CHAPTER 11

FUNDAMENTAL THEOREMS

Reduction was introduced in order to analyze β-conversion. In part II we have seen already several applications of the Church–Rosser theorem and the standardization theorem. These and other important results on β-reduction will be proved in this chapter.

11.1. The Church–Rosser theorem

There are many ways to prove the Church–Rosser theorem. In §3.2 a short proof due to Tait and Martin-Löf was given. In this section a bit longer but more perspicuous proof is presented. These two proofs will be compared in §11.2.

In order to prove that $\twoheadrightarrow$ (i.e. $\twoheadrightarrow_\beta$) satisfies the diamond property, it suffices by lemma 3.2.2 to show that this is so for $\underset{=}{\rightarrow}$. However that is not true. The following lemma throws some light on the situation.

11.1.1. LEMMA. (i) *The relations* $\rightarrow$ *and* $\underset{=}{\rightarrow}$ *do not satisfy the diamond property.*

(ii) *The relation* $\rightarrow$ *satisfies the weak diamond property.*

PROOF. (i) Let $R \rightarrow R'$ be, say $\mathbf{II} \rightarrow \mathbf{I}$ and consider

$$(\lambda x.xx)R \rightarrow RR$$

$$(\lambda x.xx)R \rightarrow (\lambda x.xx)R'.$$

A common reduct would be $R'R'$, but this cannot be reached in one step from RR. Hence $\rightarrow$ does not satisfy the full diamond property (nor does its reflexive closure).

(ii) Suppose

$$M \rightarrow M_1, \qquad M \rightarrow M_2$$

in order to construct an M_3 such that

$$M_1 \twoheadrightarrow M_3, \qquad M_2 \twoheadrightarrow M_3.$$

Let $(\Delta_i): M \to M_i$, $i = 1,2$, with $\Delta_i \equiv (\lambda x_i . P_i) Q_i$. The possible relative positions of Δ_1 and Δ_2 in M are given in the following table.

(1) $\Delta_1 \cap \Delta_2 = \emptyset$ (i.e. Δ_1, Δ_2 disjoint)	
(2) $\Delta_1 = \Delta_2$	
(3) $\Delta_1 \subset \Delta_2$	(3.1) $\Delta_1 \subset P_2$
	(3.2) $\Delta_1 \subset Q_2$
(4) $\Delta_2 \subset \Delta_1$	(4.1) $\Delta_2 \subset P_1$
	(4.2) $\Delta_2 \subset Q_1$

Let $\Delta_i' \equiv P_i[x_i := Q_i]$.

Case 1. Then

$$M \equiv \cdots \Delta_1 \cdots \Delta_2 \cdots$$

$$M_1 \equiv \cdots \Delta_1' \cdots \Delta_2 \cdots$$

$$M_2 \equiv \cdots \Delta_1 \cdots \Delta_2' \cdots$$

Then take

$$M_3 \equiv \cdots \Delta_1' \cdots \Delta_2' \cdots .$$

Case 2. Then $M_1 \equiv M_2$ and we can take $M_3 \equiv M_1$.

Case (3.1). Then $M \equiv \cdots ((\lambda x_2 . \cdots \Delta_1 \cdots) Q_2) \cdots$, where $\cdots \Delta_1 \cdots \equiv P_2$,

$$M_1 \equiv \cdots ((\lambda x_2 . \cdots \Delta_1' \cdots) Q_2) \cdots$$

$$M_2 \equiv \cdots (\cdots \Delta_1 \cdots)[x_2 := Q_2] \cdots .$$

Take

$$M_3 \equiv \cdots (\cdots \Delta_1' \cdots)[x_2 := Q_2] \cdots .$$

Then clearly $M_1 \to M_3$ and $M_2 \to M_3$ by the substitutivity of $\boldsymbol{\beta}$.

Case (3.2). Then $M \equiv \cdots ((\lambda x_2 . P_2)(\cdots \Delta_1 \cdots)) \cdots$, where $\cdots \Delta_1 \cdots \equiv Q_2$,

$$M_1 \equiv \cdots ((\lambda x_2 . P_2)(\cdots \Delta_1' \cdots)) \cdots$$

$$M_2 \equiv \cdots (P_2[x_2 := (\cdots \Delta_1 \cdots)]) \cdots .$$

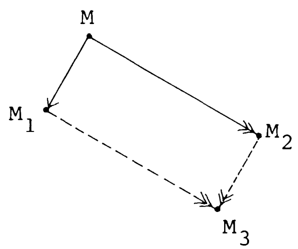
Take

$$M_3 \equiv \cdots (P_2[x_2 := (\cdots \Delta_1' \cdots)]),$$

then clearly $M_1 \to M_3$ and $M_2 \twoheadrightarrow M_3$ by remark 3.1.7 (i).

