Dependent Inductive-Coinductive Types

Category Theoretical and Syntactic Perspectives

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Joint work with Herman Geuvers

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Outline

Upcoming

Introduction

What bothered me for a while

- No type theory with arbitrary inductive-coinductive, dependent types.
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That led me to

- Extract the categorical principles behind such types.
- Build a type theory resembling the categorical principles.

Plan for this Talk

- Present dependent inductive-coinductive types as dialgebras
- Construct models for these types through polynomial functors
- Give a syntactic calculus that implements this idea
- Prove strong normalisation for that calculus
- Consider an extension with extensional identity types

Upcoming

Example

data Vec (A : **Set**) : $\mathbb{N} \to$ **Set where**
  nil : $\top \to$ Vec 0
  cons : (n : $\mathbb{N}$) $\to$ A $\times$ Vec $n \to$ Vec $(n+1)$

Interpret Vec as set family

- Lists of length n over A: $A^n = \underbrace{A \times \cdots \times A}_{n \text{ times}}$
- Vector data type: $\mathrm{Vec}\, A = \{A^n\}_{n \in \mathbb{N}}$
- $\mathrm{nil} = \{\mathrm{nil}_*\} : \{\mathbf{1}\}_{* \in \mathbf{1}} \to \{A^0\}_{* \in \mathbf{1}}$
- $\mathrm{cons} = \{\mathrm{cons}_n\}_{n \in \mathbb{N}} : \{A \times A^n\}_{n \in \mathbb{N}} \to \{A^{n+1}\}_{n \in \mathbb{N}}$

Dependent Data Types are Initial/Final Dialgebras

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Vectors as dialgebra

Let $F, G : \mathbf{Set}^{\mathbb{N}} \to \mathbf{Set}^{\mathbf{1}} \times \mathbf{Set}^{\mathbb{N}}$ be functors to the product category

$$F(X) = (\{\mathbf{1}\}, \{A \times X_n\}_{n\in\mathbb{N}}) \text{ and}$$
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Tatsuyo Hagino

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Then $(\mathsf{nil}, \mathsf{cons}) : F(\mathrm{Vec}\, A) \to G(\mathrm{Vec}\, A)$ is the initial (F, G)-dialgebra.

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Dependent, inductive data type is an initial dialgebra in a (cloven) fibration $P : \mathbf{E} \to \mathbf{B}$ with

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- An initial dialgebra $(c_1, \ldots, c_n) : F(A) \to G_u(A)$, where

$$G_u = \langle u_1^*, \cdots, u_n^* \rangle$$

  substitutes u_k in a type.

Phew, an example after this long list

Example (Vectors)

Recall that $(\mathsf{nil}, \mathsf{cons}) : F(\mathrm{Vec}\, A) \to G(\mathrm{Vec}\, A)$ is the initial initial (F, G)-dialgebra for

$$F(X) = (\{\mathbf{1}\}, \{A \times X_n\}_{n \in \mathbb{N}}) \text{ and}$$
$$G(X) = (\{X_0\}, \{X_{n+1}\}_{n \in \mathbb{N}})$$

We have $G = G_u$ with

$$u_1 = z : \mathbf{1} \to \mathbb{N} \qquad z(*) = 0$$
$$u_2 = s : \mathbb{N} \to \mathbb{N} \qquad s(n) = n + 1$$

Giving us

$$u_1^*(X) = \{X_{z(i)}\}_{i \in \mathbf{1}} = \{X_0\}_{i \in \mathbf{1}}$$
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Dualise for coinductive data types

- The functors F and G_u swap roles
- Substitutions in the domain of the destructors
- Can block application

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Example (Partial Streams)

codata PStr (A : **Set**) : $\mathbb{N}^\infty \to$ **Set where**
  hd : (n : $\mathbb{N}^\infty$) $\to$ PStr (s_∞ n) $\to$ A
  tl : (n : $\mathbb{N}^\infty$) $\to$ PStr (s_∞ n) $\to$ PStr n

where $\mathbb{N}^\infty$ are natural numbers with infinity.

$$G_u, F : \mathbf{P}_{\mathbb{N}^\infty} \to \mathbf{P}_{\mathbb{N}^\infty} \times \mathbf{P}_{\mathbb{N}^\infty}$$
$$G_u = \langle s^*_\infty, s^*_\infty \rangle \qquad F = \langle K_A^{\mathbb{N}^\infty}, \mathrm{Id} \rangle$$

How to obtain models?

