicates, which gives a convenient setting to reason about quantitative properties.

Example 16 (Quantitative Predicates). We define the category of quantitative predicates \mathbf{qPred} as follows.

$$\mathbf{qPred} = \begin{cases} \text{objects:} & \text{pairs } (X, \delta) \text{ with } X \in \mathbf{Set} \text{ and } \delta: X \rightarrow [0, 1] \\ \text{morphisms:} & f: (X, \delta) \rightarrow (Y, \gamma) \text{ if } f: X \rightarrow Y \text{ in } \mathbf{Set} \text{ and } \delta \leq \gamma \circ f \end{cases}$$

It is easy to show that the first projection $\mathbf{qPred} \rightarrow \mathbf{Set}$ gives rise to a cloven fibration, for which the reindexing functors are given for $u: X \rightarrow Y$ by

$$u^*(Y, \gamma) = (X, \lambda x. \gamma(u(x))).$$

For brevity, let us refer to an object (X, δ) in \mathbf{qPred}_X just by its underlying valuation δ . One readily checks that \mathbf{qPred} is a fibred CCC by defining the products and exponents by

$$(\delta \times \gamma)(x) = \min\{\delta(x), \gamma(x)\} \quad \text{and} \quad (\delta \Rightarrow \gamma)(x) = \begin{cases} 1, & \delta(x) \leq \gamma(x) \\ \gamma(x), & \text{otherwise} \end{cases}.$$

Fibred final objects are given by the constantly 1 valuation. Again, each fibre \mathbf{qPred}_X is a poset, hence finitely complete and so $\overline{\mathbf{qPred}}$ is a fibred CCC. \blacktriangleleft

The original motivation for the work presented in this paper was to abstract away from the details that are involved in constructing a syntactic logic for a certain coinductive relation in [4]. In [4], the author developed a first-order logic that features the later modality to reason about program equivalences. This logic was given in a very pedestrian way, since the syntax, proof system, model and proof system was constructed from scratch. The proofs often involved then something along the lines of “true because this is an index-wise interpretation of intuitionistic logic”. Thus, the aim of the following example is to show that we can just take any first-order logic L and extend it to a logic \overline{L} , in which formulas are descending chains of formulas in L . Crucially, the logic \overline{L} will have the later modality as a new formula construction, and it will get new proof rules that correspond to the morphism next , the functoriality of \blacktriangleright and construction of fixed points through the l\"ob morphism. We will also see below that quantifier can be lifted to formulas in \overline{L} , and that the later modality interacts well with conjunction, implication and quantification, cf. Thm. 8 and Lem. 9. After this long-winded motivation, let us now come to the actual example.

Example 17 (Syntactic Logic). Suppose we are given a typed calculus, for example the simply typed λ -calculus, and a first-order logic, in which the variables range over the types of the calculus. More precisely, let Γ be a context with $\Gamma = x_1 : A_1, \dots, x_n : A_n$, where the x_i are variables and the A_i are types of the

calculus. We write then $\Gamma \Vdash t : A$ if t is a term of type A in context Γ , $\Gamma \Vdash \varphi$ if φ is formula with variables in Γ , and $\Gamma \vdash \varphi$ if φ is provable in the given logic. Let us assume that the logic also features a truth formula \top , conjunction \wedge and implication \rightarrow , which are subject to the usual proof rules of intuitionistic logic. This allows us to form a fibration as follows. First, we define \mathcal{C} to be the category that has context Γ as objects and tuples t of terms as morphisms $\Delta \rightarrow \Gamma$ with $\Delta \Vdash t_i : A_i$. Next, we let L be the category that has pairs (Γ, φ) with $\Gamma \Vdash \varphi$ as objects, and a morphism $(\Delta, \psi) \rightarrow (\Gamma, \varphi)$ in L is given by a morphism $t : \Delta \rightarrow \Gamma$ in \mathcal{C} if $\Delta \vdash \psi \rightarrow \varphi[t]$, where $\varphi[t]$ denotes the substitution of t in the formula φ . The functor $p : L \rightarrow \mathcal{C}$ that maps (Γ, φ) to Γ is then easily seen to be a cloven (even split) fibration, see for example [19].

We note that p has fibred finite products and exponents, as the logic that we started with has \top , conjunction and implication with the necessary proof rules. Moreover, since each fibre is a pre-ordered set, equalisers are again trivial. Hence, \bar{p} is also a fibred CCC. Explicitly, for chains φ, ψ of formulas in \bar{p}_A above the constant chain K_A for a type A , the exponent $\psi \Rightarrow \varphi$ in \bar{p} is given by

$$(\psi \Rightarrow \varphi)_n = \bigwedge_{m \leq n} \psi_m \rightarrow \varphi_n,$$

where \bigwedge is a shorthand for a finite number of conjunctions. \blacktriangleleft

The last example that we discuss is motivated by dependent type theories. It is, in fact, also an example, where the fibres are for once not merely pre-orders but rather proper categories.

Example 18 (Set Families).

We lift now the constructions from the last Sec. 3 to the fibres of $\bar{\mathbf{E}}$.

Theorem 19. *For each $c \in \mathbf{B}$, there is a fibred functor $\blacktriangleright^c : \bar{\mathbf{E}}_c \rightarrow \bar{\mathbf{E}}_c$ given by*

$$\begin{aligned} (\blacktriangleright^c \sigma)_0 &= \mathbf{1}_{c_0} \\ (\blacktriangleright^c \sigma)_{n+1} &= c(n \leq n+1)^*(\sigma_n). \end{aligned}$$

Moreover, \blacktriangleright preserves fibred finite products and if p is a bifibration then \blacktriangleright preserves all fibred limits. Finally, there is a natural transformation $\text{next}^c : \text{Id} \Rightarrow \blacktriangleright^c$, given by $\text{next}_{\sigma,0}^c = ! : \sigma_0 \rightarrow \mathbf{1}_{c_0}$ and $\text{next}_{\sigma,n+1}^c = \sigma(n \leq n+1)$.