Cases (4.1) and (4.2) can be treated analogously to cases (3.1) and (3.2). □

Due to the existence of infinite reduction paths it does not automatically follow from lemma 11.1.1 that $\boldsymbol{\beta}$ is CR, i.e. that $\twoheadrightarrow$ satisfies the diamond lemma (see exercise 3.5.10). The diamond property for $\twoheadrightarrow$ does follow, however, from the following "strip lemma", which is a strengthening of lemma 11.1.1:



The idea of the proof of this lemma is as follows. Suppose $M \xrightarrow{\Delta} M_1$. If one keeps track of what happens with Δ during the reduction $M \twoheadrightarrow M_2$, then by reducing all "residuals" of Δ in M_2 one obtains M_3. In order to do the necessary bookkeeping, it is convenient to mark some redexes. For this it is sufficient to give an index to the first lambda of a redex. (For applications later on several indices will be allowed.) For these reasons the following auxiliary extension of Λ is introduced.

11.1.2. DEFINITION. (i) Λ' is a set of words over the following alphabet

$v_0, v_1, \ldots$	variables,
$\lambda, \lambda_0, \lambda_1, \ldots$	lambdas,
(,)	parentheses.

(ii) Λ' is inductively defined as follows.

$$x \in \Lambda',$$
$$M \in \Lambda' \Rightarrow (\lambda x.M) \in \Lambda',$$
$$M, N \in \Lambda' \Rightarrow (MN) \in \Lambda',$$
$$M, N \in \Lambda' \Rightarrow ((\lambda_i x.M)N) \in \Lambda' \quad \text{for all } i \in \mathbb{N}$$

(x denotes an arbitrary variable).

(iii) If $M \in \Lambda'$, then $|M| \in \Lambda$ is obtained from M by leaving out all indices. For example, $|(\lambda_1 x.x)((\lambda_2 x.x)(\lambda x.x))| \equiv \mathbf{I}(\mathbf{II})$.

The elements of Λ' are called λ'-terms. The same conventions are adopted as for λ-terms. The notion of reduction $\boldsymbol{\beta}$ is extended to $\boldsymbol{\beta}'$ on Λ' as follows.

11.1.3. DEFINITION. (i) Substitution on Λ' is defined in the obvious way. In particular

$$((\lambda_i x.M)N)[z:=L] \equiv (\lambda_i x.M[z:=L])(N[z:=L]).$$

(ii) The notion of reduction $\boldsymbol{\beta}'$ on Λ' is defined by $\boldsymbol{\beta}' = \boldsymbol{\beta}_0 \cup \boldsymbol{\beta}_1$ where $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}_1$ are defined by the following contraction rules:

$$\boldsymbol{\beta}_0 : (\lambda_i x.M)N \to M[x:=N],$$

$$\boldsymbol{\beta}_1 : (\lambda x.M)N \to M[x:=N],$$

where $i \in \mathbb{N}$ and $M, N \in \Lambda'$.

(iii) By remark 3.1.7(ii) the notion $\boldsymbol{\beta}'$ generates relations $\to_{\beta'}$ and $\twoheadrightarrow_{\beta'}$ on Λ': $M \to_{\beta'} N$ iff for some (indexed) context $C[\ \ ]$ with one hole and some $(P, Q) \in \boldsymbol{\beta}'$

$$M \equiv C[P] \quad \text{and} \quad N \equiv C[Q],$$

$\twoheadrightarrow_{\beta'}$ is the reflexive transitive closure of $\to_{\beta'}$.

In the next section the notion of reduction $\boldsymbol{\beta}_0$ will play an important role. For the purpose of this section the set Λ' and the notion $\boldsymbol{\beta}'$ could have been given simpler (using just one index).

11.1.4. DEFINITION. Let $M \in \Lambda'$. Define $\varphi(M) \in \Lambda$ by induction on the structure of M as follows:

$$\varphi(x) \equiv x,$$

$$\varphi(PQ) \equiv \varphi(P)\varphi(Q) \quad \text{if } P \not\equiv \lambda_i x.P',$$

$$\varphi(\lambda x.P) \equiv \lambda x.\varphi(P),$$

$$\varphi((\lambda_i x.P)Q) \equiv \varphi(P)[x:=\varphi(Q)].$$

In other words, φ contracts all the redexes with an index (from the inside to the outside; in section 11.2 it will be shown that other ways of contracting all the indexed redexes always lead to the same result).

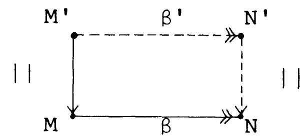
11.1.5. NOTATION. If $|M| \equiv N$ or $\varphi(M) \equiv N$, then this will be denoted by

$$M \underset{|\ |}{\to} N \quad \text{or} \quad M \underset{\varphi}{\to} N$$

respectively. This is convenient for a schematical formulation of statements.

11.1.6. LEMMA.

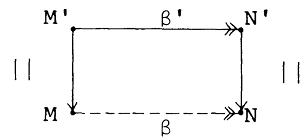
(i)



$M', N' \in \Lambda'$,

$M, N \in \Lambda$.

(ii)



$M', N' \in \Lambda'$,

$M, N \in \Lambda$.