Theorem

Dependent, strictly positive data types can be interpreted in a locally Cartesian closed category with dependent products and disjoint coproducts if it has initial algebras for polynomial functors.

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Proof ingredients

- Reduce dialgebras to algebras/coalgebras
- Dependent polynomial functors are closed under taking fixed points
- Fixed points of dependent polynomials can be constructed from non-dependent
- Final coalgebras of polynomials can be constructed from initial algebras

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For details see

Dependent Inductive and Coinductive Types are Fibrational Dialgebras, H.B., FICS 2015.

Upcoming

Handling dependencies

Parameter contexts and instantiations

$$\Theta \mid \Gamma_1 \vdash A : \Gamma_2 \multimap * ,$$

Suppose that $\Gamma_2 = x_1 : B_1, \ldots, x_n : B_n$ and $\Gamma_1 \vdash t_k : B_k$, then

$$\Theta \mid \Gamma_1 \vdash A \mathbin{@} t_1 \mathbin{@} \cdots \mathbin{@} t_n : * .$$

Type constructor variables

$$X : \Gamma_2 \multimap * \mid \Gamma_1 \vdash X : \Gamma_2 \multimap * ,$$

which we can instantiate to

$$X : \Gamma_2 \multimap * \mid \Gamma_1 \vdash X \mathbin{@} t_1 \mathbin{@} \cdots \mathbin{@} t_n : * .$$

Forming types

$$\frac{k = 1, \ldots, n \qquad \sigma_k : \Gamma_k \to \Gamma \qquad \Theta, X : \Gamma \multimap * \mid \Gamma_k \vdash A_k : *}{\Theta \mid \emptyset \vdash \rho(X : \Gamma \multimap * ; \vec{\sigma}; \vec{A}) : \Gamma \multimap *}$$

- $n \in \mathbb{N}$
- $\rho \in \{\mu, \nu\}$
- $\sigma_k : \Gamma_k \to \Gamma$ substitution of terms in context Γ_k for variables in Γ
- A_k type with extra free variable

Examples I/II

Example (Vectors)

data Vec (A : **Set**) : $\mathbb{N} \to$ **Set where**
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$$\mathsf{Vec}\, A := \mu(X : \Gamma \rightarrowtriangle * ; (\sigma_1, \sigma_2); (\mathbf{1}, A \times X \mathbin{@} k))$$

$$\Gamma = n : \mathsf{Nat} \quad \text{and} \quad \Gamma_1 = \emptyset \quad \text{and} \quad \Gamma_2 = k : \mathsf{Nat}$$

$$\sigma_1 = (0) : \Gamma_1 \to (n : \mathsf{Nat}) \quad \text{and} \quad \sigma_2 = (s \mathbin{@} k) : \Gamma_2 \to (n : \mathsf{Nat})$$

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Examples II/II

- Π is a coinductive type
- Existential quantification is its dual
- All of intuitionistic predicate logic can be represented
- Partial streams

Forming terms

For inductive types

$$\frac{\vdash \mu(X : \Gamma \to * ; \vec{\sigma}; \vec{A}) : \Gamma \to * \qquad 1 \leq k \leq n}{\vdash \alpha_k^{\mu(X:\Gamma\to *;\vec{\sigma};\vec{A})} : (\Gamma_k, y : A_k[\mu/X]) \to \mu \mathbin{@} \sigma_k}\ \textbf{(Ind-I)}$$

$$\frac{\vdash C : \Gamma \to * \qquad \Delta, \Gamma_k, y_k : A_k[C/X] \vdash g_k : (C \mathbin{@} \sigma_k) \qquad \forall k = 1, \ldots, n}{\Delta \vdash \mathsf{rec}\ \overrightarrow{(\Gamma_k, y_k).\, g_k} : (\Gamma, y : \mu \mathbin{@} \mathsf{id}_\Gamma) \to C \mathbin{@} \mathsf{id}_\Gamma}$$