Proof. We define $\blacktriangleright \sigma$ on morphisms by case distinction as follows.

$$\begin{aligned} (\blacktriangleright \sigma)(0 \leq 0) &= \text{id} : \mathbf{1}_{c_0} \rightarrow \mathbf{1}_{c_0} \\ (\blacktriangleright \sigma)(0 \leq n+1) &= c(n \leq n+1)^*(\sigma_n) \xrightarrow{c(n \leq n+1) \sigma_n} \sigma_n \xrightarrow{!} \mathbf{1}_{c_0} \\ (\blacktriangleright \sigma)(m+1 \leq n+1) &= \text{mediating morphism in the following diagram} \end{aligned}$$

$$\begin{array}{ccc} c(n \leq n+1)^*(\sigma_n) & \longrightarrow & \sigma_n \\ \downarrow (\blacktriangleright \sigma)(m+1 \leq n+1) & & \downarrow \sigma(m \leq n) \\ c(m \leq m+1)^*(\sigma_m) & \longrightarrow & \sigma_m \end{array} \xrightarrow{p} \begin{array}{ccc} c_{n+1} & \longrightarrow & c_n \\ \downarrow & & \downarrow \\ c_{m+1} & \longrightarrow & c_m \end{array}$$

Note that the right diagram commutes by functoriality of c . It is clear that $\bar{p}(\blacktriangleright\sigma) = c$ by the above definition, and so $\blacktriangleright\sigma$ is an object in $\bar{\mathbf{E}}_c$. Defining \blacktriangleright on morphisms is a straightforward, as it is to check functoriality. That \blacktriangleright^c is preserved by reindexing, that is, for $f: c \rightarrow d$ in \mathbf{B} one has $f^* \circ \blacktriangleright^c \cong \blacktriangleright_d \circ f^*$, is given by the properties of a cloven fibration. That \blacktriangleright^c preserves products is a simple calculation. The preservation of all fibred limits if p is a bifibration is given by the fact that \blacktriangleright^c then has a fibred left adjoint \blacktriangleleft_c given by $(\blacktriangleleft_c \sigma)_n = \sum_{c(n \leq n+1)} \sigma_{n+1}$. Finally, naturality of next^c is given as before. \square

Let us briefly stop to discuss the perspective on the later modality that arises canonically from the development in the previous section.

Remark 20. We note that we can instantiate all the results from Sec. 3 to \mathbf{E} as follows. Suppose that \mathbf{B} is a finitely complete CCC and \mathbf{E} also has a global finite limits and exponents, such that the corresponding adjunctions are given by maps of fibrations. This means, for instance, that for all $X \in \mathbf{E}$ there are adjunctions $(-) \times X \dashv (-)^X$ and $(-) \times pX \dashv (-)^{pX}$ on \mathbf{E} and \mathbf{B} , respectively, such that $((-) \times pX, (-) \times X)$ and $((-)^{pX}, (-)^X)$ are maps of fibrations. This structure gives us that $\bar{p}: \bar{\mathbf{E}} \rightarrow \bar{\mathbf{B}}$ has global exponents. Moreover, one can show that $(\blacktriangleright, \blacktriangleright): \bar{p} \rightarrow \bar{p}$ is a map of fibrations and that the next - and L\"ob -operations are preserved by \bar{p} : $\bar{p}(\text{next}) = \text{next}$ and $\bar{p}(\text{l\"ob}) = \text{l\"ob}$. However, we will not make use of these results here, as their use is vastly more complicated than the fibred approach. For example, the predicate fibration has global exponents given by

$$(P \subseteq X)^{(Q \subseteq Y)} = \{f: Y \rightarrow X \mid \forall y \in X. f(y) \in P\} \subseteq X^Y.$$

The problem is that we would need to show that solutions of certain morphism obtained through using l\"ob are vertical, as we often want to prove the set inclusion of predicates. Since formulating and proving such conditions seem to very hard and since they do not even seem to be useful, we will refrain from pursuing the global Cartesian structure on \bar{p} further here. \blacktriangleleft

As we mentioned above, if p has a global final object, then we can instantiate Sec. 3 to the fibration p . This gives us a map of fibration $(\blacktriangleright, \blacktriangleright)$ on \bar{p} . Since the fibred final objects $\mathbf{1}_I$ in \mathbf{E}_I are related to the final object $\mathbf{1}$ of \mathbf{E} by $\mathbf{1}_I \cong !_I^*(\mathbf{1})$, we obtain that the global and local later modalities are intrinsically related

Lemma 21. *For all $\sigma \in \bar{\mathbf{E}}_I$, we have $\blacktriangleright^c \sigma \cong \text{next}_c^*(\blacktriangleright\sigma)$.*

Due to Lem. 14, we can apply many construction easily point-wise to chains with constant index. For instance, we can lift products and coproducts in the following sense.

Theorem 22. *If for $f: I \rightarrow J$ in \mathbf{B} the coproduct $\coprod_f: \mathbf{E}_I \rightarrow \mathbf{E}_J$ along f exists, then the coproduct $\coprod_{\bar{f}}: \bar{\mathbf{E}}_I \rightarrow \bar{\mathbf{E}}_J$ along \bar{f} is given by $\overline{\coprod_f}$. Similarly, the product $\prod_{\bar{f}}$ along \bar{f} is given by $\overline{\prod_f}$.*

Proof. By Lem. 14 and Lem. 1, we obtain that an adjunction $\coprod \dashv f$ lifts to an adjunction $\overline{\coprod_f} \dashv \bar{f}$. Hence, the coproduct along \bar{f} is given by $\overline{\coprod_f}$. \square

Example 23. Both \mathbf{Pred} and $\mathbf{Fam}(\mathbf{Set})$ are well known to have products and coproducts along any function in \mathbf{Set} . We note that also \mathbf{qPred} has products along all functions $f: X \rightarrow Y$, given by

$$\prod_f (\delta: X \rightarrow [0, 1])(y) = \inf\{\delta(x) \mid x \in X, f(x) = y\}.$$

Finally, in a syntactic logic, as in Ex. 17, one has that $L \rightarrow \mathcal{C}$ obtains products and coproducts along projections $(\Gamma, x: A) \rightarrow \Gamma$ from universal and existential quantification over A , respectively. To have arbitrary (co)products, one additionally needs an equality relation in the logic, cf. [19]. By Thm. 22, all these products and coproducts lift to the corresponding fibration of descending chains. \blacktriangleleft

Let us denote for $I \in \mathbf{B}$ the later modality $\blacktriangleright^{K_I}$ on $\overline{\mathbf{E}}_I$ by \blacktriangleright^I . We can then establish the following essential properties about the interaction of the later modalities and (co)products, which are analogue to those in [8, cf. Thm. 2.7]. This theorem allows one to distribute in proofs quantifiers over the later modality.

