Self-evaluation in Lambda Calculus

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3.1. PROPOSITION. *There is no $Q \in \Lambda$ such that $QM = \ulcorner M \urcorner$.*

PROOF. By the effectiveness of $\#$ there exists a lambda term $F_{\text{app?}}$ such that

$$\begin{array}{rcl} F_{\text{app?}} \ulcorner x \urcorner & = & \mathbf{false} \\ F_{\text{app?}} \ulcorner PQ \urcorner & = & \mathbf{true} \\ F_{\text{app?}} \ulcorner \lambda x.P \urcorner & = & \mathbf{false} \end{array}$$

Suppose such a Q exists. Then one has

$$\begin{array}{rcl} \mathsf{I}x = x & \Rightarrow & Q(\mathsf{I}x) = Qx \\ & \Rightarrow & \ulcorner \mathsf{I}x \urcorner = \ulcorner x \urcorner \\ & \Rightarrow & F_{\text{app?}} \ulcorner \mathsf{I}x \urcorner = F_{\text{app?}}(\ulcorner x \urcorner) \\ & \Rightarrow & \mathbf{true} = \mathbf{false}, \end{array}$$

contradicting the Church-Rosser theorem. Hence Q doesn't exist. ■

3.2. Definition. $\Lambda^{\emptyset} = \{M \in \Lambda \mid \mathrm{FV}(M) = \emptyset\}$.

3.3. Theorem. *There exists a term* $\mathsf{E} \in \Lambda^{\emptyset}$ *such that for all* $M \in \Lambda^{\emptyset}$

$$\mathsf{E}\ulcorner M \urcorner = M. \text{ ``\textit{Lambda calculus understands its own meaning.}''}$$

Proof. Let $\rho \in \Lambda$. We think of it as a 'valuation', a map that assigns to variables x_i a lambda term $\rho(x_i) \triangleq \rho\,\mathbf{c}_i$. As usual in model theory such valuations may need modification. Define

$$\rho[x_i \mapsto X],$$

as the $\rho' \in \Lambda$ defined by

$$\begin{aligned} \rho'(x_i) &= X \\ \rho'(x_j) &= \rho(x_j), \qquad \text{if } i \neq j. \end{aligned}$$

Note that there exists a term H such that

$$H\mathbf{c}_i \rho X = \rho[x_i \mapsto X].$$

There exists a $\mathsf{D}\in\Lambda^{\emptyset}$ satisfying

$$\begin{aligned} \mathsf{D}\,\rho\ulcorner x_i\urcorner &= \rho(x_i) \\ \mathsf{D}\,\rho\ulcorner PQ\urcorner &= (\mathsf{D}\,\rho\ulcorner P\urcorner)(\mathsf{D}\,\rho\ulcorner Q\urcorner) \\ \mathsf{D}\,\rho\ulcorner \lambda x_i.P\urcorner &= \lambda y.(\mathsf{D}\,\rho[x_i \mapsto y]\ulcorner P\urcorner). \end{aligned}$$

By induction on $P = P(x_1, \cdots, x_k)\in\Lambda$ one can show that

$$\mathsf{D}\,\rho\ulcorner P\urcorner = P[x_1 := \rho(x_1), \cdots, x_k := \rho(x_k)]$$

Here $P[x_1 := \rho(x_1), \cdots, x_k := \rho(x_k)]$ denotes simultaneous substitution.

Now define $\mathsf{E} \triangleq \mathsf{D}\,\rho_0$, with $\rho_0 = \mathsf{K}\,\mathsf{I}$. This E is in $\Lambda^{\emptyset}$.

But then for $P\in\Lambda^{\emptyset}$ one has

$$\mathsf{E}\ulcorner P\urcorner = \mathsf{D}\,\rho_0\ulcorner P\urcorner = P. \ \blacksquare$$

1. Define for combinatory terms built up from variables, I, and K an 'effective' quoting function.

2. Define the analogon E_{CL} for combinatory terms.

3. Show that (terms range over $\Lambda^{\emptyset}$)

$$\begin{array}{lcrcl}
\exists F \forall X & & F\ulcorner X \urcorner & = & XX \\
\forall F \exists X & & F\ulcorner X \urcorner & = & XX \\
\exists F \forall X & & F\ulcorner X \urcorner & = & \ulcorner X \urcorner X \\
\forall F \exists X & & F\ulcorner X \urcorner & = & X\ulcorner X \urcorner
\end{array}$$

4. Add to the lambda calculus constants δ, ϵ and the reduction rule $\delta XX \to \epsilon$

Show that the resulting reduction relation satisfies the weak CR property:

$$[M \to N_1 \ \& \ M \to N_2] \Rightarrow \exists L.[N_1 \twoheadrightarrow L \ \& \ N_2 \twoheadrightarrow L]$$

Klop [1980] shows that it does not satisfy the CR property.

Exercises (cont'nd)

Give types to $\lambda xy.xxy$, $\lambda xy.yxx$.