PROOF. (i) By lemma 6.5.3.
(ii) As ever. □

Note that $\mathbf{Y}^1 \twoheadrightarrow \mathbf{\Theta}$.

6.5.6. DEFINITION (Gödel numbering). (i) It is easy to define, by standard techniques, an effective one-one map $\sharp : \Lambda \to \mathbb{N}$. In the rest of this book $\sharp$ denotes one such map.

(ii) $\sharp M$ is called the Gödel number of M.

The following convention will be used throughout:

6.5.7. CONVENTION. Notions about sets of integers, sequences of integers, etc., are translated to terms via the mapping $\sharp$.

EXAMPLES. (i) A set $\mathcal{A} \subset \Lambda$ is recursive iff $\sharp\mathcal{A} = \{\sharp M \mid M \in \mathcal{A}\}$ is recursive.

(ii) A sequence of terms $M_0, M_1, \ldots$ is recursive iff $\lambda\!\!\!\lambda n.\sharp M_n$ is a recursive sequence.

In fact the convention was already used in the proof of theorem 6.3.13 where it was shown that $\{(M, N) | \lambda \vdash M = N\}$ is an r.e. set.

6.5.8. NOTATION. $\ulcorner M \urcorner \equiv \ulcorner \sharp M \urcorner$.

There is little danger of confusion between the $\ulcorner\ \urcorner$ of definitions 6.2.9 and 6.5.8. The first is defined on $\mathbb{N}$, the second on Λ.

6.5.9. SECOND FIXED POINT THEOREM.

$$\forall F\ \exists X \quad F \ulcorner X \urcorner = X.$$

PROOF. By the effectiveness of $\sharp$, there are recursive functions Ap and Num such that Ap $(\sharp M, \sharp N) = \sharp MN$ and Num$(n) = \sharp \ulcorner n \urcorner$. Let Ap and Num be λ-defined by **Ap** and **Num** $\in \Lambda^0$. Then

$$\mathbf{Ap} \ulcorner M \urcorner \ulcorner N \urcorner = \ulcorner MN \urcorner, \qquad \mathbf{Num} \ulcorner n \urcorner = \ulcorner \ulcorner n \urcorner \urcorner;$$

hence in particular

$$\mathbf{Num} \ulcorner M \urcorner = \ulcorner \ulcorner M \urcorner \urcorner.$$

Now define

$$W \equiv \lambda x. F(\mathbf{Ap}\, x (\mathbf{Num}\, x)), \qquad X \equiv W \ulcorner W \urcorner.$$

Then

$$X \equiv W^{\ulcorner} W^{\urcorner} = F(\mathbf{Ap}\ulcorner W \urcorner (\mathbf{Num}\ulcorner W \urcorner))$$

$$= F\ulcorner W \ulcorner W \urcorner\urcorner \equiv F\ulcorner X \urcorner . \quad \square$$

As for ordinary fixed points one has $X \twoheadrightarrow F\ulcorner X \urcorner$. Moreover the construction is uniform in the following sense.

6.5.10. COROLLARY. $\exists \boldsymbol{\Theta}_2 \in \Lambda^0\ \forall F \in \Lambda^0\ \boldsymbol{\Theta}_2 \ulcorner F \urcorner \twoheadrightarrow F\ulcorner \boldsymbol{\Theta}_2 \ulcorner F \urcorner\urcorner$.

PROOF. See exercise 8.5.6. $\square$

6.6. Undecidability results

6.6.1. DEFINITION Let $\mathcal{A} \subset \Lambda$.

(i) $\mathcal{A}$ is *non-trivial* if $\mathcal{A} \neq \emptyset$, $\mathcal{A} \neq \Lambda$.

(ii) $\mathcal{A}$ is *closed under equality* if

$$\forall M, N \in \Lambda[\, M \in \mathcal{A} \text{ and } M = N \Rightarrow N \in \mathcal{A}].$$

Recall that two sets of integers $\mathcal{A}, \mathcal{B}$ are recursively separable iff there exists a recursive set $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \cap \mathcal{C} = \emptyset$.

The following theorem is due to Scott [1963]. It is related to Rice's theorem in recursion theory, cf. Rogers [1967], p. 324.

6.6.2. THEOREM. (i) Let $\mathcal{A}, \mathcal{B} \subset \Lambda$ be non empty sets closed under equality. Then $\mathcal{A}$ and $\mathcal{B}$ are not recursively separable.

(ii) Let $\mathcal{A} \subset \Lambda$ be a non-trivial set closed under equality. Then $\mathcal{A}$ is not recursive.