Theorem 24. *The following holds for fibred products and coproducts in $\overline{\mathbf{p}}$.*

- There is an isomorphism $\blacktriangleright^J \circ \prod_{\overline{f}} \cong \prod_{\overline{f}} \circ \blacktriangleright^I$.
- There is a natural transformation $\iota: \prod_{\overline{f}} \circ \blacktriangleright^I \Rightarrow \blacktriangleright^J \circ \prod_{\overline{f}}$. Moreover, if f is inhabited, that is, has a section $g: J \rightarrow I$, then ι has a section ι^g .

Proof. Establishing the sought-after isomorphism and ι is straightforward. The section ι^g of ι for a given $g: J \rightarrow I$ is can be defined by

$$\begin{aligned} \iota_{\sigma,0}^g &= \mathbf{1}_J \xrightarrow{\mathbf{1}_g} \mathbf{1}_I \xrightarrow{\eta} f^* \prod_f \mathbf{1}_I \xrightarrow{\overline{f} \prod_f \mathbf{1}_I} \prod_f \mathbf{1}_I \\ \iota_{\sigma,n+1}^g &= \prod_f \sigma_n \xrightarrow{\text{id}} \prod_f \sigma_n. \end{aligned}$$

That this is a right-inverse of ι follows from finality if $\mathbf{1}_J$. \square

Remark 25. It should be possible to establish in $\overline{\mathbf{E}}$ fibred products and coproducts along general morphisms of $\overline{\mathbf{B}}$. However, this is a much more difficult task, which will use ideas similar to those used in Thm. 5. Intuitively, the products that we established correspond to universal quantifiers over fixed sets, while general products would correspond to universal quantification over variable sets. The difference is analogous to that in Kripke models of (intuitionistic) first-order logic: Suppose $\mathcal{M} = (W, \leq, U)$ is a model, where \leq is a partial order on W and U an interpretation for the quantification domain. If U is merely a set, then the satisfaction \models relation is defined for universal quantification by

$$w, \rho \models \forall x. \varphi \iff \forall u \in U. w, \rho[x \mapsto u] \models \varphi.$$

However, if U is a family $U: W \rightarrow \mathbf{Set}$, then the interpretation of universal quantification involves a quantification over all successor worlds:

$$w, \rho \models \forall x. \varphi \iff \forall w \leq v. \forall u \in U(v). v, \rho[x \mapsto u] \models \varphi.$$

This means that if we want to lift products to general chains, then the fibred products will involve again a quantification over morphisms in the index category, and the product must also be given by an end, as we used it in the construction of exponents in Thm. 5. Since this construction is fairly involved and not necessary for our current purposes, we will leave such a construction aside for now. \blacktriangleleft

We finish this section by lifting also the construction of fixed points for contractive maps to fibrations.

Theorem 26. *For every $\sigma \in \overline{\mathbf{E}}_c$ there is a unique map in $\overline{\mathbf{E}}_c$, dinatural in σ ,*

$$\text{löb}_\sigma^c : \sigma^{\blacktriangleright\sigma} \rightarrow \sigma,$$

such that for all g -contractive maps f the map $\text{löb}_\sigma^c \circ \lambda_g$ is a solution for f .

Proof. We define $\text{löb}_n^c : (\sigma^{\blacktriangleright\sigma})_n \rightarrow \sigma_n$ again by iteration on n . For 0, we put

$$\text{löb}_0 := (\sigma^{\blacktriangleright\sigma})_0 \xrightarrow{\langle \text{id}, ! \rangle} (\sigma^{\blacktriangleright\sigma})_0 \times \mathbf{1}_{c_0} = (\sigma^{\blacktriangleright\sigma})_0 \times (\blacktriangleright\sigma)_0 \xrightarrow{\text{ev}_0} \sigma_0,$$

where ev is the counit of $(-)\times\blacktriangleright\sigma \dashv (-)^{\blacktriangleright\sigma}$. In the iteration step, we first define a morphism step_n as the mediating morphism in the following diagram.

$$\begin{array}{ccc} (\sigma^{\blacktriangleright^c\sigma})_{n+1} & \xrightarrow{(\sigma^{\blacktriangleright^c\sigma})(n \leq n+1)} & (\sigma^{\blacktriangleright^c\sigma})_n \\ \downarrow \text{step}_n & & \downarrow \text{löb}_n^c \\ (\blacktriangleright^c\sigma)_{n+1} = c(n \leq n+1)^* \sigma_n & \xrightarrow{c(n \leq n+1)\sigma_n} & \sigma_n \end{array} \xrightarrow{p} \begin{array}{ccc} c_{n+1} & \xrightarrow{c(n \leq n+1)} & c_n \\ \parallel & & \parallel \\ c_{n+1} & \xrightarrow{c(n \leq n+1)} & c_n \end{array}$$

The map löb_{n+1}^c is then given by

$$\text{löb}_{n+1}^c := (\sigma^{\blacktriangleright\sigma})_{n+1} \xrightarrow{\langle \text{id}, \text{step}_n \rangle} (\sigma^{\blacktriangleright\sigma})_{n+1} \times (\blacktriangleright^c\sigma)_{n+1} \xrightarrow{\text{ev}_{n+1}} \sigma_{n+1},$$

That löb^c is vertical, i.e., $\bar{p}(\text{löb}^c) = \text{id}$ is clear from the definition. The other properties follows like in Thm. 11. \square

5 The Final Chain and Up-To Techniques

Having laid the ground work, we come now to the actual objects of interest: coinductive predicates. We will proceed again in two steps, in that we first present coinductive predicates over arbitrary categories and then move to fibrations. The following captures the usual construction of the final chain.