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Dual for coinductive types

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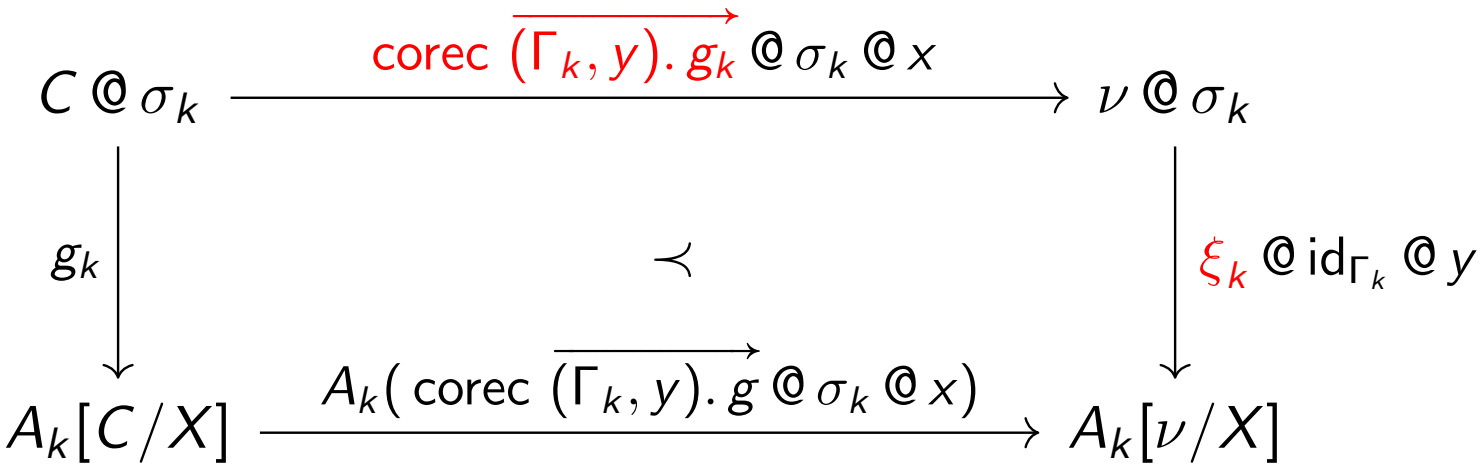
Computations

Reduction

- Defined as closure of contraction relation
- Essentially follows homomorphism diagrams
- Preserves types