PROOF. (i) Let $M_0 \in \mathcal{A}$, $M_1 \in \mathcal{B}$. Suppose $\mathcal{C}$ is a recursive set such that $\mathcal{A} \subset \mathcal{C}, \mathcal{B} \cap \mathcal{C} = \emptyset$. The characteristic function of $\#\mathcal{C}$ is recursive and hence λ-defined by some F. Hence

$$M \in \mathcal{C} \Rightarrow F\ulcorner M \urcorner = \ulcorner 0 \urcorner,$$

$$M \notin \mathcal{C} \Rightarrow F\ulcorner M \urcorner = \ulcorner 1 \urcorner.$$

Now define

$$G \equiv \lambda x.\ \textit{If}\ \mathbf{Zero}(Fx)\ \textit{then}\ M_1\ \textit{else}\ M_0.$$

Then

$$M \in \mathcal{C} \Rightarrow G \ulcorner M \urcorner = M_1,$$

$$M \notin \mathcal{C} \Rightarrow G \ulcorner M \urcorner = M_0.$$

By the second fixed point theorem, $G \ulcorner X \urcorner = X$ for some X. But then

$$X \in \mathcal{C} \Rightarrow X = G \ulcorner X \urcorner = M_1 \in \mathcal{B} \Rightarrow X \notin \mathcal{C},$$

$$X \notin \mathcal{C} \Rightarrow X = G \ulcorner X \urcorner = M_0 \in \mathcal{A} \Rightarrow X \in \mathcal{C},$$

a contradiction.

(ii) If $\mathcal{A} \subset \Lambda$ is a nontrivial set closed under equality, then (i) applies to $\mathcal{A}$ and its complement. Hence $\mathcal{A}$ cannot be recursive. □

Theorem 6.6.2 is false for the λI-calculus. Take $\mathcal{A} = \{M \in \Lambda_I \mid FV(M) = \{x\}\}$. This set is recursive and in the λI-calculus closed under equality, see lemma 9.1.2(iv). The following relativization of theorem 6.6.2 also holds for the λI-calculus.

6.6.3. Definition. Let $\mathcal{A} \subset \Lambda^0$. $\mathcal{A}$ is *closed under equality of combinators* if

$$\forall M, N \in \Lambda^0 [M \in \mathcal{A} \text{ and } M = N \Rightarrow N \in \mathcal{A}].$$

6.6.4. Corollary. *Theorem* 6.6.2 *holds for* $\mathcal{A}, \mathcal{B} \subset \Lambda^0$ *which are closed under equality of combinators.*

Proof. Same as for theorem 6.6.2. □

Church [1936] gave one of the first examples of an r.e. set which is not recursive:

6.6.5. Theorem. $\{M \mid M$ *has a* nf$\}$ *is an* r.e. *set which is not recursive.*

Proof. The set is r.e. since

$$M \text{ has } a \text{ nf} \quad \text{iff} \quad \exists N\ N \text{ is in nf and } \lambda \vdash M = N.$$

By theorem 6.6.2 the set is not recursive. □

A theory $\mathcal{T}$ is called *essentially undecidable* iff $\mathcal{T}$ is consistent and has no consistent recursive extension. The following was noticed by Grzegorczyk.

6.6.6. THEOREM. *The* λ*-calculus (i.e. the theory* $\boldsymbol{\lambda}$*) is essentially undecidable.*

PROOF. Suppose $\mathfrak{T}$ is a consistent extension of $\boldsymbol{\lambda}$. Then $\{M \mid \mathfrak{T} \vdash M = I\}$ is closed under equality and nontrivial. Hence by theorem 6.6.2 not recursive. Therefore $\mathfrak{T}$ is not recursive. □

6.7 Digression: Self-referential sentences and the recursion theorem

To conclude this section it will be shown how the constructions of Gödel's self-referential sentence and the recursion theorem can be interpreted as applications of the fixed point combinator.

Let P be first-order Peano arithmetic. If s is a syntactic object of P, then $\sharp s$ is its Gödel number ("code") and $\ulcorner s \urcorner \equiv \ulcorner \sharp s \urcorner$ the corresponding numeral in P.

I. THEOREM (Gödel). *Let* $A(x)$ *be a formula of* P *with* $\mathrm{FV}(A(x)) = \{x\}$. *Then there exists a sentence* B *of* P *such that*

$$P \vdash B \leftrightarrow A(\ulcorner B \urcorner).$$

(*B says: "I have property A."*)

PROOF. Define for $n_1, n_2 \in N$

$$n_1 \sim n_2 \quad \text{iff} \quad \text{for some sentences } A_1, A_2 \text{ of } P \quad n_i = \sharp A_i \text{ and } P \vdash A_1 \leftrightarrow A_2.$$

It is sufficient to show that given $A(x)$ one has

$$(1) \qquad \exists n \quad n \sim \sharp A(\ulcorner n \urcorner).$$

For then, taking B such that $n = \sharp B$,

$$P \vdash B \leftrightarrow A(\ulcorner n \urcorner) = A(\ulcorner B \urcorner).$$

In order to prove (1) make the following abbreviations.

(i) For a variable x and $n \in \mathbb{N}$

$$\lambda x.n = \langle \sharp x, n \rangle,$$

where $\langle\ ,\ \rangle$ is a primitive recursive pairing function with inverses $(\ )_0$, $(\ )_1$.