Definition 27. *Let \mathbf{C} be a category with a final object and $\Phi : \mathbf{C} \rightarrow \mathbf{C}$ a functor. We define a chain $\overleftarrow{\Phi} \in \overline{\mathbf{C}}$ by*

$$\overleftarrow{\Phi}_0 = \mathbf{1} \quad \text{and} \quad \overleftarrow{\Phi}(m \leq n) = \begin{cases} !: \overleftarrow{\Phi}_n \rightarrow \mathbf{1}, & m = 0 \text{ or } n = 0 \\ \Phi(\overleftarrow{\Phi}(m' \leq n')), & m = m' + 1, n = n' + 1 \end{cases}$$

The following theorem will play a central role in recursive proofs, as it allows us to unfold $\overleftarrow{\Phi}$ and thereby to make progress in a recursive proof. Additionally, it tells us that $\overleftarrow{\Phi}$ is a fixed point of the functor $\blacktriangleright \circ \overline{\Phi}$, cf. [8, Thm. 2.14].

Theorem 28. *We have that $\overleftarrow{\Phi} = \blacktriangleright(\overline{\Phi}\overleftarrow{\Phi})$.*

Just as important as unfolding $\overleftarrow{\Phi}$ is the ability to remove contexts, use transitivity of relations etc. in a proof. Such properties can be captured through so-called compatible up-to techniques [10,27].

Theorem 29. *Let T and Φ be functors $\mathbf{C} \rightarrow \mathbf{C}$. If there is a natural transformation $\rho: T\Phi \Rightarrow \Phi T$, then there is a map $\overleftarrow{\rho}: \overline{T}\overleftarrow{\Phi} \rightarrow \overleftarrow{\Phi}$ in $\overline{\mathbf{C}}$.*

Proof. We define $\overleftarrow{\rho}_n$ by iteration on n :

$$\begin{aligned} \overleftarrow{\rho}_0 &= T\mathbf{1} \xrightarrow{!} \mathbf{1} \\ \overleftarrow{\rho}_{n+1} &= T\overleftarrow{\Phi}_n \xrightarrow{\rho^{\overleftarrow{\Phi}_n}} \Phi T\overleftarrow{\Phi}_n \xrightarrow{\overleftarrow{\Phi}\overleftarrow{\rho}_n} \overleftarrow{\Phi}\overleftarrow{\Phi}_n = \overleftarrow{\Phi}_{n+1} \end{aligned}$$

That $\overleftarrow{\rho}$ is a morphism in $\overline{\mathbf{C}}$ follows easily by induction, and by using naturality of ρ functoriality of $\overleftarrow{\Phi}$. \square

Remark 30. Pous and Rot [24] prove a result similar to Thm. 29, namely that a monotone function T on a complete lattice is below the companion of Φ if and only if there is a map $\overline{T}\overleftarrow{\Phi} \rightarrow \overleftarrow{\Phi}$. This result is equivalent to Thm. 29 because the companion itself is compatible. \blacktriangleleft

Let $\Delta_n: \mathbf{C} \rightarrow \mathbf{C}^n$ be the diagonal functor and put $\Phi^{\times n} = \underbrace{\Phi \times \dots \times \Phi}_{n\text{-times}}$. We

then obtain the following corollary of Thm. 29, which allows us its application to compatible up-to techniques that have n arguments. For example, the transitive closure of a relation requires 2 arguments, see [10] for details.

Corollary 31. *Let $n \in \mathbb{N}$ and $T: \mathbf{C}^n \rightarrow \mathbf{C}$ be a functor. If there is a natural transformation $\rho: T\Phi^{\times n} \Rightarrow \Phi T$, then there is a map $\overleftarrow{\rho}: \overline{T}(\Delta_n\overleftarrow{\Phi}) \rightarrow \overleftarrow{\Phi}$ in $\overline{\mathbf{C}}$.*

Let us now move to the setting of fibrations. For the remainder of this section, we assume to be given a functor $F: \mathbf{B} \rightarrow \mathbf{B}$ that describes the behaviour of coalgebras, and a lifting $G: \mathbf{E} \rightarrow \mathbf{E}$ of F that describes a predicate on F -coalgebras, see [17] for a more detailed introduction.

Assumption 32. *We assume to be given a map of fibrations $(F, G): p \rightarrow p$ and a coalgebra $c: X \rightarrow FX$ in \mathbf{B} . Moreover, we require that \mathbf{B} has a final object.*

Under these assumption, we can define a functor $\Phi: \mathbf{E}_X \rightarrow \mathbf{E}_X$ by

$$\Phi := c^* \circ G: \mathbf{E}_X \rightarrow \mathbf{E}_X,$$

which describes, what is often called, a *predicate transformer*. A coalgebra for Φ is then referred to as a Φ -*invariant*. One can now talk about up-to techniques for G and for Φ . Both kinds are related by the following result, which allows us to obtain compatible up-to techniques on fibres from global ones.

Theorem 33. *Let $T: \mathbf{E} \rightarrow \mathbf{E}$ be a lifting of the identity $\text{Id}_{\mathbf{E}}$. If there is natural transformation $\rho: TG \Rightarrow GT$ with $P\rho = \text{id}F: F \Rightarrow F$, then there is a natural transformation $\rho^c: T\Phi \Rightarrow \Phi T$ with $P\rho^c = \text{id}: \text{Id} \Rightarrow \text{Id}$.*

Similarly, one obtains also a descending chain for G .