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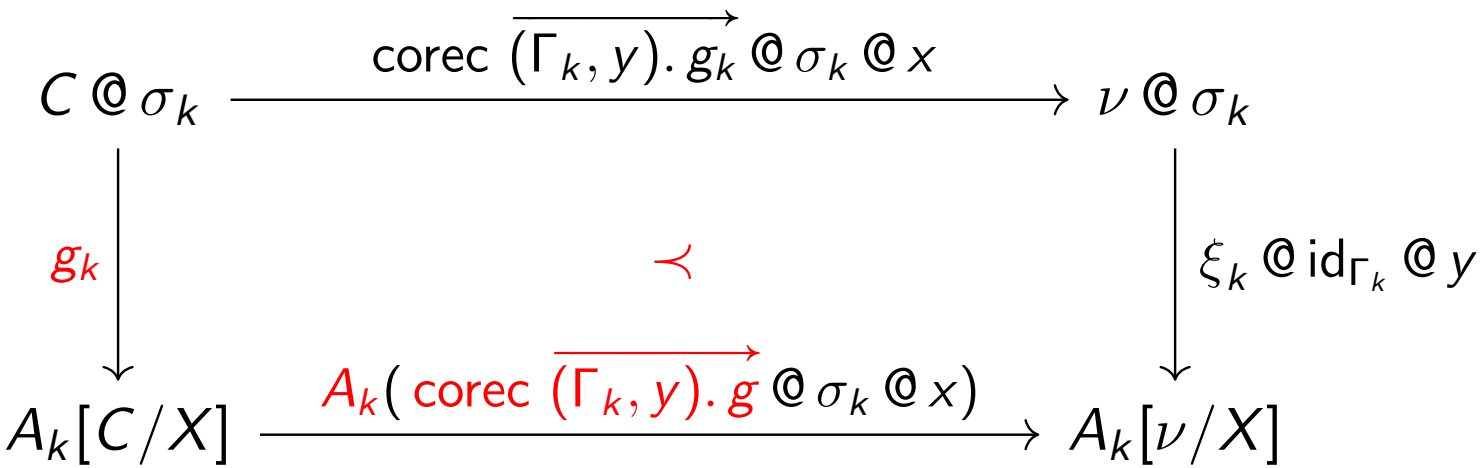

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For details see

'Type Theory based on Dependent Inductive and Coinductive Types', H.B. and H. Geuvers, in LICS'16, 2016 and arXiv:CS.LO/1605.02206 for proofs.

Upcoming

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The Fix

- Blend Observational Type Theory, Cubical TT and Dependent Inductive-Coinductive Types (inspired by Conor McBride)
- See my talk at TYPES'16 and, hopefully, my thesis

Blending OTT, Cubical TT and Dependent Inductive-Coinductive Types

Thank you for the inspiration Conor![1]

[1] McBride, that is. *A Cubical Crossroads* @ Agda Implementors' Meeting XXIII.

Plan

- Remove possibility to introduce propositional equality
- Paths between types through abstraction over points in interval (Cubical)
- Coercion operator (OTT)
- Heterogeneous path type (OTT)
- Coinduction: Constructor for paths

Removing Equality

Easy: Replace ...

$$\frac{k = 1, \ldots, n \qquad \sigma_k : \Gamma_k \to \Gamma \qquad \Theta, X : \Gamma \rightarrowtriangle * \mid \Gamma_k \vdash A_k : *}{\Theta \mid \emptyset \vdash \rho(X : \Gamma \rightarrowtriangle * ; \vec{\sigma}; \vec{A}) : \Gamma \rightarrowtriangle *}$$

... by

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Interval, Paths between Types and Coercion

$$\frac{}{\Gamma \vdash 0 : \mathbb{I}} \qquad \frac{}{\Gamma \vdash 1 : \mathbb{I}} \qquad \frac{i \in \Delta}{\Gamma \mid \Delta \vdash i : \mathbb{I}} \qquad \frac{\vdash p, q, r : \mathbb{I}}{\vdash p(q - r) : \mathbb{I}}$$

$$\frac{\Gamma \vdash S : * \qquad \Gamma \vdash T : *}{\Gamma \vdash S \sim T : *} \qquad \frac{\vdash Q : S \sim T \qquad \vdash p : \mathbb{I}}{\vdash Q\, p : *}$$

$$\frac{\Gamma \mid \Delta, i : \mathbb{I} \vdash M : * \qquad M[0/i] \equiv S \qquad M[1/i] \equiv T}{\Gamma \mid \Delta \vdash \langle i \rangle .\, M : S \sim T}$$

$$\frac{\vdash Q : S \sim T \qquad \vdash p, q : \mathbb{I} \qquad \vdash s : Q\, p}{\vdash s\, [p \mid Q \mid q\rangle : Q\, q}$$

Paths between terms

$$\frac{\vdash s : S \qquad \vdash t : T}{\vdash s \sim t : *} \qquad \frac{\vdash s, t : T \qquad \vdash Q : s \sim t \qquad \vdash p : \mathbb{I}}{\Gamma \vdash Q\, p : T}$$