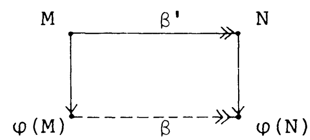
PROOF. (i) First suppose $M \twoheadrightarrow_\beta N$ is a one step reduction. Then N is obtained by contracting a redex in M and N' can be obtained by contracting the corresponding redex in M'. The general statement follows by transitivity.

(ii) Similar but easier: just leave out all indices from a reduction path from M' to N'. □

11.1.7. LEMMA. (i) *Let* $M, N \in \Lambda'$. *Then*

$$\varphi(M[x:=N]) \equiv \varphi(M)[x:=\varphi(N)].$$

(ii)

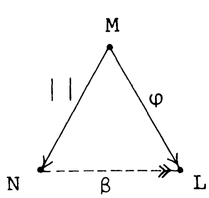


$M, N \in \Lambda'$.

PROOF. (i) By induction on the structure of M, using the substitution lemma in case $M \equiv (\lambda_i y.P)Q$. The conditions for the substitution lemma may be assumed to hold by the variable convention 2.1.13.

(ii) By induction on the generation of $\twoheadrightarrow_{\beta'}$, using (i). □

11.1.8. LEMMA.

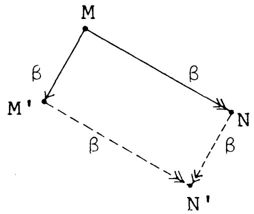


$M \in \Lambda'$,

$(N, L \in \Lambda)$.

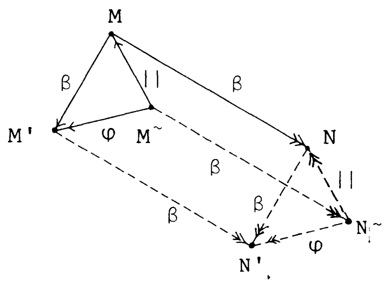
PROOF. By induction on the structure of M. □

11.1.9. STRIP LEMMA.



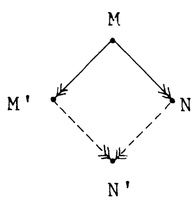
M,M',N,N' ∈ Λ.

PROOF. Let M' be the result of contracting the redex occurrence Δ in M. Let $\tilde{M} \in \Lambda'$ be obtained from M by indexing Δ. Then $|\tilde{M}| \equiv M$ and $\varphi(\tilde{M}) \equiv M'$. By lemmas 11.1.6 (i), 11.1.7 (ii) and 11.1.8 we can erect the following diagram



which proves the strip lemma. □

11.1.10. CHURCH–ROSSER THEOREM. ***β*** *is* CR. *That is,*

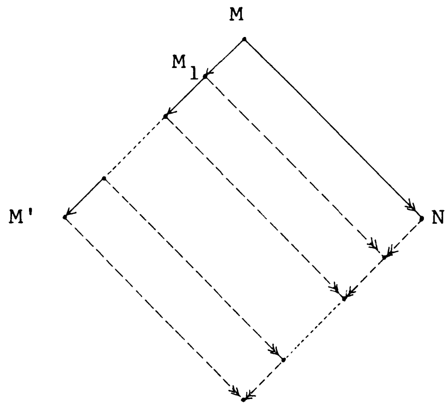


M,M',N,N' ∈ Λ.

PROOF. If $M \twoheadrightarrow M'$, then

$$M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \equiv M'.$$

Hence the diamond property follows from the strip lemma and a simple diagram chase:



□

In chapter 3 we have proved that $\boldsymbol{\beta\eta}$ is CR. It is instructive to notice that the proof given for $\boldsymbol{\beta}$ does not generalize immediately to $\boldsymbol{\beta\eta}$ (see exercise 11.5.4).

11.2. The finiteness of developments

The main theorem in this section states that for each $M \in \Lambda$ the so called developments (a special kind of reduction starting with M) are always finite. This theorem (denoted by FD) has important consequences, among them being the CR theorem, the conservation theorem for the restricted theory and the standardization theorem.

FD was first proved by Church and Rosser [1936] for the λI-calculus in order to prove the CR theorem for that theory. For the full λ-calculus FD was proved by Schroer [1965] and independently by Hyland [1973] and Hindley [1978]. The proof given below is taken from Barendregt et al. [1976] and is a simplification of that of Hyland.

One formulation of FD is simply

$$\mathrm{SN}(\boldsymbol{\beta}_0),$$

that is, reductions on Λ' contracting only indexed redexes are always finite. Often this theorem is formulated in terms of "residuals". It is convenient to introduce this terminology.

11.2.1. LEMMA (Projecting). *Let σ' be a $\boldsymbol{\beta}'$-reduction starting with $M' \in \Lambda'$, say*

$$\sigma' : M' \equiv M_0' \underset{\beta'}{\overset{\Delta_0'}{\to}} M_1' \underset{\beta'}{\overset{\Delta_1'}{\to}} \cdots .$$