Lemma 34. *Let (F, G) be a lifting to $p: \mathbf{E} \rightarrow \mathbf{B}$. Then $\overleftarrow{G} \in \overline{\mathbf{E}}_{\overleftarrow{F}}$.*

The global chain \overleftarrow{G} is again related to the local one $\overleftarrow{\Phi}$ as follows. From the coalgebra $c: X \rightarrow FX$, we define a morphism $\overleftarrow{c}: K_X \rightarrow \overleftarrow{F}$ in $\overline{\mathbf{E}}$ iteratively by

$$\overleftarrow{c}_0 = !_X: X \rightarrow \mathbf{1} \quad \text{and} \quad \overleftarrow{c}_{n+1} = X \xrightarrow{c} FX \xrightarrow{F\overleftarrow{c}_n} \overleftarrow{F}_n.$$

Using \overleftarrow{c} , we can relate the global and local chains.

Proposition 35. *In $\overline{\mathbf{E}}_X$, we can find isomorphisms*

- $\overleftarrow{c}^*(\overleftarrow{G}) \cong \overleftarrow{\Phi}$ and
- $(\text{next} \circ \overleftarrow{c})^*(\blacktriangleright \overleftarrow{G}) \cong \blacktriangleright^X \overleftarrow{\Phi}$.

From Prop. 35, we can obtain an alternative proof of one of the central results (Thm. 3.7.i) by Hasuo et al. [17].

Corollary 36. *We have $\lim \overleftarrow{\Phi} \cong c_\omega^*(\lim \overleftarrow{G})$, where $c_\omega: X \rightarrow \lim \overleftarrow{F}$ is the unique map induced by \overleftarrow{c} and the limit property.*

Proof. We have

$$\begin{aligned} \lim \overleftarrow{\Phi} &\cong \lim(\overleftarrow{c}^* \overleftarrow{G}) && \text{(By Prop. 35)} \\ &\cong \lim(\overleftarrow{\pi} \circ c_\omega^* \overleftarrow{G}) && \text{(Def. of } c_\omega) \\ &\cong \lim(\overleftarrow{c}_\omega^*(\overleftarrow{\pi}^* \overleftarrow{G})) && \text{(Cloven fibration)} \\ &\cong c_\omega^*(\lim(\overleftarrow{\pi}^* \overleftarrow{G})) && \text{Fibred limits} \\ &\cong c_\omega^*(\lim \overleftarrow{G}) && \text{([17, Lem. 3.5])} \end{aligned}$$

□

If the chain $\overleftarrow{\Phi}$ converges in ω steps, then we obtain soundness and completeness for proofs given over $\overleftarrow{\Phi}$. This result is a trivial reformulation of the usual construction of final coalgebra. However, the present formulation is more convenient in the context of the recursive proofs that we construct by appealing to the later modality, as those will be maps in $\overline{\mathbf{E}}_X$.

Proposition 37. *Suppose $\nu\Phi$ is a coinductive predicate, that is, there is a final coalgebra $\nu\Phi \xi: \nu\Phi \rightarrow \Phi(\nu\Phi)$. If Φ preserves ω^{op} -limits, then maps $A \rightarrow \nu\Phi$ in \mathbf{E}_X are given equivalently by maps $K_A \rightarrow \overleftarrow{\Phi}$ in $\overline{\mathbf{E}}_X$.*

6 Examples

In this last section, we demonstrate how the framework that we developed can be used to obtain recursive proofs for coinductive predicates over different kinds of first-order logic. The first example is thereby in the setting of set-based predicates.

Example 38. In this example, we define a predicate on streams that expresses that a real-valued stream is greater than 0 everywhere and use the developed framework to prove that a certain stream is in the predicate. This example is fairly straightforward, but still has all the ingredients to illustrate the framework.

Let $F: \mathbf{Set} \rightarrow \mathbf{Set}$ and $G: \mathbf{Pred} \rightarrow \mathbf{Pred}$ be given by $F = \mathbb{R} \times \text{Id}$ and $G(X, P) = (FX, \{(a, x) \mid a > 0 \wedge x \in P\})$. It is easy to show that G is a lifting of F , and we obtain the predicate of streams that are larger than 0 everywhere as the final coalgebra of the functor $\Phi: \mathbf{Pred}_{\mathbb{R}^\omega} \rightarrow \mathbf{Pred}_{\mathbb{R}^\omega}$ with $\Phi = \langle \text{hd}, \text{tl} \rangle^* \circ G$.

Next, we define for $a \in \mathbb{R}$ the constant stream a^ω by the following stream differential equation (SDE) [16].

$$a_0^\omega = a \quad (a^\omega)' = a^\omega$$

Similarly, we can define the point-wise addition of streams by

$$(s \oplus t)_0 = s_0 + t_0 \quad (s \oplus t)' = s' \oplus t'.$$

Finally, let $s \in \mathbb{R}^\omega$ be given by the following SDE.

$$s_0 = 1 \quad s' = 1^\omega \oplus s.$$

Our goal is to prove that s is greater than 0 everywhere, that is, we want to prove that s is in the final coalgebra $\nu\Phi$ of the above Φ . Since the tail s' of s defined of $1^\omega \oplus -$, the following up-to technique will be handy. Let us define $C: \mathbf{Pred}_{\mathbb{R}^\omega} \rightarrow \mathbf{Pred}_{\mathbb{R}^\omega}$ to be

$$C(P) = \{1^\omega \oplus t \mid t \in P\}.$$

One easily shows that C is Φ -compatible, that is, $C\Phi \subseteq \Phi C$. In fact, this follows from point-wise addition being causal, see [26,24]. Thus, we have by Thm. 29 that $\overline{C\Phi} \subseteq \overleftarrow{\Phi}$, where \subseteq is the point-wise inclusion of indexed predicates.

