The Arithmetical Hierarchy

Extra Material

∀, ∃ rules

1.1. Proposition.

$$
\begin{aligned}
\neg\forall &\Leftrightarrow \exists\neg \\
\neg\exists &\Leftrightarrow \forall\neg \\
(\forall x.P) \vee (\forall y.Q) &\Leftrightarrow \forall x \forall y.[P(x) \vee Q(y)] \\
(\exists x.P) \vee (\exists y.Q) &\Leftrightarrow \exists x \exists y.[P(x) \vee Q(y)]
\end{aligned}
$$

Beware

$$
\begin{aligned}
[\forall x.P(x)] \;\Rightarrow\; [\forall x Q(x)] &\Leftrightarrow \neg\forall y.P(y) \vee \forall y.Q(y) \\
&\Leftrightarrow \exists x.\neg P(x) \vee \forall y.Q(y)
\end{aligned}
$$

$\forall, \exists$ rules in arithmetic

1.2. Proposition.

$$
\begin{aligned}
&1.\ \forall\forall && \Leftrightarrow && \forall\\
&2.\ \exists\exists && \Leftrightarrow && \exists\\
&3.\ \forall x{<}n\exists y && \Leftrightarrow && \exists y\forall x{<}n\\
&4.\ \exists x{<}n\forall y && \Leftrightarrow && \forall y\exists x{<}n
\end{aligned}
$$

Proof.

$$
\begin{aligned}
1.\ \forall x\forall y.R(e,x,y) &\Leftrightarrow \forall z.R(e,(z)_0,(z)_1)\\
2.\ \exists x\exists y.R(e,x,y) &\Leftrightarrow \exists z.R(e,(z)_0,(z)_1)\\
3.\ \forall x{<}n\exists y.R(e,x,y) &\Leftrightarrow \exists y_0,\ldots,\exists y_{n-1}\forall x{<}n.R(e,x,y_x)\\
&\Leftrightarrow \exists y\forall x{<}n.R(e,x,(y)_x)\\
4.\ \exists x{<}n\forall y &\Leftrightarrow \neg\neg\exists x{<}n\forall y\\
&\Leftrightarrow \neg\forall x{<}n\exists y\neg\\
&\Leftrightarrow \neg\exists y\forall x{<}n\neg\\
&\Leftrightarrow \forall y\exists x{<}n\neg\neg\\
&\Leftrightarrow \forall y\exists x{<}n \quad \blacksquare
\end{aligned}
$$

1.3. PROPOSITION. *Let* $R(x_1, \ldots, x_n)$ *and* $f_1, \ldots, f_n$ *computable*

Define

$$Q(y_1, \ldots, y_m) \overset{\triangle}{\iff} R(f_1(\vec{y}), \ldots, f_n(\vec{y})).$$

Then

$$R \in \Pi^0_k \quad \Rightarrow \quad Q \in \Pi^0_k$$

$$R \in \Sigma^0_k \quad \Rightarrow \quad Q \in \Sigma^0_k$$

PROOF. Let $R(\vec{x}) \Leftrightarrow \forall z_1 \exists z_2 \ldots .P(\vec{x}, \vec{z})$. Then

$$Q(\vec{y}) \Leftrightarrow R(f_1(\vec{y}), \ldots, f_n(\vec{y})) \Leftrightarrow \forall z_1 \exists z_2 \ldots .P(f_1(\vec{y}), \ldots, f_n(\vec{y}), \vec{z})$$

which is in the same class as R. ■

1.4. LEMMA. $A \leq_m B \in \Pi^0_k \Rightarrow A \in \Pi^0_k$

Similarly for the Σ*-classes.*

Disjoint union

1.5. LEMMA. *Define* $A \cup^* B = \{< a, 0 >|\ a \in A\} \cup \{< b, 1 >|\ b \in B\}$.

Note that $A \leq_m A \cup^* B$ and $B \leq_m A \cup^* B$.

From this one shows $(K \cup^* \overline{K}) \in \Delta^0_2$ but not in Π^0_1 or in Σ^0_1.

Similarly, $(K_n \cup^* \overline{K}_n) \in \Delta^0_{n+1}$ but not lower.