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$$\frac{\vdash s : S \qquad \vdash t : T}{\vdash s \sim t : *} \qquad \frac{\vdash s, t : T \qquad \vdash Q : s \sim t \qquad \vdash p : \mathbb{I}}{\Gamma \vdash Q\, p : T}$$

$$\frac{\vdash t : T}{\vdash \mathsf{refl}_t : t \sim t}$$

Coinduction (for non-dependent types):

$$R : (x : \nu, y : \nu) \multimap *$$

$$\frac{x_k : \nu, y_k : \nu, z_k : R \mathbin{@} x_k \mathbin{@} y_k \vdash g_k : \overline{A_k}[R][\xi_k \mathbin{@} x_k / x, \xi_k \mathbin{@} y_k / y]}{\vdash \mathsf{coind}\ \overrightarrow{(x_k, y_k, z_k).\, g_k} : (x : \nu, y : \nu, z : R \mathbin{@} x \mathbin{@} y) \multimap x \sim y}$$

Paths between terms

$$\frac{\vdash s : S \qquad \vdash t : T}{\vdash s \sim t : *} \qquad \frac{\vdash s, t : T \qquad \vdash Q : s \sim t \qquad \vdash p : \mathbb{I}}{\Gamma \vdash Q\, p : T}$$

$$\frac{\vdash t : T}{\vdash \mathsf{refl}_t : t \sim t}$$

Coinduction (for non-dependent types):

$$R : (x : \nu, y : \nu) \to *$$

$$\frac{x_k : \nu, y_k : \nu, z_k : R \mathbin{@} x_k \mathbin{@} y_k \vdash g_k : \overline{A_k}[R][\xi_k \mathbin{@} x_k / x, \xi_k \mathbin{@} y_k / y]}{\vdash \mathsf{coind}\ \overrightarrow{(x_k, y_k, z_k).\, g_k} : (x : \nu, y : \nu, z : R \mathbin{@} x \mathbin{@} y) \to x \sim y}$$

Remark

Using bisimulations is rather brute-force. Maybe better: later modality.

Reductions

$$0(p - q) \succ p$$

$$1(p - q) \succ q$$

$$(\langle i \rangle . M)\, p \succ M[p/i]$$

$$\text{If } Q : s \sim t, \text{then } Q\,0 \succ s$$

$$\text{If } Q : s \sim t, \text{then } Q\,1 \succ t$$

$$s\,[0 \mid Q \mid 0\rangle \succ s$$

$$s\,[1 \mid Q \mid 1\rangle \succ s$$

$$(\alpha_k \mathbin{@} t)[0 \mid \langle i \rangle . \mu(X : * ; \overrightarrow{A[i]}) \mid 1\rangle \succ \alpha_k \mathbin{@} (t[0 \mid \langle i \rangle . A_k[i] \mid 1\rangle)$$

$$\left(\text{corec } \overrightarrow{(y_k) . g_k} \mathbin{@} t\right)[0 \mid \langle i \rangle . \nu(X : * ; \overrightarrow{A[i]}) \mid 1\rangle \succ (\text{corec } \overrightarrow{(y_k) . g'_k} \mathbin{@} t')$$

Example (Function Extensionality)

Define

$$R : (h : A \to B, k : A \to B) \rightarrowtriangle *$$
$$R := (h).\,(k).\,\Pi x : A.h\,x \sim k\,x$$

and

$$\frac{\dfrac{x : A, h, k : A \to B, z : R\,@\,h\,@\,k \vdash z : \Pi x : A.h\,x \sim k\,x}{x : A, h, k : A \to B, z : R\,@\,h\,@\,k \vdash z\,x : h\,x \sim k\,x}}{f, g, p : \Pi x : A.f\,x \sim g\,x \vdash \mathsf{coind}\,((x, h, k, z).\,z\,x)\,@\,f\,@\,g\,@\,p : f \sim g}$$

Thank you very much for your attention!