Given an indexed predicate $\sigma \in \overline{\mathbf{Pred}}_X$, we define

$$\vdash \sigma := \overline{\mathbf{I}}_X \subseteq \sigma.$$

Hence, $\vdash \sigma$ holds if there is a morphism $\overline{\mathbf{I}}_X \rightarrow \sigma$ in $\overline{\mathbf{Pred}}_X$. Given $x \in X$, we define the predicate $x \overline{\in} \sigma$ in $\overline{\mathbf{Pred}}_X$ to be the following exponential in $\overline{\mathbf{Pred}}_X$.

$$x \overline{\in} \sigma := \sigma^{K_{\{x\}}}.$$

Spelling out these definitions, one easily finds that

$$\vdash x \overline{\in} \sigma \iff \forall n \in \mathbb{N}. x \in \sigma_n.$$

For brevity, let us write $\varphi := s \bar{\in} \overleftarrow{\Phi}$ and \blacktriangleright for $\blacktriangleright^{\mathbb{R}^\omega}$. Using the previous results, we now obtain a proof for $\vdash \varphi$ as follows, where each proof step is given applying the indicated construction in $\overline{\text{Pred}}_{\mathbb{R}^\omega}$.

$$\begin{array}{c}
\frac{}{\blacktriangleright \varphi \vdash \blacktriangleright (s \bar{\in} \overleftarrow{\Phi})} \text{ (Identity)} \\
\frac{}{\blacktriangleright \varphi \vdash \blacktriangleright (1^\omega \oplus s \bar{\in} \overline{C}(\overleftarrow{\Phi}))} \text{ (Def. } C) \\
\frac{}{\blacktriangleright \varphi \vdash \blacktriangleright (1^\omega \oplus s \bar{\in} \overleftarrow{\Phi})} \text{ (} C \text{ compatible)} \\
\frac{}{\blacktriangleright \varphi \vdash \blacktriangleright (s' \bar{\in} \overleftarrow{\Phi})} \text{ (Def. of } s) \\
\frac{s_0 > 0}{\blacktriangleright \varphi \vdash \blacktriangleright (s' \bar{\in} \overleftarrow{\Phi})} \text{ (} \blacktriangleright \text{ pres. products)} \\
\frac{}{\blacktriangleright \varphi \vdash \blacktriangleright (s \bar{\in} \overline{\Phi}(\overleftarrow{\Phi}))} \text{ (} \blacktriangleright \text{ functor)} \\
\frac{}{\blacktriangleright \varphi \vdash s \bar{\in} \blacktriangleright (\overline{\Phi}(\overleftarrow{\Phi}))} \text{ (Step, Thm. 28)} \\
\frac{}{\vdash \varphi} \text{ (L\"ob)}
\end{array}$$

Thus, we have obtained a proof that s is greater than 0 everywhere purely by applying the category theoretical constructions presented in this paper. \blacktriangleleft

The next example shows that the same category theoretical setup that we used to prove something above, can also be used to define functions.

Example 39. Given a set A , we define a functor F and a lifting G^A to the family fibration $\text{Fam}(\mathbf{Set}) \rightarrow \mathbf{Set}$ as follows.

$$\begin{array}{ll}
F: \mathbf{Set} \rightarrow \mathbf{Set} & G^A: \text{Fam}(\mathbf{Set}) \rightarrow \text{Fam}(\mathbf{Set}) \\
F = \mathbf{1} + \text{Id} & G^A(I, X)_{u \in \mathbf{1} + I} = \begin{cases} \mathbf{1}, & u = \kappa_1 \langle \rangle \\ A \times X_v, & u = \kappa_2 v \end{cases}
\end{array}$$

F has as final coalgebra the predecessor function $\text{pred}: \mathbb{N}^\infty \rightarrow \mathbf{1} + \mathbb{N}^\infty$ on the natural numbers extended with one element that indicates infinity. The family of so-called partial streams PStr_A [4] is the final coalgebra of $\Phi^A = \text{pred}^* \circ G^A$. Our goal is now to define for a given $f: A \rightarrow B$ a map $\text{PStr}(f): \text{PStr}_A \rightarrow \text{PStr}_B$. Unfortunately, the results in [17] do not apply here. But one can still show that Φ^A preserves ω^{op} -limits, hence maps into PStr_A are equivalently given by maps into the chain $\overleftarrow{\Phi}^A$. Hence, we can obtain $\text{PStr}(f)$ equivalently as a map $\overleftarrow{\Phi}^A \rightarrow \overleftarrow{\Phi}^A$ in $\overline{\text{Fam}}(\mathbf{Set})_{\mathbb{N}^\infty}$. Denoting by \Rightarrow the exponential in this fibre, we can construct the desired map by applying the following “proof” steps, where we write $u \mid \text{pred}u = t \vdash X \rightarrow Y$ if we construct a map in $\text{Fam}(\mathbf{Set})_{\mathbb{N}^\infty}$ with the

constraint that the index $u \in \mathbb{N}^\infty$ fulfils $\text{pred}(u) = t$.

$$\begin{array}{c}
\frac{u \mid \text{pred } u = \kappa_1 \langle \rangle \vdash \blacktriangleright! : (\blacktriangleright \overleftarrow{\Phi^A} \Rightarrow \blacktriangleright \overleftarrow{\Phi^B})_u \times \overleftarrow{\Phi^A}_u \rightarrow \blacktriangleright \mathbf{1}}{(*)} \text{Cases for } G \\
\frac{u \vdash (\blacktriangleright \overleftarrow{\Phi^A} \Rightarrow \blacktriangleright \overleftarrow{\Phi^B})_u \times \overleftarrow{\Phi^A}_u \rightarrow \blacktriangleright (\overleftarrow{\Phi^A} \overleftarrow{\Phi^B})_u}{\text{Index abstraction}} \\
\frac{(\blacktriangleright \overleftarrow{\Phi^A} \Rightarrow \blacktriangleright \overleftarrow{\Phi^B}) \times \overleftarrow{\Phi^A} \rightarrow \blacktriangleright \overleftarrow{\Phi^A} \overleftarrow{\Phi^B}}{\text{Step}} \\
\frac{(\blacktriangleright \overleftarrow{\Phi^A} \Rightarrow \blacktriangleright \overleftarrow{\Phi^B}) \times \overleftarrow{\Phi^A} \rightarrow \overleftarrow{\Phi^B}}{\text{Abstraction}} \\
\frac{(\blacktriangleright \overleftarrow{\Phi^A} \Rightarrow \blacktriangleright \overleftarrow{\Phi^B}) \rightarrow (\overleftarrow{\Phi^A} \Rightarrow \overleftarrow{\Phi^B})}{\blacktriangleright \text{ functor}} \\
\frac{\blacktriangleright (\overleftarrow{\Phi^A} \Rightarrow \overleftarrow{\Phi^B}) \rightarrow (\overleftarrow{\Phi^A} \Rightarrow \overleftarrow{\Phi^B})}{\text{Löb}} \\
\frac{\mathbf{1} \rightarrow (\overleftarrow{\Phi^A} \Rightarrow \overleftarrow{\Phi^B})}{\text{Uncurry}} \\
\overleftarrow{\Phi^A} \rightarrow \overleftarrow{\Phi^B}
\end{array}$$

The step (*) is thereby given as follows, where we write S for $\blacktriangleright \overleftarrow{\Phi^A} \Rightarrow \blacktriangleright \overleftarrow{\Phi^B}$.

$$\begin{array}{c}
\blacktriangleright (f \circ \pi_1 \circ \pi_2) : S_u \times (\blacktriangleright A \times \blacktriangleright \overleftarrow{\Phi^A}_v) \rightarrow \blacktriangleright B \quad \text{and} \\
\text{ev} \circ (\text{id} \times \pi_2) : S_u \times (\blacktriangleright A \times \blacktriangleright \overleftarrow{\Phi^A}_v) \rightarrow \blacktriangleright \overleftarrow{\Phi^B}_v \\
\frac{u \mid \text{pred } u = \kappa_2 v \vdash S_u \times (\blacktriangleright A \times \blacktriangleright \overleftarrow{\Phi^A}_v) \rightarrow \blacktriangleright B \times \blacktriangleright \overleftarrow{\Phi^B}_v}{\text{Pairing}} \\
\frac{u \mid \text{pred } u = \kappa_2 v \vdash S_u \times (\blacktriangleright A \times \blacktriangleright \overleftarrow{\Phi^A}_v) \rightarrow \blacktriangleright (B \times \overleftarrow{\Phi^B}_v)}{\blacktriangleright \text{ pres. } \times} \\
\frac{u \mid \text{pred } u = \kappa_2 v \vdash S_u \times \overleftarrow{\Phi^A}_u \rightarrow \blacktriangleright (B \times \overleftarrow{\Phi^B}_v)}{\text{Unfold}}
\end{array}$$

7 Conclusion and Future Work

In this paper, we have established a framework that allows us to reason about coinductive predicates in many cases by using recursive proofs. At the heart of this approach sits the so-called later modality, which was comes from provability logic [6,32,33] but was later used to obtain guarded recursion in type theories [2,3,9,21] and in domain theory [7,8]. This modality allows us to control the recursion steps in a proof without having to invoke parity or similar conditions [12,14,28,31], as we have seen in the examples in Sec. 6. Moreover, even though similar Birkedal et al. [8] obtained similar results, their framework is limited to **Set**-valued presheaves, while our results are applicable in a much wider range of situations, see the examples in Sec. 4.

So what is there left to do? For once, we have not touched upon how to automatically extract a syntactic logic and models from the fibration $\overline{L} \rightarrow \overline{C}$ obtained in Ex. 17. This would subsume and simplify much of the development in [4]. Next, we discussed already in Rem. 25 that the construction of fibre products for general morphisms in fibrations of descending chains is fairly involved. However, such a construction would be useful, for example, to obtain Kripke models abstractly. Finally, also a closer analysis of the relation to proof systems obtained through parameterised coinduction, the companion or cyclic proof systems would be interesting.

References

1. Abel, A., Pientka, B.: Wellfounded recursion with copatterns: A unified approach to termination and productivity. In: ICFP. pp. 185–196 (2013), <http://doi.acm.org/10.1145/2500365.2500591>
2. Appel, A.W., Melliès, P.A., Richards, C.D., Vouillon, J.: A very modal model of a modern, major, general type system. In: POPL. pp. 109–122. ACM (2007)
3. Atkey, R., McBride, C.: Productive coprogramming with guarded recursion. In: ICFP. pp. 197–208. ACM (2013), <http://bentnib.org/productive.pdf>
4. Basold, H.: Mixed Inductive-Coinductive Reasoning: Types, Programs and Logic. Phd thesis, Radboud University, Nijmegen (To be published), <http://www.cs.ru.nl/~hbasold/publications/Thesis.pdf>
5. Basold, H., Pous, D., Rot, J.: Monoidal Company for Accessible Functors. In: CALCO 2017. LIPIcs, vol. 72. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik (2017), <http://www.cs.ru.nl/~hbasold/publications/MonoidalCompanyAccessible.pdf>
6. Beklemishev, L.D.: Parameter Free Induction and Provably Total Computable Functions. *Theor. Comput. Sci.* 224(1-2), 13–33 (1999), [http://dx.doi.org/10.1016/S0304-3975\(98\)00305-3](http://dx.doi.org/10.1016/S0304-3975(98)00305-3)
7. Birkedal, L., Møgelberg, R.E.: Intensional Type Theory with Guarded Recursive Types qua Fixed Points on Universes. In: LICS. pp. 213–222. IEEE Computer Society (2013)
8. Birkedal, L., Møgelberg, R.E., Schwinghammer, J., Støvring, K.: First steps in synthetic guarded domain theory: Step-indexing in the topos of trees. *Logical Methods in Computer Science* 8(4) (2012), [http://dx.doi.org/10.2168/LMCS-8\(4:1\)2012](http://dx.doi.org/10.2168/LMCS-8(4:1)2012)
9. Bizjak, A., Grathwohl, H.B., Clouston, R., Møgelberg, R.E., Birkedal, L.: Guarded Dependent Type Theory with Coinductive Types. In: FoSSaCS. *Lecture Notes in Computer Science*, vol. 9634, pp. 20–35. Springer (2016), <https://arxiv.org/abs/1601.01586>
10. Bonchi, F., Petrişan, D., Pous, D., Rot, J.: Coinduction Up-to in a Fibrational Setting. In: Proc. of CSL-LICS '14. pp. 20:1–20:9. ACM, New York, USA (2014), <http://doi.acm.org/10.1145/2603088.2603149>
11. Brotherston, J.: Cyclic Proofs for First-Order Logic with Inductive Definitions. In: Beckert, B. (ed.) *Proceedings of TABLEAUX 2005. Lecture Notes in Computer Science*, vol. 3702, pp. 78–92. Springer (2005), http://dx.doi.org/10.1007/11554554_8
12. Brotherston, J., Simpson, A.: Complete Sequent Calculi for Induction and Infinite Descent. In: *Proceedings of LICS 2007*. pp. 51–62. IEEE Computer Society (2007), <http://dx.doi.org/10.1109/LICS.2007.16>
13. Dax, C., Hofmann, M., Lange, M.: A Proof System for the Linear Time μ -Calculus. In: Arun-Kumar, S., Garg, N. (eds.) *Proceedings of FSTTCS 2006. LNCS*, vol. 4337, pp. 273–284. Springer (2006), http://dx.doi.org/10.1007/11944836_26
14. Fortier, J., Santocanale, L.: Cuts for circular proofs: Semantics and cut-elimination. In: CSL. pp. 248–262 (2013), <http://dblp.uni-trier.de/rec/bibtex/conf/csl/FortierS13>
15. Giménez, E.: Codifying Guarded Definitions with Recursive Schemes. In: *Selected Papers from the TYPES '94 Workshop*. pp. 39–59. Springer-Verlag, London, UK (1995)
16. Hansen, H.H., Kupke, C., Rutten, J.: Stream Differential Equations: Specification Formats and Solution Methods. *LMCS* 13(1) (2017), [https://doi.org/10.23638/LMCS-13\(1:3\)2017](https://doi.org/10.23638/LMCS-13(1:3)2017)

17. Hasuo, I., Cho, K., Kataoka, T., Jacobs, B.: Coinductive Predicates and Final Sequences in a Fibration. *Electronic Notes in Theoretical Computer Science* 298, 197–214 (Nov 2013), <http://www.sciencedirect.com/science/article/pii/S1571066113000601>
18. Hur, C.K., Neis, G., Dreyer, D., Vafeiadis, V.: The Power of Parameterization in Coinductive Proof. In: *Proceedings of POPL 2013*. pp. 193–206. POPL '13, ACM, New York, NY, USA (2013), <http://doi.acm.org/10.1145/2429069.2429093>
19. Jacobs, B.: *Categorical Logic and Type Theory*. No. 141 in *Studies in Logic and the Foundations of Mathematics*, North Holland, Amsterdam (1999)
20. Kelly, M.: *Basic Concepts of Enriched Category Theory*. No. 64 in *Lecture Notes in Mathematics*, Cambridge University Press, reprints in theory and applications of categories, no. 10 (2005) edn. (1982)
21. Nakano, H.: A Modality for Recursion. In: *LICS*. pp. 255–266. IEEE Computer Society (2000), <http://www602.math.ryukoku.ac.jp/~nakano/papers/modality-lics00.ps>
22. Niwinski, D., Walukiewicz, I.: Games for the μ -Calculus. *Theor. Comput. Sci.* 163(1&2), 99–116 (1996), [http://dx.doi.org/10.1016/0304-3975\(95\)00136-0](http://dx.doi.org/10.1016/0304-3975(95)00136-0)
23. Pous, D.: Coinduction All the Way Up. In: Grohe, M., Koskinen, E., Shankar, N. (eds.) *Proceedings of LICS '16*. pp. 307–316. ACM (2016), <http://doi.acm.org/10.1145/2933575.2934564>
24. Pous, D., Rot, J.: Companions, Codensity, and Causality. In: *Proceedings of FOSSACS 2017* (2017)
25. Roşu, G., Lucanu, D.: Circular Coinduction: A Proof Theoretical Foundation. In: *CALCO*. *Lecture Notes in Computer Science*, vol. 5728, pp. 127–144. Springer (2009)
26. Rot, J.: *Enhanced Coinduction*. Phd, University Leiden, Leiden (2015)
27. Rot, J., Bonchi, F., Bonsangue, M., Pous, D., Rutten, J., Silva, A.: *Enhanced Coalgebraic Bisimulation*. *MSCS To Appear* (2014), <http://www.liacs.nl/~jrot/up-to.pdf>
28. Santocanale, L.: A Calculus of Circular Proofs and Its Categorical Semantics. In: *FoSSaCS*. pp. 357–371 (2002), <http://dblp.uni-trier.de/rec/bibtex/conf/fossacs/Santocanale02>
29. Santocanale, L.: μ -Bicomplete Categories and Parity Games. *RAIRO - ITA* 36(2), 195–227 (2002), <http://dblp.uni-trier.de/rec/bibtex/journals/ita/Santocanale02>
30. Shulman, M.: *Exponentials of cartesian closed categories* (2012), https://ncatlab.org/nlab/show/cartesian+closed+category#exponentials_of_cartesian_closed_categories
31. Simpson, A.: Cyclic Arithmetic is Equivalent to Peano Arithmetic. In: *Proceedings of FoSSaCS'17* (2017)
32. Smoryński, C.: *Self-Reference and Modal Logic*. Universitext, Springer-Verlag (1985)
33. Solovay, R.M.: Provability interpretations of modal logic. *Israel Journal of Mathematics* 25(3), 287–304 (1976), <http://dx.doi.org/10.1007/BF02757006>
34. Street, R., Verity, D.: The comprehensive factorization and torsors. *Theory and Applications of Categories* 23(3), 42–75 (2010)
35. Streicher, T.: *Fibred Categories à la Jean Bénabou*. arXiv:math.CT/1801.02927 (2018)